Problem Set #2
Due: Thursday, 20 September 2012

Students registered in MATH 401 should submit solutions to three of the following problems. Students in MATH 801 should submit solutions to all five.

1. For a positive integer $n$, the **boolean lattice** $BL_n$ is the graph whose vertex set is the set of all subsets of $\{1, 2, \ldots, n\}$ where two subsets are adjacent if their **symmetric difference** has precisely one element. Show that the $n$-cube $Q_n$ and the boolean lattice $BL_n$ are isomorphic.

2. For positive integers $m$ and $n$, the **Kneser graph** $KG_{n,m}$ has vertex set consisting of the $m$-subset of $\{1, \ldots, n\}$ and two vertices are adjacent if the corresponding subsets are disjoint.
   (a) Show that $KG_{n,1} \cong K_n$.
   (b) Show that $KG_{n,2}$ is isomorphic to the complement of line graph $L(K_n)$.

3. The **Desargues set system** $(V, \mathcal{F})$ has $V := \{0, 1, \ldots, 9\}$ and
   $$ \mathcal{F} := \{\{0, 4, 7\}, \{0, 5, 8\}, \{0, 6, 9\}, \{1, 2, 3\}, \{1, 4, 5\}, \{1, 7, 8\}, \{2, 4, 6\}, \{2, 7, 9\}, \{3, 5, 6\}, \{3, 8, 9\}\}.$$
   Show that the following three graphs are isomorphic:
   - the intersection graph of the Desargues set system,
   - the line graph of $K_5$,
   - the complement of the Petersen graph.

4. (a) Let $G$ be a graph satisfying $e(G) > \binom{v(G)-1}{2}$. Prove that $G$ is connected.
   (b) For any positive integer $n$, find a disconnected graph $G$ such that $v(G) = n$ and $e(G) = \binom{n-1}{2}$.

5. (a) Let $G$ be a graph with minimum degree $\delta(G)$. If $\delta(G) > \frac{1}{2}(v(G) - 2)$, then show that $G$ is connected.
   (b) For any positive even number $n$, find a disconnected $\frac{1}{2}(n - 2)$-regular graph $G$ such that $v(G) = n$.

**Hint.** It suffices to show that every vertex not adjacent to one with minimal degree has a common neighbour.