1 Counting Techniques

We introduce the two types of combinatorial proofs. The first, called *double counting*, shows that two expressions are equal by demonstrating that they are just different ways of counting the elements in a single set. The second technique, called *a bijective proof*, proves that two sets have the same cardinality by exhibiting a one-to-one correspondence between them. Even with their apparent simplicity, these methods often lead to more elegant arguments and greater insight. To showcase the benefits, we assemble examples involving domino tilings, permutations, and triangulations.

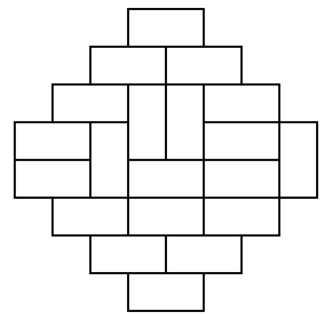

Figure 1.1: Domino tiling of an Aztec diamond

1.0 Domino Tilings

We present a class of combinatorial models that arises frequently in statistical mechanics. A ***domino*** is a polygon made from two unit squares joined edge-to-edge. When rotations are considered distinct, there are two fixed dominoes: the (1×2)-rectangle is the horizontal domino and the (2×1)-rectangle is the vertical domino.

Definition 1.0.1. A ***domino tiling*** of a planar region is a covering (with no overlaps or gaps) by dominos.

Theorem 1.0.2 (Fibonacci tiling model)**.** *For any nonnegative integer n, the Fibonacci number F_{n+1} is the number of the domino tilings of a $(2 \times n)$-rectangle.*

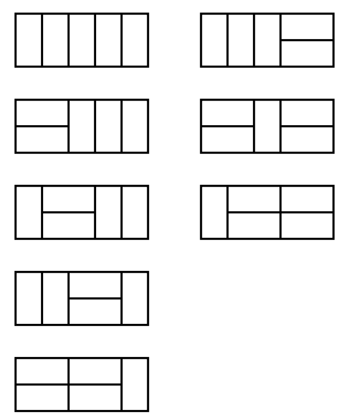

Figure 1.2: The 8 domino tilings of a (2×5)-rectangle

Inductive proof. Since no dominos provide the unique covering a (2×0)-rectangle and a vertical domino gives the unique covering a (2×1)-rectangle, the base cases hold because $F_1 = 1$ and $F_2 = 1$. Assume that, for all nonnegative integers k such that $k < n + 1$, the number of domino tilings of a $(2 \times k)$-rectangle is F_{k+1}. For the induction step, focus on the right edge of the $(2 \times n)$-rectangle. There are two possible configurations.

- When a vertical domino covers the right edge, the induction hypothesis shows that the complementary $\big(2 \times (n-1)\big)$-rectangle can be tiled in F_n ways.
- When two horizontal tiles cover the right edge, the induction hypothesis shows that the complementary $\big(2 \times (n-2)\big)$-rectangle can be tiled in F_{n-1} ways.

Using the Fibonacci recurrence, we see that the total number of domino tilings of a $(2 \times n)$-rectangle is $F_n + F_{n-1} = F_{n+1}$. □

Using this combinatorial model, we give a more attractive proof for an earlier identity; see Problem 0.3.2.

Problem 1.0.3. For any nonnegative integer n, demonstrate that

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1\,.$$

Double-Counting solution. On a $\big(2 \times (n+1)\big)$-rectangle, how many domino tilings use at least one horizontal domino?

Answer 1: Applying the Fibonacci tiling model, there are F_{n+2} domino tilings of the $\big(2 \times (n+1)\big)$-rectangle. Excluding the tiling having all vertical dominos leaves $F_{n+2} - 1$ tilings with at least one horizontal domino.

Answer 2: Focus on the pair of horizontal dominos closest to the right edge. For some $1 \leqslant k \leqslant n$, this pair is $k-1$ units from the left edge of the rectangle and $n-k$ units from the right. The Fibonacci tiling model establishes that there are F_k tilings of the left $\big(2 \times (k-1)\big)$-rectangle; the right $\big(2 \times (n-k)\big)$-rectangle must be tiled by vertical dominos. Since we have $F_0 = 0$, the total number of tilings with at least one horizontal domino is

$$F_0 + \sum_{k=1}^{n} F_k = \sum_{k=0}^{n} F_k = F_0 + F_1 + \cdots + F_n\,. \qquad \square$$

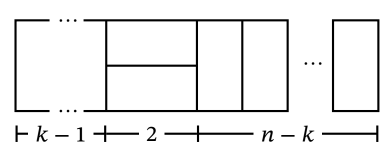

Figure 1.3: Rightmost pair of horizontal dominos

Each feature of a domino tiling leads to a combinatorial identity. For instance, a domino tiling of a $(2 \times n)$-rectangle is called ***m-decomposable***, for some $0 \leqslant m \leqslant n$, if it is the union of two tilings: one covering the left $(2 \times m)$-subrectangle and the other covering the right $\big(2 \times (n-m)\big)$-subrectangle. Hence, a domino tiling is not m-decomposable precisely when there is a horizontal domino that is $m-1$ units from the left edge of the rectangle.

Problem 1.0.4. For all nonnegative integers m and n, prove that

$$F_{m+n+1} = F_{m+1}\,F_{n+1} + F_m\,F_n\,.$$

Double-Counting Solution. On a $\big(2 \times (m+n)\big)$-rectangle, how many domino tilings are there?

Answer 1: The Fibonacci tiling model implies that there are F_{m+n+1} such domino tilings.

Answer 2: Focusing on m-decomposability gives 2 possibilities.

- If the tiling is m-decomposable, then it is the union of a tiling a $(2 \times m)$-rectangle and a tiling of a $(2 \times n)$-rectangle. The Fibonacci tiling model shows that the first rectangle has F_{m+1} tilings and the second has F_{n+1}, so there are $F_{m+1}\,F_{n+1}$ decomposable tilings.
- If the tiling is not m-decomposable, then it has a pair of horizontal dominos that are $m-1$ units from the left edge and $n-1$ units from the right. In this case, the Fibonacci tiling

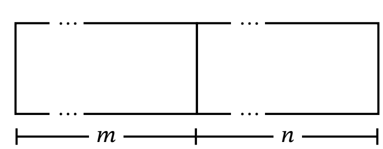

Figure 1.4: m-decomposable

model shows that there are F_m tilings of the $\big(2 \times (m-1)\big)$-rectangle and F_n tilings of the $\big(2 \times (n-1)\big)$-rectangle, which means that there are $F_m F_n$ indecomposable tilings.

Therefore, we have a total of $F_{m+1} F_{n+1} + F_m F_n$ tilings of an $\big(1 \times (m+n)\big)$-rectangle. □

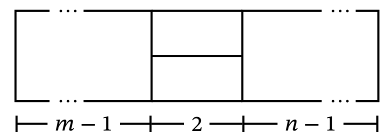

Figure 1.5: m-indecomposable

Sometimes it is more convenient to interpret the sides of an identity as the cardinality of different sets.

Problem 1.0.5. For any nonnegative integer n, established that $F_{2n+2} = F_{n+2}^2 - F_n^2$.

Bijective Solution.

Set 1: Consider the set of all domino tilings of a $\big(2 \times (2n+1)\big)$-rectangle. The Fibonacci tiling model shows that there are F_{2n+2} such domino tilings.

Set 2: Consider pairs of tilings of a $\big(2 \times (n+1)\big)$-rectangle where at least one has a vertical domino adjacent to its right edge. Since there are F_{n+2}^2 pairs (with no condition) and F_n^2 pairs having a horizontal domino covering the right edge in both, there are $F_{n+2}^2 - F_n^2$ pairs having the desired form.

Correspondence: The map sending a single tiling S to the pair (T_1, T_2) is a piecewise function. If S is n-decomposable, then the first tiling T_1 is the left $(2 \times n)$-rectangle in S with a vertical domino adjoined on the right and the second tiling T_2 is the complementary right $\big(2 \times (n+1)\big)$-rectangle in S. If S is not n-decomposable, then T_1 is the left $\big(2 \times (n+1)\big)$-rectangle in S and T_2 is the complementary right $(2 \times n)$-rectangle in S with a vertical domino adjoined on the right. Conversely, if T_1 has a vertical domino covering its right edge, then we join the left $(2 \times n)$-rectangle in T_1 with T_2. If T_1 does not have a vertical domino covering its right edge, then T_2 does and we join T_1 with the left $(2 \times n)$-rectangle in T_2. By construction, these operations are mutual inverses.

Since there is a bijection between the given sets, we conclude that $F_{2n+2} = F_{n+2}^2 - F_n^2$. □

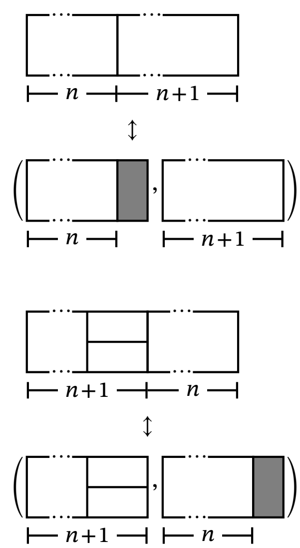

Figure 1.6: Correspondence for the identity $F_{2n+2} = F_{n+2}^2 - F_n^2$

Exercises

Problem 1.0.6. For all $n \in \mathbb{N}$, demonstrate that the Fibonacci number F_{n+2} is equal to each the following:

(i) the number of binary n-tuples having no consecutive 0's;
(ii) the number of subsets of the set $[n] := \{1, 2, \dots, n\}$ that contain no consecutive integers.

Problem 1.0.7. For all $n \in \mathbb{N}$, demonstrate that the Fibonacci number F_{n+1} is equal to each the following:

(i) the number of matchings in the path graph P_n;
(ii) the number of perfect matchings in the graph $P_2 \times P_n$.

Problem 1.0.8. For all $m, n \in \mathbb{N}$, exhibit a bijection between the domino tilings of the $(m \times n)$-rectangle and the perfect matchings in the grid graph $P_m \times P_n$.

Problem 1.0.9. For all $n \in \mathbb{N}$, use double-counting arguments to verify the following identities among Fibonacci numbers:
(i) $F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$,
(ii) $F_2 + F_4 + F_6 + \cdots + F_{2n+2} = F_{2n+3} - 1$,
(iii) $F_0^2 + F_1^2 + F_2^2 + \cdots + F_{n+2}^2 = F_{n+2} F_{n+3}$,
(iv) $F_n^2 + F_{n+1}^2 = F_{2n+1}$,
(v) $2(F_0^2 + F_2^2 + \cdots + F_n^2) + F_{n+1}^2 = F_{2n+2}$.

Problem 1.0.10. For all $n \in \mathbb{N}$, use a bijective argument to verify that $F_{2n+3} = F_{n+2} F_{n+3} - F_n F_{n+1}$.

Problem 1.0.11. By definition, the ***Jacobsthal numbers*** satisfy $J_0 := 0, J_1 := 1$, and $J_n := J_{n-1} + 2 J_{n-2}$ for all $n \geqslant 2$. Hence, this sequence begins with $0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, \ldots$. For all $n \in \mathbb{N}$, demonstrate that J_{n+1} is the number of tilings of an $(3 \times n)$-rectangle with (1×1)- and (2×2)-square tiles.

1.1 Permutations

Permutations arise in almost every branch of mathematics and in many other fields of science. There are at least three distinct ways of thinking about the permutations of the set $[n] := \{1, 2, \ldots, n\}$.

The permutations of $[n]$ are functions that can be composed with each other, forming the symmetric group $\mathfrak{S}_n$.

Definition 1.1.1. A ***permutation*** of the set $[n]$ is a bijective map from $[n]$ to itself. Each permutation of $[n]$ is also identified with
- an arrangment of the n distinct elements in the set $[n]$, and
- a directed graph with n labelled vertices such that every vertex is the head of one arrow and the tail of one arrow.

The ***two-line notation*** for a permutation is a $(2 \times n)$-table thats lists the elements of the set $[n]$ in the first row and the corresponding image in the second:

$$\sigma := \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}.$$

The ***one-line notation*** for permutation just records the second row: $\sigma = \sigma(1) \;\; \sigma(2) \;\; \sigma(3) \;\; \cdots \;\; \sigma(n)$. The direct graph has an arrow from the vertex i to vertex $\sigma(i)$ for all $1 \leqslant i \leqslant n$. Since the function $\sigma : [n] \to [n]$ is surjective, every vertex is the head of one arrow.

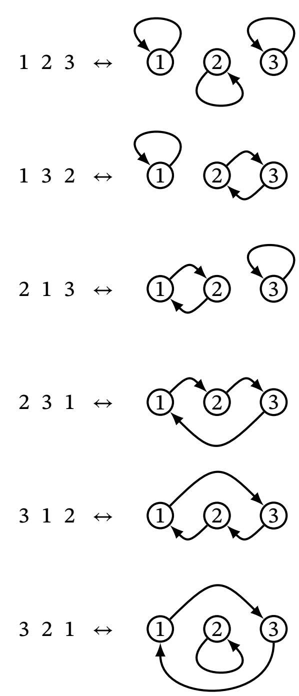

Figure 1.7: The 6 permutations of the set [3]

Notation 1.1.2. For any nonnegative integer n, the ***factorial*** of n is

$$n! := n(n-1)(n-2)\cdots(2)(1) = \prod_{j=1}^{n} k.$$

This sequence begins $1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \ldots$.

Proposition 1.1.3. *For any nonnegative integer n, the number of permutations of the set $[n]$ is $n!$.*

Inductive proof. Since there is a unique map from the empty set to any other set, there is a unique permutation of $[0] = \varnothing$, so the base case holds. For all $k < n$, assume that number of permutations of $[k]$ is $k!$. When constructing a permutation of $[n]$, there are n choice for the first element in the order. By the induction hypothesis, there are $(n-1)!$ ways to order the complementary set of $n-1$ elements. Therefore, the total number of permutations is $n\big((n-1)!\big) = n!$. □

Problem 1.1.4. For any nonnegative integer n, demonstrate that

$$\sum_{k=0}^{n} (k)(k!) = (n+1)! - 1\,.$$

Double-Counting Solution. How many permutations of the finite set $[n+1]$ are there when we exclude the identity $1\ 2\ 3\ \cdots\ n+1$?

Answer 1: Since there are $(n+1)!$ permutations of the set $[n+1]$, excluding one gives $(n+1)! - 1$ having the desired form.

Answer 2: Focus on the first number not in its natural position. For some $1 \leqslant k \leqslant n$, how many permutations have $n-k+1$ as the first number to differ from its canonical position? Such a permutation begins $1\ 2\ 3\ \cdots\ n-k$ and is followed by selecting one of the k numbers from the set $\{n-k+2, n-k+3, \dots, n+1\}$ The remaining k number, now including the number $n-k+1$, can be arranged in $k!$ ways. Thus, there are $(k)(k!)$ ways for $n-k+1$ to be the first misplaced number. Summing over all feasible values of k yields

$$\sum_{k=1}^{n} (k)(k!) = \sum_{k=0}^{n} (k)(k!)\,.$$ □

$k=1$	$k=2$	$k=3$
1 2 4 3		
	1 3 2 4	
	1 3 4 2	
	1 4 2 3	
	1 4 3 2	
		2 1 3 4
		2 1 4 3
		2 3 1 4
		2 3 4 1
		2 4 1 3
		2 4 3 1
		3 1 2 4
		3 1 4 2
		3 2 1 4
		3 2 4 1
		3 4 1 2
		3 4 2 1
		4 1 2 3
		4 1 3 2
		4 2 1 3
		4 2 3 1
		4 3 1 2
		4 3 2 1

Figure 1.8: The permutations of the set $[3+1]$ excluding $1\ 2\ 3\ 4$

Exercises

Problem 1.1.5. For all $n \in \mathbb{N}$, the ***double factorial*** is

$$n!! := \prod_{k=0}^{\lceil n/2 \rceil - 1} (n - 2k)\,.$$

This sequence begins with $1, 1, 2, 3, 8, 15, 48, 105, 384, 945, \dots$.

(i) For all $n \in \mathbb{N}$, show that $(2n)!!$ is the number of ways to seat n couples in a row such that everyone is next to their partner.

(ii) For all $n \in \mathbb{N}$, use a double-counting argument to prove

$$\sum_{j=0}^{n} (2j+1)(2j)!! = (2n+2)!! - 1\,.$$

Problem 1.1.6. The ***permanent*** of a square matrix is defined by the same expansion as the determinant except that each term of the permanent is given the plus sign (unlike the determinant alternates signs). Specifically, if $A = [a_{j,k}]$ is an $(n \times n)$-matrix, then we have

$$\begin{aligned} \det(A) &:= \sum_{\sigma \in \mathfrak{S}_n} \left(\operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)} \right), \\ \operatorname{per}(A) &:= \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{n} a_{j,\sigma(j)} . \end{aligned}$$

For any positive integer n, compute the determinant and permanent of the matrix

$$B := \begin{bmatrix} (1)(1) & (1)(2) & (1)(3) & \cdots & (1)(n) \\ (2)(1) & (2)(2) & (2)(3) & \cdots & (2)(n) \\ (3)(1) & (3)(2) & (3)(3) & \cdots & (3)(n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n)(1) & (n)(2) & (n)(3) & \cdots & (n)(n) \end{bmatrix}$$

1.2 Catalan Numbers

A ***polygon*** is a plane figure described by a finite number of line segments (whose interiors do not intersect) that are connected to form a simple closed curve. The line segments in this piecewise-linear curve are the ***edges*** of the polygon and the points where two edges meet are its ***vertices***. A polygon with n vertices is called an ***n-gon***. A polygon is ***convex*** if the line segment joining any two points on the curve lies in the interior of the simple closed curve. A ***diagonal*** is a line segment joining two vertices that are not on the same edge.

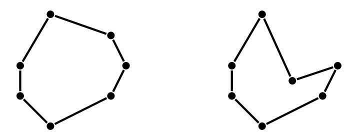

Figure 1.9: A convex and non-convex 7-gon

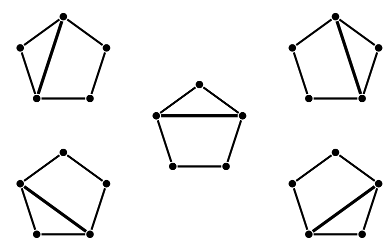

Figure 1.10: The 5 diagonals in a convex 5-gon

Problem 1.2.1. For any nonnegative integer n, prove that a convex $(n+3)$-gon has $(n+3)n/2$ distinct diagonals.

Solution. Each diagonal has two distinct vertices. There are $n+3$ choices for a first vertex. By avoiding the original choice and its neighbours, there $(n+3)-3$ choices for the second. Since there are 2 ways to order the vertices, we conclude that there are $(n+3)n/2$ distinct diagonals. □

A ***triangulation*** of a convex n-gon is a set of $n-3$ diagonal with disjoint interiors. These diagonals subdivide the interior of the polygon into $n-2$ triangles.

Definition 1.2.2. For all $n \in \mathbb{N}$, the ***Catalan number*** C_n counts the triangulations of a convex $(n+2)$-gon. When $n = 0$, we declare that the line segment has one triangulation. This sequence begins $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, \ldots$.

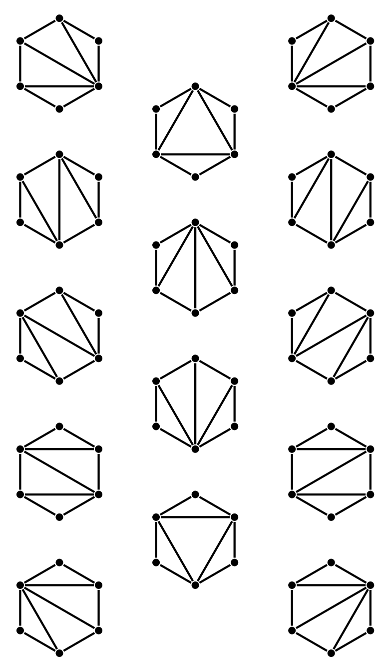

Figure 1.11: The 14 triangulations of a convex 5-gon

Proposition 1.2.3 (Catalan Ratio). *For any nonnegative integer n, the Catalan numbers satisfy $2(2n+1)C_n = (n+2)C_{n+1}$ or equivalently*

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}\,.$$

Bijective Proof.

Set 1: Given a triangulation of the $(n+2)$-gon, choose and orient one of its $(n+2)+(n-1) = 2n+1$ line segments (edges and diagonals). The definition of the Catalan numbers shows that there are $2(2n+1)C_n$ triangulations with an oriented line segment.

Set 2: Consider an $(n+3)$-gon to be resting on one of its edges called the 'base'. Given a triangulation of the $(n+3)$-gon, we mark one of the edges other than the base. From the definition of the Catalan numbers, we see that there are $(n+2)C_{n+1}$ such marked triangulations.

Correspondence: Collapsing the marked side in a triangulated $(n+3)$-gon and orienting the remaining edge with an arrow pointing towards the vertices that were identified produces a triangulation of an $(n+2)$-gon with an oriented line segment. For the opposite direction, expanding the oriented edge in $(n+2)$-gon into a triangle (by doubling the vertices at the head) and marking its new edge (joining the duplicated vertices) gives a marked triangulation of an $(n+3)$-gon; the new edge is transverse to the base. By construction, these maps compose, in either order, to the identity.

Since there is a bijection between the given sets, we conclude that $2(2n+1)C_n = (n+2)C_{n+1}$. □

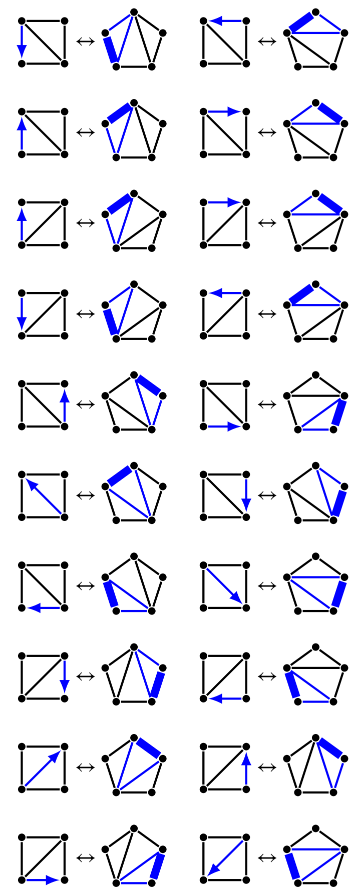

Figure 1.12: Catalan ratio correspondence for $n = 2$

This immediately gives a closed formula for these numbers.

Corollary 1.2.4. *For any nonnegative integer n, the n-th Catalan number is*

$$C_n = \prod_{k=2}^{n} \frac{n+k}{k} = \frac{(2n)!}{n!(n+1)!}\,.$$

Proof. The Catalan ratio gives

$$\begin{aligned}
C_n &= \left(\frac{C_n}{C_{n-1}}\right)\left(\frac{C_{n-1}}{C_{n-2}}\right)\cdots\left(\frac{C_1}{C_0}\right)C_0 \\
&= \left(\frac{2(2n-1)}{n+1}\right)\left(\frac{2(2n-3)}{n}\right)\cdots\left(\frac{2(1)}{2}\right)(1) \\
&= \left(\frac{2n(2n-1)}{n(n+1)}\right)\left(\frac{2(n-1)(2n-3)}{(n-1)n}\right)\cdots\left(\frac{2(1)(1)}{(1)(2)}\right) \\
&= \frac{(2n)(2n-1)\cdots(1)}{(n)(n-1)\cdots(1)(n+1)(n)\cdots(1)} = \frac{(2n)!}{n!(n+1)!} \\
&= \frac{(2n)(2n-1)\cdots(n+2)}{(n)(n-1)\cdots(2)} = \prod_{k=2}^{n}\frac{n+k}{k}\,.
\end{aligned}$$

□

Proposition 1.2.5 (Catalan Recurrence)**.** *For any nonnegative integer n, we have*

$$C_{n+1} = \sum_{k=0}^{n} C_k \, C_{n-k} \, .$$

Bijective Proof. To orient the edges in a polygon, assume that we traverse the boundary counterclockwise.

Set 1: The definition of the Catalan numbers implies that there are C_{n+1} triangulations of a convex $(n+3)$-gon.

Set 2: For all $0 \leqslant k \leqslant n$, consider a pair of triangulated convex polygons where the first has $(k+2)$ vertices and the second has $(n-k+2)$-vertices. It follows that the total number of such pairs is the sum $\sum_{k=0}^{n} C_k \, C_{n-k}$.

Correspondence: Removing an edge e from a triangulated convex $(n+3)$-gon produces two smaller triangulated polygons that share a common vertex. To order the doubleton, assume that the second polygon contains the edge of the $(n+3)$-gon preceding e in the counterclockwise order. For some $0 \leqslant k \leqslant n$, the first smaller polytope has $k+2$ vertices, so its partner has $n-k+2$ vertices. It is possible that either smaller polytope is just a 2-gon; this happens when the triangle containing the removed edge contains an additional edge. Conversely, consider a pair of triangulated polygons having $k+2$ and $n-k+2$ vertices respectively. Choosing a vertex in each of the polytopes to identify and joining the succeeding vertex in the first polytope with the preceding vertex in the second polytope with a new edge, we obtain a triangulated $(n+3)$-gon. By construction, these operations are mutual inverses.

Since there is a bijection between the given sets, we conclude that Catalan recurrence holds. □

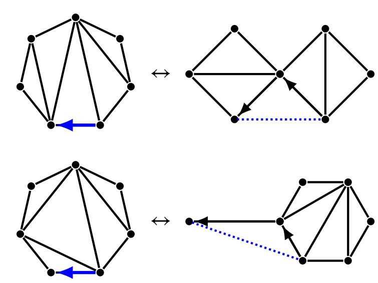

Figure 1.13: Catalan recurrence correspondence for $n = 4$

Exercises

Problem 1.2.6. The ***central polygonal numbers*** are defined by $a_0 := 0$ and $a_n := a_{n-1} + n$ for all $n \geqslant 1$. This sequences starts $1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, \dots$. Demonstrate that a_n equals the maximal number of regions defined by n lines in the plane.

Problem 1.2.7. For any nonnegative integer n, let a_n denote the number of diagonals in a convex $(n+3)$-gon. This sequence begins $0, 2, 5, 9, 14, 20, 27, 35, 44, 54, 65, 77, 90, 104, 119, 135, 152, \dots$.

(i) Show that this sequence satisfies $a_n = a_{n-1} + n + 1$ for all $n \geqslant 1$.

(ii) Prove that a_n is the maximal number of pieces that can be obtained by cutting an annulus with n straight lines.

Problem 1.2.8. For all $n \in \mathbb{N}$, the ***superfactorial***

$$\operatorname{sf}(n) := \prod_{k=1}^{n} k!$$

is the product of the first n factorials. Prove that the fraction

$$\frac{\operatorname{sf}(2n)}{(n+1)\big(\operatorname{sf}(n)\big)^4}$$

is an integer.

Problem 1.2.9. For all $n \in \mathbb{N}$, consider the integral

$$C_n := \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}}\, dx \,.$$

(i) For all $n \in \mathbb{N}$, show that $C_n := \dfrac{2^{2(n+1)}}{\pi} \displaystyle\int_0^1 y^{2n} \sqrt{1-y^2}\, dy$.
(ii) Compute C_0.
(iii) Demonstrate that $2(2n+1)C_n = (n+2)C_{n+1}$.

Problem 1.2.10. For all $n, k \in \mathbb{N}$, the ***Fuss-Catalan number*** $C_{n,k}$ is the number of subdivisions (using diagonals that don't intersect in their interiors) of a convex $(kn+2)$-gon into regions that are $(k+2)$-gons.

(i) Prove Fuss-Catalan ratio

$$\left(\prod_{j=1}^{k} (kn+j+1) \right) C_{n+1,k} = (k+1) \left(\prod_{j=1}^{k} (kn+n+j) \right) C_{n,k} \,.$$

(ii) Establish that

$$C_{n,k} = \frac{1}{kn+1} \binom{(k+1)n}{n}.$$