3.2 Stirling Duality

Stirling subset numbers and Stirling cycle numbers are, from the correct perspective, two sides of the same coin. To see this duality, we declare that the addition formulas for both kinds of Stirling numbers are valid for all nonnegative integers n and k. Stipulating that $\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} = 0$ for all negative integers n leads to a unique solution for these recurrences. Informally, knowing the Stirling numbers at two of the three pairs $(n-1, k-1)$, $(n-1, k)$, and (n, k) allows one to recursively solve for the third. Formalizing this procedure links the two kinds of Stirling numbers.

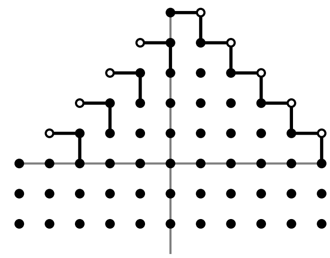

Figure 3.6: Visualizing the informal recursion

Theorem 3.2.1 (Negation). *For all integers n and k, we have*

$$\begin{Bmatrix} -k \\ -n \end{Bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

Inductive proof. For all integers n and k, we have

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \begin{Bmatrix} 0 \\ k \end{Bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

so the base cases of this double induction hold. Whenever m and j are nonnegative integers satisfying $m + j \leqslant n + k$, assume that $\begin{Bmatrix} -j \\ -m \end{Bmatrix} = \begin{bmatrix} m \\ j \end{bmatrix}$ and $\begin{bmatrix} -m \\ -j \end{bmatrix} = \begin{Bmatrix} j \\ m \end{Bmatrix}$. The addition formulas [3.0.3, 3.1.4] for Stirling numbers and the induction hypotheses give

$$\begin{Bmatrix} -k \\ -n-1 \end{Bmatrix} = \begin{Bmatrix} -k+1 \\ -n \end{Bmatrix} - (-n)\begin{Bmatrix} -k \\ -n \end{Bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + n\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ k \end{bmatrix},$$
$$\begin{bmatrix} -n \\ -k-1 \end{bmatrix} = \begin{bmatrix} -n+1 \\ -k \end{bmatrix} - (-n-1)\begin{bmatrix} -n \\ -k \end{bmatrix} = \begin{Bmatrix} k \\ n-1 \end{Bmatrix} + (n+1)\begin{Bmatrix} k \\ n \end{Bmatrix} = \begin{Bmatrix} k+1 \\ n \end{Bmatrix}.$$

Similarly, whenever $m, j \in \mathbb{N}$ satisfy $m + j \leqslant n + k$, assume that $\begin{Bmatrix} -j \\ m \end{Bmatrix} = \begin{bmatrix} -m \\ j \end{bmatrix}$. This time, the addition formulas for Stirling numbers and the induction hypotheses give

$$\begin{Bmatrix} -k-1 \\ n \end{Bmatrix} = \frac{1}{n}\left(\begin{Bmatrix} -k \\ n \end{Bmatrix} - \begin{Bmatrix} -k-1 \\ n-1 \end{Bmatrix}\right) = \frac{1}{n}\left(\begin{bmatrix} -n+1 \\ k+1 \end{bmatrix} - \begin{bmatrix} -n \\ k \end{bmatrix}\right) = \begin{bmatrix} -n \\ k+1 \end{bmatrix},$$
$$\begin{bmatrix} -n-1 \\ k \end{bmatrix} = \frac{1}{(-n-1)}\left(\begin{bmatrix} -n \\ k \end{bmatrix} - \begin{bmatrix} -n-1 \\ k-1 \end{bmatrix}\right) = \frac{1}{(n+1)}\left(\begin{Bmatrix} -k+1 \\ n+1 \end{Bmatrix} - \begin{Bmatrix} -k \\ n \end{Bmatrix}\right) = \begin{Bmatrix} -k \\ n+1 \end{Bmatrix},$$

which completes the induction steps. □

Remark 3.2.2. For all nonnegative integers n and k except for $(n, k) = (0, 0)$, strengthening the induction step to shows that

$$\begin{Bmatrix} -n \\ k \end{Bmatrix} = 0 = \begin{bmatrix} -n \\ k \end{bmatrix}.$$

As an upshot, this duality ensures that the identities for Stirling numbers are twinned. For instance, the analogues of the Upper Sum identity [2.0.8] are the following pair.

n	$\left\{ {n \atop -5} \right\}$	$\left\{ {n \atop -4} \right\}$	$\left\{ {n \atop -3} \right\}$	$\left\{ {n \atop -2} \right\}$	$\left\{ {n \atop -1} \right\}$	$\left\{ {n \atop 0} \right\}$	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$	$\left\{ {n \atop 4} \right\}$	$\left\{ {n \atop 5} \right\}$
−5	1	0	0	0	0	0	0	0	0	0	0
−4	10	1	0	0	0	0	0	0	0	0	0
−3	35	6	1	0	0	0	0	0	0	0	0
−2	50	11	3	1	0	0	0	0	0	0	0
−1	24	6	2	1	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0
2	0	0	0	0	0	0	1	1	0	0	0
3	0	0	0	0	0	0	1	3	1	0	0
4	0	0	0	0	0	0	1	7	6	1	0
5	0	0	0	0	0	0	1	15	25	10	1

Table 3.1: Matrix of all Stirling numbers

Proposition 3.2.3 (Upper sum). *For any nonnegative integer n and any integer k, we have*

$$\text{(i)} \quad \left\{ {n+1 \atop k+1} \right\} = \sum_{j \in \mathbb{Z}} \binom{n}{j} \left\{ {j \atop k} \right\}, \qquad \text{(ii)} \quad \left[{n+1 \atop k+1} \right] = \sum_{j \in \mathbb{Z}} \left[{n \atop j} \right] \binom{j}{k}.$$

Double-counting proof for (i). How many partitions of the set $[n+1]$ have $k+1$ blocks?

Answer 1: Applying the definition of Stirling subset numbers, we see that there are $\left\{ {n+1 \atop k+1} \right\}$ partitions of the set $[n+1]$ with $k+1$ blocks.

Answer 2: Focus on the blocks that do not contain the element $n+1$. For some $0 \leqslant j \leqslant n$, there are $\binom{n}{j}$ ways to choose the elements that are not contained in the same block as $n+1$. This chosen j-subset of $[n]$ can be partitioned into k blocks in $\left\{ {j \atop k} \right\}$ ways. Hence, there are $\sum_{j \in \mathbb{Z}} \binom{n}{j} \left\{ {j \atop k} \right\}$ ways to partition the set $[n+1]$ into $k+1$ blocks. □

Double-counting proof for (ii). How many permutations of the set $[n+1]$ have exactly $k+1$ cycles?

Answer 1: The definition of Stirling cycle numbers implies that there are $\left[{n+1 \atop k+1} \right]$ permutations of the set $[n+1]$ with $k+1$ cycles.

Answer 2: Focus on the cycles that do not contain $n+1$. For all $k \leqslant j \leqslant n$, there are $\left[{n \atop j} \right]$ permutations of $[n]$ with j cycles and there are $\binom{j}{k}$ ways to select k of these cycles. We claim that a cycle containing $n+1$ corresponds to the complementary $j-k$ cycles. More precisely, the product of cycles $(C_1)(C_2)\cdots(C_{j-k})$ maps to the single cycle $(n+1 \;\; C_1 \;\; C_2 \;\; \cdots \;\; C_{j-k})$. Conversely, given a cycle $(n+1 \;\; a_1 \;\; a_2 \;\; \cdots \;\; a_m)$ for some $0 \leqslant m \leqslant n$, let $j-k$ denote the length of the longest increasing subsequence $a_1 < a_{i_2} < a_{i_3} < \cdots < a_{i_{j-k}}$ where $1 < i_2 < i_3 < \cdots < i_{j-k}$. When there are more than one such sequence choose the first in the lexicographic order. Hence, the given cycle maps to

$$(a_1 \;\; a_2 \;\; \cdots \;\; a_{i_2-1})(a_{i_2} \;\; a_{i_2+1} \;\; \cdots \;\; a_{i_3-1}) \cdots (a_{i_{j-k}} \;\; a_{i_{j-k}+1} \;\; \cdots \;\; a_m).$$

By construction, the maps compose to the identity thereby establishing the claim. Thus, the number of permutations of the set $[n+1]$ with $k+1$ cycles is $\sum_{j \in \mathbb{Z}} \left[{n \atop j} \right] \binom{j}{k}$. □

$(5\;4\;3)(7\;2\;6)(8\;1) \leftrightarrow (9\;\underline{5}\;4\;3\;\underline{7}\;2\;6\;\underline{8}\;1)$
$(7\;1\;5\;2)(8\;3\;6\;4) \leftrightarrow (9\;\underline{7}\;1\;5\;2\;\underline{8}\;3\;6\;4)$
$(3\;2)(5\;1)(6\;4)(8\;7) \leftrightarrow (9\;\underline{3}\;2\;\underline{5}\;1\;\underline{6}\;4\;\underline{8}\;7)$

Exercises

Problem 3.2.4.

(i) For all nonnegative integers n and all integers k, demonstrate that $\begin{bmatrix} n \\ k \end{bmatrix} \geqslant \begin{Bmatrix} n \\ k \end{Bmatrix}$.

(ii) For all nonnegative integers n, determine for which integers k the equality $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix}$ holds.

Problem 3.2.5. For all nonnegative integers m and n, prove the following identities via a double-counting argument.

(i) $\begin{Bmatrix} n+1 \\ m+1 \end{Bmatrix} = \sum_{k=0}^{n} \begin{Bmatrix} k \\ m \end{Bmatrix} (m+1)^{n-k}$.

(ii) $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} k \\ m \end{bmatrix} n^{\underline{n-k}}$.

Problem 3.2.6. For all nonnegative integers m and n, prove the following identities via a double-counting argument.

(i) $\begin{Bmatrix} m+n+1 \\ m \end{Bmatrix} = \sum_{k=0}^{m} k \begin{Bmatrix} n+k \\ k \end{Bmatrix}$.

(ii) $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^{m} (n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}$.

Problem 3.2.7. For any nonnegative integer n and any integer k, the ***Lah number*** $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor$ is the number of partitions of the set $[n]$ into k nonempty lists; also see Problem 3.0.9.

(i) For any nonnegative integers n, use a double-counting argument to prove that

$$x^{\overline{n}} = \sum_{k \in \mathbb{Z}} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor x^{\underline{k}} .$$

(ii) Combine identities to deduce that $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \sum_{j \in \mathbb{Z}} \begin{bmatrix} n \\ j \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix}$.

3.3 Eulerian Numbers

Besides counting cycles, there are other prominent numerical statistics on permutations. In this subsection, we examine two.

Definition 3.3.1. Let n be a nonnegative integer. For a permutation σ of the set $[n]$, an index j satisfying $1 \leqslant j \leqslant n-1$ is an ***ascent*** when $\sigma(j) < \sigma(j+1)$ and it is a ***descent*** when $\sigma(j) > \sigma(j+1)$.

Definition 3.3.2. For any nonnegative integer n and any integer k, the ***Eulerian number*** $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ is the number of permutations of the set $[n]$ with k ascents. The angle brackets suggest "less than" and "greater than" signs.

Some special values are easy to determine.

- For all $k < 0$ or all $k \geqslant n$, we have $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = 0$ because the number of ascents is nonnegative and at most $n-1$.
- For any nonnegative integer n, we have $\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle = 1$ because the unique permutation with no ascents is $n \ \ (n-1) \ \ \cdots \ \ 3 \ \ 2 \ \ 1$.
- For any positive integer n, we have $\left\langle \begin{matrix} n \\ n-1 \end{matrix} \right\rangle = 1$ because the unique permutation with $n-1$ ascents is $1 \ \ 2 \ \ 3 \ \ \cdots \ \ n$.

0	1	2	3
4 3 2 1			
	1 4 3 2		
	2 1 4 3		
	2 4 3 1		
	3 1 4 2		
	3 2 1 4		
	3 2 4 1		
	3 4 2 1		
	4 1 3 2		
	4 2 1 3		
	4 2 3 1		
	4 3 1 2		
		2 3 4 1	
		3 4 1 2	
		1 3 4 2	
		2 4 1 3	
		4 1 2 3	
		1 4 2 3	
		1 2 4 3	
		2 3 1 4	
		3 1 2 4	
		1 3 2 4	
		2 1 3 4	
			1 2 3 4

Figure 3.7: The permutations of the set [4] partitioned by the number of ascents

From Table 3.7, we see that $\left\langle \begin{matrix} 4 \\ 0 \end{matrix} \right\rangle = 1$, $\left\langle \begin{matrix} 4 \\ 1 \end{matrix} \right\rangle = 11$, $\left\langle \begin{matrix} 4 \\ 2 \end{matrix} \right\rangle = 11$, and $\left\langle \begin{matrix} 4 \\ 3 \end{matrix} \right\rangle = 1$.

In contrast with Stirling numbers, the Eulerian numbers have a natural symmetry; compare this relation with the symmetry identity [2.0.5] for binomial coefficients.

Proposition 3.3.3 (Symmetry). *For any nonnegative integer n and any integer k, we have*

$$\left\langle {n+1 \atop k} \right\rangle = \left\langle {n+1 \atop n-k} \right\rangle .$$

Bijective proof.

Set 1: Consider all of the permutations of the set $[n+1]$ with k ascents. The definition of the Eulerian numbers implies that this set has cardinality $\left\langle {n+1 \atop k} \right\rangle$.

Set 2: Consider the permutations of the set $[n+1]$ with $n-k$ ascents. The definition of the Eulerian numbers implies that this set has cardinality $\left\langle {n+1 \atop n-k} \right\rangle$.

Correspondence: The permutation $\sigma(1) \;\; \sigma(2) \;\; \cdots \;\; \sigma(n+1)$ of the set $[n+1]$ has k ascents and $n-k$ descents if and only if the permutation $\sigma(n+1) \;\; \sigma(n) \;\; \cdots \;\; \sigma(1)$ of the set $[n+1]$ has $n-k$ ascents and k descents. Hence, the involution defined by reversing the one-line notation defines a bijection between permutations with k ascents and those with $n-k$ ascents. □

1 2 3 4	↔	4 3 2 1
1 2 4 3	↔	3 4 2 1
1 3 2 4	↔	4 2 3 1
1 3 4 2	↔	2 4 3 1
1 4 2 3	↔	3 2 4 1
1 4 3 2	↔	2 3 4 1
2 1 3 4	↔	4 3 1 2
2 1 4 3	↔	3 4 1 2
2 3 1 4	↔	4 1 3 2
2 4 1 3	↔	3 1 4 2
3 1 2 4	↔	4 2 1 3
3 2 1 4	↔	4 1 2 3

Figure 3.8: Eulerian symmetry on permutations of the set [4]

Once more, we have a two-term recurrence.

Proposition 3.3.4 (Addition). *For any nonnegative integer n and any integer k, we have*

$$\left\langle {n+1 \atop k+1} \right\rangle = (n-k)\left\langle {n \atop k} \right\rangle + (k+2)\left\langle {n \atop k+1} \right\rangle .$$

Double-counting proof. Among the permutations of the set $[n+1]$, how many have $k+1$ ascents?

Answer 1: The definition of Eulerian numbers implies that the number of permutations of $[n+1]$ with $k+1$ ascents is $\left\langle {n+1 \atop k+1} \right\rangle$.

Answer 1: Focus on the largest element $n+1$. Each permutation $\tau(1) \;\; \tau(2) \;\; \cdots \;\; \tau(n)$ of the set $[n]$ leads to $n+1$ permutations of the set $[n+1]$ by inserting $n+1$ in all possible ways. Suppose that $n+1$ is inserted into position j to obtain the permutation

$$\sigma := \tau(1) \quad \tau(2) \quad \cdots \quad \tau(j-1) \quad n{+}1 \quad \tau(j) \quad \cdots \quad \tau(n)\,.$$

When $j = n$ or $\tau(j-1) > \tau(j)$, the number of ascents in σ is one more than in τ, so there are $\big((n-k-1)+1\big)\left\langle {n \atop k} \right\rangle$ permutations of $[n+1]$ with $k+1$ ascents having this form. When $j = 1$ or $\tau(j-1) < \tau(j)$, the number of ascents in σ is the same as the number in τ, so there are $(k+2)\left\langle {n \atop k+1} \right\rangle$ permutations of $[n+1]$ with $k+1$ ascents having this form. Thus, there are $(n-k)\left\langle {n \atop k} \right\rangle + (k+2)\left\langle {n \atop k+1} \right\rangle$ permutations of the set $[n+1]$ with $k+1$ ascents. □

n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$	$\left\langle {n \atop 6} \right\rangle$
0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1	4	1	0	0	0	0
4	1	11	11	1	0	0	0
5	1	26	66	26	1	0	0
6	1	57	302	302	57	1	0

Figure 3.9: Matrix of Eulerian numbers

The Eulerian version of the binomial theorem [2.1.6] may be viewed as a change of basis for univariate polynomials.

Theorem 3.3.5 (Worpitzky)**.** *For any nonnegative integer n, we have*

$$x^n = \sum_{k \in \mathbb{Z}} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}.$$

Double-counting proof. It suffices to verify the identity when x is a sufficient large integer. How many integer n-tuples $(a_1, a_2, \dots, a_n)$ where $1 \leqslant a_j \leqslant x$ for all $1 \leqslant j \leqslant n$ are there?

Answer 1: Each entry a_j may be chosen independently from the x possibilities, so there are x^n well-formed n-tuples.

Answer 2: Rearrange the n-tuple so that $a_{\sigma(1)} \geqslant a_{\sigma(2)} \geqslant \cdots \geqslant a_{\sigma(n)}$ and, for all $1 \leqslant j < n$, $a_{\sigma(j)} = a_{\sigma(j+1)}$ implies that $\sigma(j) < \sigma(j+1)$. The permutation $\sigma := \sigma(1) \;\; \sigma(2) \;\; \cdots \;\; \sigma(n)$ of the set $[n]$ is uniquely determined by an n-tuple $(a_1, a_2, \dots, a_n)$. It suffices to prove that each permutation having k ascents and $n-1-k$ descents arises from exactly $\binom{x+k}{n}$ n-tuples. If the permutation σ has a descent at the index j, then we have $\sigma(j) > \sigma(j+1)$ and $a_{\sigma(j)} > a_{\sigma(j+1)}$. For example, any $(a_1, a_2, \dots, a_9)$ giving rise to the permutation $5 \;\; 8 \;\; 3 \;\; 2 \;\; 4 \;\; 6 \;\; 7 \;\; 9 \;\; 1$ must satisfy

$$x \geqslant a_5 \geqslant a_8 > a_3 > a_2 \geqslant a_4 \geqslant a_6 \geqslant a_7 \geqslant a_9 > a_1 \geqslant 1,$$

because this permutation has descents at the indices 2, 3, and 8. This chain of inequalities is equivalent to

$$x + 5 \geqslant a_5 + 5 > a_8 + 4 > a_3 + 4 > a_2 + 4 > a_4 + 3 > a_6 + 2 > a_7 + 1 > a_9 > a_1 \geqslant 1,$$

and the definition of the binomial coefficient implies that there are $\binom{x+5}{9}$ choices for the subset

$$\{a_1, a_9, a_7 + 1, a_6 + 2, a_4 + 3, a_2 + 4, a_3 + 4, a_8 + 4, a_5 + 5\} \subseteq [x+5].$$

Generalizing this argument, we see that each permutation with k ascents will be obtained from $\binom{x+k}{n}$ n-tuples. □

Inductive proof. When $n = 0$, we have $x^0 = 1 = \left\langle {0 \atop 0} \right\rangle \binom{x+k}{0}$, so the base case holds. For some nonnegative integer n, assume that

$$x^n = \sum_{k \in \mathbb{Z}} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}.$$

Using the induction hypothesis, the absorption identity 2.1.3 (twice), the addition formula 2.0.6 for binomial coefficients, and the addition formula 3.3.4 for Eulerian numbers gives

$$\begin{aligned} x^{n+1} = (x)(x^n) &= x \left[\sum_{k \in \mathbb{Z}} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \right] \\ &= \sum_{k \in \mathbb{Z}} \left\langle {n \atop k} \right\rangle \left[(x+k+1) \binom{x+k}{n} - (k+1) \binom{x+k}{n} \right] \\ &= \sum_{k \in \mathbb{Z}} \left\langle {n \atop k} \right\rangle \left[(n+1) \binom{x+k+1}{n+1} - (k+1) \binom{x+k+1}{n+1} + (k+1) \binom{x+k}{n+1} \right] \end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{k\in\mathbb{Z}} (n-k)\left\langle {n \atop k} \right\rangle \binom{x+k+1}{n+1}\right] + \left[\sum_{k\in\mathbb{Z}} (k+1)\left\langle {n \atop k} \right\rangle \binom{x+k}{n+1}\right] \\
&= \left[\sum_{k\in\mathbb{Z}} (n-k+1)\left\langle {n \atop k-1} \right\rangle \binom{x+k}{n+1}\right] + \left[\sum_{k\in\mathbb{Z}} (k+1)\left\langle {n \atop k} \right\rangle \binom{x+k}{n+1}\right] \\
&= \sum_{k\in\mathbb{Z}} \left[(n-k+1)\left\langle {n \atop k-1} \right\rangle + (k+1)\left\langle {n \atop k} \right\rangle\right] \binom{x+k}{n+1} \\
&= \sum_{k\in\mathbb{Z}} \left\langle {n+1 \atop k} \right\rangle \binom{x+k}{n+1}. \qquad \square
\end{aligned}
$$

Exercises

Problem 3.3.6. An ***excedance*** of a permutation σ of the set $[n]$ is index j such that $\sigma(j) > j$. For all nonnegative integers n and k, give a bijective proof that the number of permutations of the set $[n]$ with k excedances coincides with the number of permutations of the set $[n]$ with k descents.

Problem 3.3.7. For any nonnegative integer n, a ***Stirling permutation*** is a permutation of the multiset $M_n := \{1^2, 2^2, \dots, n^2\}$ such that, for each element j appearing in the permutation, the elements between the two copies of j are larger than j. For any nonnegative integers n and any integer k, the ***Eulerian number of the second kind***, denoted $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$, counts the number of Stirling permutations of the multiset M_n that have k ascents.

(i) For any nonnegative integer n, prove by induction that the number of Stirling permutations of M_{n+1} is $(2n+1)!!$.

(ii) For all nonnegative integers n and k, verify the additive formula for Eulerian number of the second kind via double-counting:

$$\left\langle\!\left\langle {n+1 \atop k+1} \right\rangle\!\right\rangle = (2n-k)\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle + (k+2)\left\langle\!\left\langle {n \atop k+1} \right\rangle\!\right\rangle.$$

(iii) For all integer n and k satisfying $0 \leqslant n, k \leqslant 5$, compute the matrix whose (n,k)-entry is $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$.

1 1 2 2 3 3, 1 1 2 3 3 2,
1 1 3 3 2 2, 1 3 3 1 2 2,
3 3 1 1 2 2, 1 2 2 1 3 3,
1 2 2 3 3 1, 1 2 3 3 2 1,
1 3 3 2 2 1, 3 3 1 2 2 1,
2 2 1 1 3 3, 2 2 1 3 3 1,
2 2 3 3 1 1, 2 3 3 2 1 1,
3 3 2 2 1 1.

Figure 3.10: The 15 Stirling permutations of M_3

From Table 3.10, we see that $\left\langle\!\left\langle {3 \atop 0} \right\rangle\!\right\rangle = 1$, $\left\langle\!\left\langle {3 \atop 1} \right\rangle\!\right\rangle = 8$, and $\left\langle\!\left\langle {3 \atop 2} \right\rangle\!\right\rangle = 6$.

4
Combinatorial Models

We feature some of the most common combinatorial gadgets. Although integer partitions, trees, and lattice paths link combinatorics to representation theory, computer science, and probability theory, we confine our exploration to their enumerative aspects.

4.0 Integer Partitions

Despite the semantic overload, the word 'partition' has a second mathematical meaning.

Definition 4.0.1. A ***partition*** of an integer n is an integer sequence $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_j, \dots)$ such that $|\lambda| := \sum_{j \in \mathbb{N}} \lambda_j = n$ and

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_j \geqslant \cdots \geqslant 0 .$$

The notation $\lambda \vdash n$ means that λ is an integer partition of n.

We do not distinguish between sequences that differ only the number of trailing zeros. For example, we identify the sequences $(3, 3, 2, 1, 0, 0, \dots)$, $(3, 3, 2, 1, 0, 0, 0)$, and $(3, 3, 2, 1)$ as partitions of 9. The nonzero terms λ_j are the ***parts*** of λ and the number of parts of λ is its ***length***. When the partition λ has m_j parts equal to j, we may write $\lambda = (\dots, j^{m_j}, \dots, 2^{m_2}, 1^{m_1})$, so $(4, 4, 2, 2, 2, 1) = (4^2, 2^3, 1)$.

There 7 partitions of the integer 5:

$$\begin{array}{ll} (1,1,1,1,1)\,, & (2,1,1,1)\,, \\ (2,2,1)\,, & (3,1,1)\,, \\ (3,2)\,, & (4,1)\,, \\ (5)\,. & \end{array}$$

Hence, we have $p(5) = 7$, $p_5(5) = 1$, $p_4(5) = 1$, $p_3(5) = 2$, $p_2(5) = 2$, $p_1(5) = 1$, and $p_0(5) = 0$.

Definition 4.0.2. For any nonnegative integer n and any integer k, let $p_k(n)$ be the number of partitions of the integer n with k parts. The ***partition function*** $p(n)$ counts the total number of partitions of the nonnegative integer n, so $p(n) = \sum_{k \in \mathbb{Z}} p_k(n)$.

Some special values are easy to determine.

- For any nonnegative integer n, we have $p_n(n) = 1$ because (1^n) is the unique partition with n parts. Notice that $p_0(0) = p(0) = 1$ corresponds to the empty partition.
- For all $k > n$, we have $p_k(n) = 0$ because each part is a nonnegative integer.
- For any positive integer n, we have $p_0(n) = 0$ because every partition of a positive integer has at least one part.
- For all integers n such that $n \geqslant 2$, we have $p_{n-1}(n) = 1$ because $n = 2 + \sum_{j=1}^{n-2} 1$ is the unique partition with $n - 1$ parts.
- For any positive integer n, we have $p_1(n) = 1$ because (n) is the unique partition of n with one part.
- For any positive integer n, we have $p_2(n) = \lfloor n/2 \rfloor$ because the integer partitions with two parts are $(n - j, j)$ where $1 \leqslant j \leqslant n/2$.

The function $p_k(n)$ satisfies a two-term recurrence which deviates slightly from our established pattern.

Proposition 4.0.3 (Addition). *For any nonnegative integer n and any integer k, we have $p_{k+1}(n+1) = p_k(n) + p_{k+1}(n-k)$.*

Double-counting proof. How many $(x_1, x_2, \dots, x_{k+1}) \in \mathbb{N}^{k+1}$ satisfy $n+1 = x_1 + x_2 + \cdots + x_{k+1}$ and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{k+1} \geqslant 1$?

Answer 1: By definition, the number of partitions of the integer $n+1$ with $k+1$ parts is $p_{k+1}(n+1)$.

Answer 2: Focus on x_{k+1}. If $x_{k+1} = 1$, then there are $p_k(n)$ tuples. When $x_{k+1} > 1$, there is a bijection between the $(k+1)$-tuples $(x_1, x_2, \dots, x_{k+1}) \in \mathbb{N}^k$ satisfying both $n+1 = x_1 + x_2 + \cdots + x_{k+1}$ and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k > 1$, and the $(y_1, y_2, \dots, y_{k+1}) \in \mathbb{N}^{k+1}$ satisfying $n-k = y_1 + y_2 + \cdots + y_{k+1}$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{k+1} \geqslant 1$ defined by $y_j := x_j - 1$ for all $1 \leqslant j \leqslant k+1$. Hence, there are $p_{k-1}(n-k)$ solutions in this second case. We conclude that the total number of $(x_1, x_2, \dots, x_{k+1}) \in \mathbb{N}^{k+1}$ satisfying $n+1 = x_1 + x_2 + \cdots + x_{k+1}$ and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{k+1} \geqslant 1$ is $p_k(n) + p_{k+1}(n-k)$. □

Combining the addition formula for the restricted partition function with the boundary cases above, we may compute the matrix whose (n,k)-entry is $p_k(n)$; see Table 4.1.

n	$p_0(n)$	$p_1(n)$	$p_2(n)$	$p_3(n)$	$p_4(n)$	$p_5(n)$	$p_6(n)$	$p_7(n)$	$p_8(n)$	$p(n)$
0	1	0	0	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0	1
2	0	1	1	0	0	0	0	0	0	2
3	0	1	1	1	0	0	0	0	0	3
4	0	1	2	1	1	0	0	0	0	5
5	0	1	2	2	1	1	0	0	0	7
6	0	1	3	3	2	1	1	0	0	11
7	0	1	3	4	3	2	1	1	0	15
8	0	1	4	5	5	3	2	1	1	22

Table 4.1: Matrix of partition numbers

Collections of integer partitions subject to various restrictions have interesting characterizations.

Problem 4.0.4. For any nonnegative integer n, demonstrate that the number of partitions of the integer n into distinct parts equals the number of partitions of the integer n with odd parts.

Bijective Proof.

Set 1: Consider the set of partitions of n with odd parts.

Set 2: Consider the set of partitions of n into distinct parts.

Correspondence: To exhibit the required bijections, the key idea is that every positive integer can be express uniquely as a product of an odd positive integer and a power of 2. A partition of the integer n with odd parts maps to a partition of the integer n with distinct parts by replacing the multiplicity of each odd part with its binary expansion. Conversely, a partition of the integer n into distinct parts maps to a partition with odd parts by expressing each part as a product of a power of 2 and an odd positive integer, and then consolidating the odd integers. For example, we have

$$(7^3, 5^2, 3) \leftrightarrow (14, 10, 7, 3) \qquad \text{and}$$
$$(9^5, 5^{12}, 3^2, 1^3) \leftrightarrow (40, 36, 20, 9, 6, 2, 1)$$

because

$$\begin{aligned} 34 &= (3)(7) + (2)(5) + (1)(3) \\ &= (2^0 + 2^1)(7) + (2^1)(5) + (2^0)(3) \\ &= (2^1)(7) + (2^1)(5) + (2^0)(7) + (2^0)(3) \\ &= 14 + 10 + 7 + 3\,, \\ 114 &= (5)(9) + (12)(5) + (2)(3) + (3)(1) \\ &= (2^0 + 2^2)(9) + (2^2 + 2^3)(5) + (2^1)(3) + (2^0 + 2^1)(1) \\ &= (2^3)(5) + (2^2)(9) + (2^2)(5) + (2^0)(9) + (2^1)(3) + (2^1)(1) + (2^0)(1) \\ &= 40 + 36 + 20 + 9 + 6 + 2 + 1\,. \end{aligned}$$

These maps compose, in either order, to the identity map. Given the bijections, we see that the number of partitions of the integer n into distinct parts equals the number of partitions of the integer n with odd parts. □

Exercises

Problem 4.0.5. For all $m, n \in \mathbb{N}$, show that

$$\sum_{j=0}^{m} p_j(n) = p_k(n + m)\,.$$

Problem 4.0.6. Let $q_k(n)$ denote the number of partitions of the integer $n \in \mathbb{N}$ into k distinct parts. For all $m, n \in \mathbb{N}$, show that $q_m\big(n + \binom{m}{2}\big) = p_m(n)$.

Problem 4.0.7. Let $p(n)$ denote the number of partitions of the integer n. Express the number of partitions of n with no part equal to 1 as a linear combination of values $p(k)$ for some $k \in \mathbb{N}$.

4.1 Trees and Catalan Numbers

Among the myriad of combinatorial interpretations for Catalan numbers, Richard Stanley singles out a few as being the most fundamental. We highlight two these fundamental interpretations related to special graphs. A ***tree*** is a graph in which any two vertices are connected by a unique path, or equivalently a connected acyclic graph. A ***rooted tree*** is a tree in which one vertex has been designated as the root.