2
Subsets and Multisets

Counting subsets, both with and without repetitions, spawn some of the most pervasive combinatorial models. Relying largely on the double-counting arguments, we develop the theory of binomial coefficients. Among the vast number of identities, we emphasize those that appear most frequently in applications.

Notation 2.0.1. The set of integers is $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$.

2.0 Subsets Of A Fixed Cardinality

Enumerating all the subsets of a finite set is straightforward.

Proposition 2.0.2. *For any nonnegative integer n, there are 2^n distinct subsets contained in the set $[n] := \{1, 2, \dots, n\}$.*

Inductive proof. When $n = 0$, we have $[0] = \varnothing$. Since $\varnothing$ is the unique subset of $[0]$, the base case holds. Assume that $[n]$ has 2^n subsets. To count the subsets of $[n+1]$, we subdivide them into two classes.

- The subsets containing $n+1$ are a union of singleton $\{n+1\}$ and a subset of the set $[n]$. By the induction hypothesis, there are 2^n subsets of the set $[n]$, so the number of subsets containing the element $n+1$ is 2^n.
- The subsets that do not contain $n+1$ may be identified with the subsets of $[n]$. Again, the induction hypothesis implies that there are 2^n subsets of this form.

Therefore, the set $[n+1]$ has $2^n + 2^n = 2^{n+1}$ subsets. □

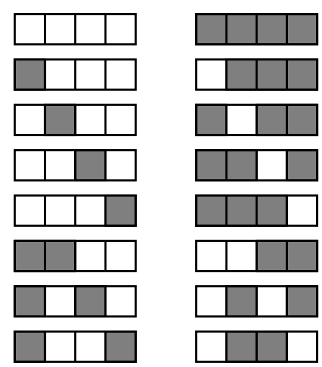

Figure 2.1: The 16 subsets of [4]

The sequence $1, 2, 4, 8, 16, 32, 64, 128, \dots$, listing the powers of 2, has another common combinatorial interpretation.

Bijective proof of Proposition 2.0.2.
Set 1: Consider the set of all subsets of $[n]$.
Set 2: Consider the set of all binary n-tuples. Since each entry is either 0 or 1, there are 2^n such vectors.
Correspondence: Send the subset $\mathcal{A} \subseteq [n]$ to its indicator vector whose i-th coordinate is 1 if $i \in \mathcal{A}$ and is 0 otherwise. Conversely, the binary n-tuple $\mathbf{v}$ is mapped to $\{i \mid \mathbf{v}_i = 1\} \subseteq [n]$. These operations are mutual inverses.

Since there is a bijection between the given sets, we conclude that there are 2^n subsets of the set $[n]$. □

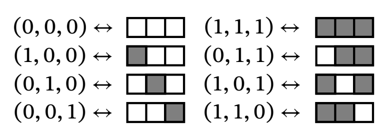

Figure 2.2: Binary 3-tuples and subsets of [3] correspondence

Counting subsets of a finite set having a fixed cardinality turns out to be much more interesting.

Definition 2.0.3. For all nonnegative integers n and all integers k, the ***binomial coefficient*** $\binom{n}{k}$ counts the subsets of the finite set $[n] := \{1, 2, \dots, n\}$ having cardinality k.

Our notation for binomial coefficients was first used in 1826 by Andreas von Ettingshausen.

Some special values are easy to determine.

- For all $k < 0$ and all $k > n$, we have $\binom{n}{k} = 0$ because there are no subsets of $[n]$ having cardinality k.
- For any nonnegative integer n, we have $\binom{n}{0} = 1$ because the empty set is the unique set with no elements.
- For any nonnegative integer n, we have $\binom{n}{n} = 1$ because the unique subset of $[n]$ having cardinality n is the set $[n]$ itself.
- For any nonnegative integer n, we have $\binom{n}{2} = n(n-1)/2$ because there are n ways to choose the first element, $n-1$ ways to choose a different element for the second, and 2 ways to order them.

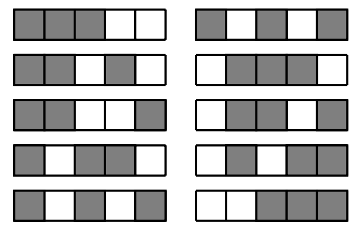

Figure 2.3: The 10 subsets of $[5]$ having cardinality 3

Although poorly suited for numerical computations, binomial coefficients have a compact expression involving factorials.

Proposition 2.0.4 (Factorial Formula)**.** *For all nonnegative integers n and all integers k such that $0 \leqslant k \leqslant n$, the binomial coefficients satisfy*

$$n! = \binom{n}{k} k!\,(n-k)! \quad \Leftrightarrow \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,.$$

Double-counting proof. How many permutations of the set $[n]$ are there?

Answer 1: Proposition 1.1.3 shows that the number of permutations of the set $[n]$ is $n!$.

Answer 2: Focus on the first k numbers in the one-line notation for a permutation of the set $[n]$. The definition of binomial coefficients implies that there are $\binom{n}{k}$ ways to choose the numbers that turn up in the initial k entries. Once these k numbers are chosen, there are $k!$ ways to arrange them. Similarly, there are $(n-k)!$ ways to arrange the complementary $n-k$ elements. Hence, the number of permutations of $[n]$ is $\binom{n}{k} k!\,(n-k)!$. □

Proposition 2.0.5 (Symmetry)**.** *For all nonnegative integers n and all integers k, we have*

$$\binom{n}{k} = \binom{n}{n-k}.$$

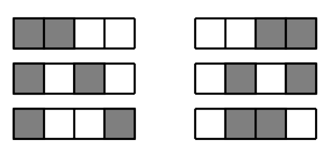

Figure 2.4: Symmetry in the 6 subsets of $[4]$ having cardinality 2

Double-counting proof. How many committees from a slate of n candidates can be formed with k members?

Answer 1: The definition of binomial coefficients implies that there are $\binom{n}{k}$ committees.

Answer 2: We may choose $n-k$ candidates to exclude from the committee, which can be done in $\binom{n}{n-k}$ ways. □

Proposition 2.0.6 (Addition). *For all nonnegative integers n and all integers k, we have*

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	2	1	0	0	0	0
3	1	3	3	1	0	0	0
4	1	4	6	4	1	0	0
5	1	5	10	10	5	1	0
6	1	6	15	20	15	6	1

Figure 2.5: Matrix of binomial coefficents

Double-counting proof. How many committees from a slate of $n + 1$ candidates can be formed with $k + 1$ members?

Answer 1: From the definition of binomial coefficients, we see that there are $\binom{n+1}{k+1}$ such committees.

Answer 2: Focus on membership of candidate $n + 1$. Since there are $\binom{n}{k+1}$ committees that exclude $n + 1$ and $\binom{n}{k}$ committees that include $n + 1$, the total number is $\binom{n}{k+1} + \binom{n}{k}$. □

Problem 2.0.7. For any nonnegative integer n, prove that

$$\sum_{k \in \mathbb{Z}} \binom{n}{k} = 2^n .$$

The sum of the entires in the n-th row of Table 2.5 is 2^n.

Double-counting solution. How many subsets of $[n]$ are there?

Answer 1: The definition for binomial coefficients implies that, for each integer k, the number of subsets of cardinality k is $\binom{n}{k}$, so there are in total $\sum_{k \in \mathbb{Z}} \binom{n}{k}$ committees.

Answer 2: Proposition 2.0.2 proves there are 2^n subsets. □

Proposition 2.0.8 (Upper sum). *For any nonnegative integers m and n, we have*

$$\sum_{j=0}^{n} \binom{j}{m} = \binom{n+1}{m+1}.$$

Sum of the first n entries in the m-th column of Table 2.5 equals the $(n+1, m+1)$-entry.

Double-counting proof. How many subsets of the set $[n + 1]$ having cardinality $m + 1$ are there?

Answer 1: The definition of binomial coefficients implies that the number of subsets of $[n + 1]$ having cardinality $m + 1$ is $\binom{n+1}{m+1}$.

Answer 2: Focus on the largest number in a given subset. For all $0 \leqslant j \leqslant n$, any subset having cardinality $m + 1$ and maximum element $j + 1$ can created by adjoining the element $j + 1$ to an subset of the set $[j]$ having cardinality m, which can be done in $\binom{j}{m}$ ways. Hence, the total number of subsets having cardinality $m + 1$ is $\sum_{j=0}^{n} \binom{j}{m}$. □

Problem 2.0.9. For any nonnegative integer n, demonstrate that

$$\sum_{k \in \mathbb{Z}} k \binom{n+1}{k} = (n+1)\, 2^n .$$

Double-counting solution. From a slate of $n + 1$ candidates, how many committees having one of member designated as the chair are there?

Answer 1: Focus on a committee with k members. From the definition of binomial coefficients, there are $\binom{n+1}{k}$ ways to choose the committee members and there are k ways to choose the chair. Adding up all the possibilities gives a total of $\sum_{k \in \mathbb{Z}} k \binom{n+1}{k}$ chaired committees.

Answer 2: First select the chair from the slate of $n+1$ candidates. From the other n candidates, there are 2^n ways to choose a subset to complete the committee. □

Problem 2.0.10. For any nonnegative integer n, show that

$$\sum_{k \in \mathbb{Z}} \binom{n+1}{2k} = 2^n \,.$$

Double-counting solution. From a slate of $n+1$ candidates, how many committees having an even number of members are there?

Answer 1: Focus on committees with $2k$ members. The definition of binomial coefficients implies that there are $\binom{n+1}{2k}$ committees with $2k$ members, so there are $\sum_{k \in \mathbb{Z}} \binom{n+1}{2k}$ committees with an even number of members.

Answer 2: The first n candidates can be freely chosen to be on or off of the committee. Once these choices are made, the fate of the candidate $n+1$ is completely determined so that the final committee has even number of members. Consequently, there are 2^n such committees. □

Exercises

Problem 2.0.11. Let F_n denote the n-th Fibonacci number. Prove each of the following identities via a double-counting argument.

(i) For all $n \in \mathbb{N}$, verify that $F_{n+1} = \sum_{k \in \mathbb{Z}} \binom{n-k}{k}$.

(ii) For all $n \in \mathbb{N}$, verify that $F_{2n} = \sum_{k \in \mathbb{Z}} \binom{n}{k} F_k$.

Problem 2.0.12. Prove each of the following identities via a double-counting argument.

(i) For all $n \in \mathbb{N}$, demonstrate that $\sum_{k \in \mathbb{Z}} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$.

(ii) For all $n \in \mathbb{N}$, show that $\sum_{k \in \mathbb{Z}} \binom{n}{2k} \binom{2k}{k} 2^{n-2k} = \binom{2n}{n}$.

2.1 Binomial Coefficients

Binomial coefficients have applications beyond their conventional combinatorial interpretation. One generalization views a binomial coefficient as a polynomial in its numerator, thereby allowing one to evaluate binomial coefficients at any real or complex number.

Definition 2.1.1. For any integer k, the ***binomial coefficient*** is

$$\binom{x}{k} := \begin{cases} \dfrac{x(x-1)(x-2)\cdots(x-k+1)}{k(k-1)(k-2)\cdots(1)} \in \mathbb{Q}[x] & \text{if } k \geqslant 0, \\ 0 & \text{if } k < 0. \end{cases}$$

When x is a nonnegative integer and k is at most x, the factorial formula [2.0.4] establishes that this new definition agrees with Definition 2.0.3.

As polynomials, the first few binomials coefficients are

$\binom{x}{0} = 1$

$\binom{x}{1} = x \,,$

$\binom{x}{2} = \frac{1}{2}x^2 - \frac{1}{2}x \,,$

$\binom{x}{3} = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x \,,$

$\binom{x}{4} = \frac{1}{24}x^4 - \frac{1}{4}x^3 + \frac{11}{24}x^2 - \frac{1}{4}x \,,$

Notwithstanding the larger context, our combinatorial methods continue to be very useful. Any nonzero polynomial in $\mathbb{Q}[x]$ has at most finitely many zeros, so it suffices to prove that polynomial identities hold for all sufficiently large integers x. For example, the addition formula [2.0.6] demonstrates that $\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$ holds, for all integers k, where "x" can be an indeterminate, a real number, or a complex number.

To probe this perspective, we show that a negative numerator in a binomial coefficient is related to positive ones by a sign.

Proposition 2.1.2 (Negation). *For any integer k, we have*

$$\binom{x}{k} = (-1)^k \binom{k-x-1}{k}.$$

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$
−4	1	−4	10	−20	35	−56	84
−3	1	−3	6	−10	15	−21	28
−2	1	−2	3	−4	5	−6	−7
−1	1	−1	1	−1	1	−1	1
0	1	0	0	0	0	0	0

Figure 2.6: Matrix of binomial coefficients with negative numerators

Algebraic proof. Since both sides vanish when k is negative, we may assume that k is a nonnegative integer. The polynomial definition gives

$$\begin{aligned}\binom{x}{k} &= \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}\\ &= (-1)^k \frac{(-x)(1-x)(2-x)\cdots(k-x-1)}{k!}\\ &= (-1)^k \frac{(k-x-1)(k-x-2)(k-x-3)\cdots(-x)}{k!}\\ &= \binom{k-x-1}{k}.\end{aligned}$$ □

Our next identity allows one to move things in and out of a binomial coefficient.

Proposition 2.1.3 (Absorption). *For any integer k, we have*

$$k\binom{x}{k} = x\binom{x-1}{k-1}.$$

When $x \in \mathbb{N}$ and $k > x$, both sides of the absorption identity are zero by Definition 2.0.3.

Double-counting proof. Since both sides vanish when $k \leqslant 0$, we may assume that k is a positive integer. A nonzero polynomial in $\mathbb{Q}[x]$ has at most finitely many zeros, so it is enough to prove this identity when $x = n$ is a sufficiently large integer. Assume that $n \geqslant k$. From a slate of n candidates, how many committees with k members and having one member designated chair are there?

Answer 1: Definition 2.0.3 implies that there are $\binom{n}{k}$ ways to choose the committee. There are k ways to select the chair, which gives a total of $k\binom{n}{k}$ chaired committees.

Answer 2: First select the chair from the slate of n candidates. From the other $n-1$ candidates, pick the remaining $k-1$ committee members. This can be done $n\binom{n-1}{k-1}$ ways. □

When dealing with products of binomial coefficients, the next identity often helps.

Proposition 2.1.4 (Trinomial revision)**.** *For all integers m and k, we have*

$$\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}.$$

When $x \in \mathbb{N}$ and $x < m$ or $m < k$, both sides of the trinomial revision identity are zero by Definition 2.0.3.

Double-counting proof. Since both sides vanish when $m < k$ or $k < 0$, we may assume that $m > k$ and k is a nonnegative integer. It is enough to prove this identity when $x = n$ is a sufficiently larger integer. Assume that $n \geqslant m$. From a slate of n candidates, how many committees with m members contain a subcommittee with k members?

Answer 1: By Definition 2.0.3, the committee can be formed in $\binom{n}{m}$ ways and the subcommittee can be formed in $\binom{m}{k}$ ways, so there are $\binom{n}{m}\binom{m}{k}$ committees with the desired structure.

Answer 2: First choose the k members who will serve on both the committee and the subcommittee. Definition 2.0.3 implies that this can be done in $\binom{n}{k}$ ways. From among the complementary $n-k$ candidates, choose the $m-k$ members who will serve on just the committee. Since there are $\binom{n-k}{m-k}$ possibilities for this second choice, there is a total of $\binom{n}{k}\binom{n-k}{m-k}$ committees with the desired structure. □

The next identity in this subsection is commonly named after Alexandre Vandermonde even though it was known to Zhu Shijie as early as 1303.

Proposition 2.1.5 (Vandermonde)**.** *For all integers k, we have*

$$\binom{x+y}{k} = \sum_{j\in\mathbb{Z}} \binom{x}{j}\binom{y}{k-j}.$$

When $(x, y) \in \mathbb{N}^2$ and $x + y < k$, Definition 2.0.3 implies that both sides of the Vandermonde identity are zero.

Double-counting proof. Since both sides vanish when $k < 0$, we may assume that k is a nonnegative integers. A nonzero univariate polynomial has at most finitely many zeros, so it is enough to prove this polynomial identity when $x = m$ and $y = n$ are both sufficiently large integers. From a crowd of $m + n$ hockey fans, consisting of m Leaf fans and n Habs fans, how many ways can one fill an arena with k fans?

As the oldest rivalry in the National Hockey League, we may safely assume that no individual is a fan of both the Toronto Maple Leafs and the Montreal Canadiens (“the Habs”).

Answer 1: Definition 2.0.3 implies that there are is $\binom{m+n}{k}$ ways.

Answer 2: Focus on the Leaf fans in the arena. First choose j Leaf fans and then $k-j$ Habs fans. Since there are $\binom{m}{j}$ ways to select the Leaf fans and $\binom{n}{k-j}$ ways to select the Habs fans, there are $\sum_{j\in\mathbb{Z}}\binom{m}{j}\binom{n}{k-j}$ ways to fill the arena. □

Our final identity for binomial coefficients is arguably the most important and is the source of the adjective “binomial”.

Theorem 2.1.6 (Binomial)**.** *For any nonnegative integer n, we have*

$$(x+y)^n = \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k\, y^{n-k}.$$

Counting proof. Consider expanding the product

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}} .$$

Every monomial in the expansion is the product of n factor, each of which is either x or y. How many different ways can one create the monomial $x^k y^{n-k}$? Each such monomial arises by choosing x from k of the factors whereas y must be chosen from the complementary $n-k$ factors. Definition 2.0.3 implies that this can be done in $\binom{n}{k}$ ways. Hence, we obtain $\sum_{k \in \mathbb{Z}} \binom{n}{k} x^k y^{n-k}$. □

The Binomial Theorem has some noteworthy specializations:

- Setting $x = y = 1$ gives $\sum_{k \in \mathbb{Z}} \binom{n}{k} = 2^n$.
- Setting $x = -1$ and $y = 1$ yields $\sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{k} = 0$.

Exercises

Problem 2.1.7. Give two proofs for each of the following identities: one using a double-counting argument and the other by relying on the key binomial identities.

(i) For all $n \geqslant 2$ and all $k \in \mathbb{Z}$, show that

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}.$$

(ii) For all $m, n \in \mathbb{N}$, show that $\sum_{k \in \mathbb{Z}} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}$.

Problem 2.1.8. For all $m, n \in \mathbb{N}$, the ***super Catalan number*** is defined to be

$$S(m,n) := \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!} .$$

(i) Show that $S(0,n) = \binom{2n}{n}$ and $\frac{1}{2} S(1,n)$ is the n-th Catalan number.

(ii) Verify that $S(m,n) = (-1)^n \, 4^{m+n} \binom{m-1/2}{m+n}$.

2.2 Multisets

Nicolaas Govert de Bruijn coined the word 'multiset' in the 1970s.

In a set, all elements are distinct. We drop this restriction in a multiset. For example, $M := \{1,1,1,2,4,4\}$ is a multiset of size 6 over the set $[4]$, where 1, 2, 3, and 4 appear with multiplicity 3, 1, 0, and 2 respectively. More formally, a ***multiset*** M over the set $[n]$ is a function $\nu \colon [n] \to \mathbb{N}$ such that $\sum_{j=1}^{n} \nu(j) < \infty$. One regards $\nu(j)$ as the number of repetitions of the number j. The integer $\sum_{j=1}^{n} \nu(j)$ is the ***size*** of the multiset M. When $a_j := \nu(j)$ for all nonnegative integers j, one sometimes writes $M = \{1^{a_1}, 2^{a_2}, \dots, n^{a_n}\}$.

$$\begin{aligned}
\{1,1,1,1\} &\leftrightarrow (4,0,0)\\
\{1,1,1,2\} &\leftrightarrow (3,1,0)\\
\{1,1,1,3\} &\leftrightarrow (3,0,1)\\
\{1,1,2,2\} &\leftrightarrow (2,2,0)\\
\{1,1,2,3\} &\leftrightarrow (2,1,1)\\
\{1,1,3,3\} &\leftrightarrow (2,0,2)\\
\{1,2,2,2\} &\leftrightarrow (1,3,0)\\
\{1,2,2,3\} &\leftrightarrow (1,2,1)\\
\{1,2,3,3\} &\leftrightarrow (1,1,2)\\
\{1,3,3,3\} &\leftrightarrow (1,0,3)\\
\{2,2,2,2\} &\leftrightarrow (0,4,0)\\
\{2,2,2,3\} &\leftrightarrow (0,3,1)\\
\{2,2,3,3\} &\leftrightarrow (0,2,2)\\
\{2,3,3,3\} &\leftrightarrow (0,1,3)\\
\{3,3,3,3\} &\leftrightarrow (0,0,4)
\end{aligned}$$

Figure 2.7: The 15 multisets over [3] of size 4

Definition 2.2.1. For any nonnegative integer n and any integer k, the ***multichoose coefficient*** $\left(\!\binom{n}{k}\!\right)$ is the number of multisets over $[n]$ of size k.

Some special values are easy to determine.

- For any nonnegative integer k, we have $\left(\!\binom{1}{k}\!\right) = 1$ because $\{1^k\}$ is the unique multiset over $[1]$ having size k.
- For any nonnegative integer n, we have $\left(\!\binom{n}{0}\!\right) = 1$ because there is a unique multiset over $[n]$ having size 0.
- For all $k < 0$, we have $\left(\!\binom{n}{k}\!\right) = 0$ because there are no multisets having negative size.

These numbers have a few other convenient interpretations.

- $\left(\!\binom{n}{k}\!\right)$ counts the ways that the k votes can be allocated to n candidates.
- $\left(\!\binom{n}{k}\!\right)$ counts the solutions $(a_1, a_2, \dots, a_n) \in \mathbb{N}^n$ to the equation $a_1 + a_2 + a_3 + \cdots + a_n = k$. The corresponding multiset over $[n]$ of size k is $\{1^{a_1}, 2^{a_2}, \dots, n^{a_n}\}$.
- $\left(\!\binom{n}{k}\!\right)$ counts the positive integer k-tuples $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ satisfying $n \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$. The corresponding multiset over $[n]$ of size k is $\{\lambda_k, \lambda_{k-1}, \dots, \lambda_1\}$.

We first show that multichoose coefficients are closely related to binomial coefficients.

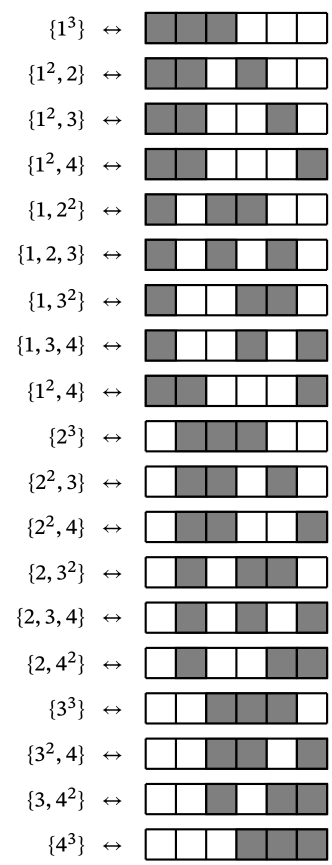

Figure 2.8: The 20 multisets over the set $[4]$ of size 3

Theorem 2.2.2 (Multichoose coefficients as binomial coefficients). *For any nonnegative integer n and any integer k, we have*

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

Double-counting proof. Since both sides vanish when $k < 0$, we may assume that k is a nonnegative integer. How ways are there to allocate k votes to n candidates?

Answer 1: By definition, the number of allocations is $\left(\!\binom{n}{k}\!\right)$.

Answer 2: We represent each allocation with 'stars and bars'. Specifically, each allocation is represented as an arrangement of k stars (the votes) and $n - 1$ bars (the dividers between the candidates). In Figure 2.8, grey squares are the 'stars' and white squares are the 'bars'. For all $1 \leqslant i \leqslant n$, the number of stars between the $(i-1)$-st and the i-th dividers is the number of votes allocated to candidate i. Each arrangement involves choosing k stars from among $n + k - 1$ symbols, so the total number of allocations is $\binom{n+k-1}{k}$. □

This link with binomial coefficients also provides a polynomial interpretation for multichoose coefficients. We declare that

$$\left(\!\!\binom{x}{k}\!\!\right) := \binom{x+k-1}{k} = \frac{x(x+1)(x+2)\cdots(x+k-1)}{k(k-1)(k-2)\cdots(1)} \in \mathbb{Q}[x].$$

This definition has another pleasant form.

Corollary 2.2.3. *For any integer k, we have* $\left(\!\!\binom{x}{k}\!\!\right) = (-1)^k \binom{-x}{k}$.

Algebraic proof. The negation identity [2.1.2] gives

$$\begin{aligned}\left(\!\!\binom{x}{k}\!\!\right) &= \binom{x+k-1}{k} \\ &= (-1)^k \binom{k-(x+k-1)-1}{k} = (-1)^k \binom{-x}{k}. \qquad \square\end{aligned}$$

Like binomial coefficients, multichoose coefficients have a natural symmetry and satisfy a linear recurrence.

Proposition 2.2.4 (Symmetry). *For all nonnegative integer n and k excluding the degenerate case $(n,k)=(0,0)$, we have $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{k+1}{n-1}\!\right)$.*

Algebraic proof. Proposition 2.2.2 and the symmetry of binomial coefficients [2.0.5] give

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \binom{n+k-1}{n-1} = \left(\!\!\binom{k+1}{n-1}\!\!\right). \qquad \square$$

n	$\left(\!\binom{n}{0}\!\right)$	$\left(\!\binom{n}{1}\!\right)$	$\left(\!\binom{n}{2}\!\right)$	$\left(\!\binom{n}{3}\!\right)$	$\left(\!\binom{n}{4}\!\right)$	$\left(\!\binom{n}{5}\!\right)$
0	1	0	0	0	0	0
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126

Figure 2.9: Matrix of multichoose coefficients

Proposition 2.2.5 (Addition). *For any integer k, we have*

$$\left(\!\!\binom{x}{k}\!\!\right) = \left(\!\!\binom{x}{k-1}\!\!\right) + \left(\!\!\binom{x-1}{k}\!\!\right).$$

Double-counting proof. Since both sides vanish when $k<0$, we may assume that k is a nonnegative integer. When $k=0$, the special values of the multichoose coefficient show that both sides equal 1, so we may further assume that k is a positive integer. A nonzero polynomial in $\mathbb{Q}[x]$ has at most finitely many zeros, so it suffices to establish this identity when $x=n$ is sufficiently large integer. How many ways can we allocate k votes to n candidates?

Answer 1: From the definition for the multichoose coefficient, we see that there are $\left(\!\binom{n}{k}\!\right)$ allocations.

Answer 2: Focus on whether the candidate n gets a vote. If they do, then there are $\left(\!\binom{n}{k-1}\!\right)$ ways to allocate the other votes, because the candidate n receives the last vote. If they don't, then there are $\left(\!\binom{n-1}{k}\!\right)$ ways to allocate the votes, because the candidate n receives no votes. Thus, there is a total of $\left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right)$ ways to allocate the votes. $\square$

Algebraic proof. Proposition 2.2.2 and the addition formula for binomial coefficients [2.0.6] give

$$\begin{aligned}\left(\!\!\binom{x}{k}\!\!\right) &= \binom{x+k-1}{k} \\ &= \binom{x+k-2}{k} + \binom{x+k-2}{k-1} = \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{x}{k-1}\!\!\right). \qquad \square\end{aligned}$$

Exercises

Problem 2.2.6. For all positive integers $n, k \in \mathbb{Z}$, a ***composition*** of n into k parts is a k-tuple $(a_1, a_2, \dots, a_k)$ of positive integers such that $a_1 + a_2 + \cdots + a_k = n$.

(i) Provide a bijective proof that the number of compositions of n into k parts is $\binom{n-1}{k-1}$.
(ii) Show that the total number of compositions of n is 2^{n-1}.
(iii) Show that $\left(\!\binom{k}{n-k}\!\right) = \binom{n-1}{k-1}$ via a double-counting argument.

Problem 2.2.7. Using a double-counting argument, prove the following identities.

(i) For all $m, n \in \mathbb{N}$, show that $\left(\!\binom{n}{2m+1}\!\right) = \sum_{k \in \mathbb{Z}} \left(\!\binom{k}{m}\!\right) \left(\!\binom{n-k+1}{m}\!\right)$.
(ii) For all $m, n, k \in \mathbb{N}$, show that $\left(\!\binom{m+n}{k}\!\right) = \sum_{j \in \mathbb{Z}} \left(\!\binom{m}{j}\!\right) \left(\!\binom{n}{k-j}\!\right)$.

Problem 2.2.8. For all $m, n \in \mathbb{N}$, prove

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{m+n+1}{m}$$

via a double-counting argument and rewrite this identity in terms of multichoose coefficients.