

part with its binary expansion. Conversely, a partition of the integer n into distinct parts maps to a partition with odd parts by expressing each part as a product of a power of 2 and an odd positive integer, and then consolidating the odd integers. For example, we have

$$(7^3, 5^2, 3) \leftrightarrow (14, 10, 7, 3) \qquad \text{and}$$
$$(9^5, 5^{12}, 3^2, 1^3) \leftrightarrow (40, 36, 20, 9, 6, 2, 1)$$

because

$$\begin{aligned} 34 &= (3)(7) + (2)(5) + (1)(3) \\ &= (2^0 + 2^1)(7) + (2^1)(5) + (2^0)(3) \\ &= (2^1)(7) + (2^1)(5) + (2^0)(7) + (2^0)(3) \\ &= 14 + 10 + 7 + 3\,, \\ 114 &= (5)(9) + (12)(5) + (2)(3) + (3)(1) \\ &= (2^0 + 2^2)(9) + (2^2 + 2^3)(5) + (2^1)(3) + (2^0 + 2^1)(1) \\ &= (2^3)(5) + (2^2)(9) + (2^2)(5) + (2^0)(9) + (2^1)(3) + (2^1)(1) + (2^0)(1) \\ &= 40 + 36 + 20 + 9 + 6 + 2 + 1\,. \end{aligned}$$

These maps compose, in either order, to the identity map. Given the bijections, we see that the number of partitions of the integer n into distinct parts equals the number of partitions of the integer n with odd parts. □

Exercises

Problem 4.0.5. For all $m, n \in \mathbb{N}$, show that

$$\sum_{j=0}^{m} p_j(n) = p_k(n+m)\,.$$

Problem 4.0.6. Let $q_k(n)$ denote the number of partitions of the integer $n \in \mathbb{N}$ into k distinct parts. For all $m, n \in \mathbb{N}$, show that $q_m\big(n + \binom{m}{2}\big) = p_m(n)$.

Problem 4.0.7. Let $p(n)$ denote the number of partitions of the integer n. Express the number of partitions of n with no part equal to 1 as a linear combination of values $p(k)$ for some $k \in \mathbb{N}$.

4.1 Trees and Catalan Numbers

Among the myriad of combinatorial interpretations for Catalan numbers, Richard Stanley singles out a few. We highlight two these fundamental interpretations related to special graphs. A ***tree*** is a graph in which any two vertices are connected by a unique path or equivalently a tree is a connected acyclic graph. A ***rooted tree*** is a tree in which one vertex has been designated as the root.

A ***binary tree*** is rooted tree where every vertex has at most two children and each child is labelled as left or right. Binary trees

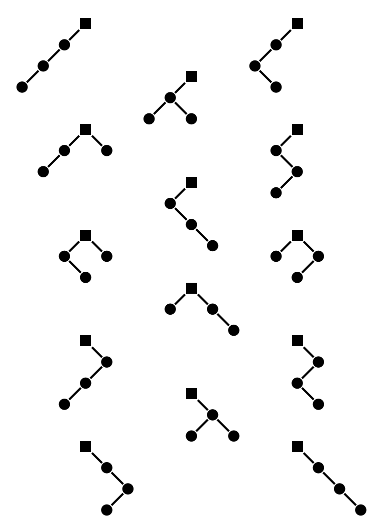

Figure 4.1: The 14 binary trees with 4 vertices

may be recursively constructed as follows: the empty set ∅ is a binary tree; if B_ℓ and B_r are binary trees, then we obtain the new binary tree B by making the root of B_ℓ the left child of a new root vertex and making the root of B_r the right child of this new root.

A ***plane tree*** is a rooted tree in which an ordering is specified for the children of each vertex—an ordering on the children is equivalent to an embedding of the tree in the plane. Plane trees are recursively constructed as follows: a single vertex is a plane tree; given a sequence $(P_1, P_2, \dots, P_m)$ of plane trees, the new plane tree P is obtained by adding edges incident to the roots in each P_j to a new root vertex.

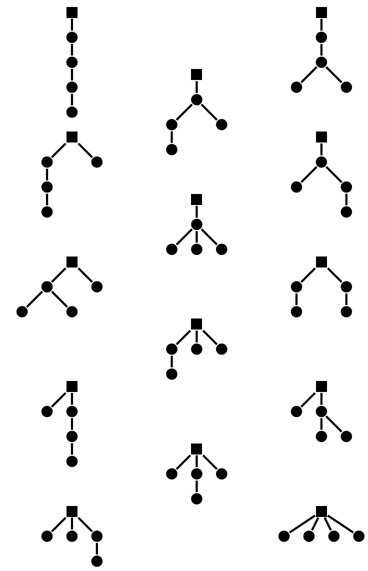

Figure 4.2: The 14 plane trees with 5 vertices

Theorem 4.1.1. *For any nonnegative integer n, the Catalan number C_n counts each of the following:*
(a) *triangulations T of a convex polygon with $n+2$ vertices,*
(b) *binary trees B with n vertices, and*
(c) *plane trees P with $n+1$ vertices.*

Bijection between (a) *and* (b).
Set 1: Consider all triangulations T of a convex $(n+2)$-gon.
Set 2: Consider all binary trees B with n vertices.
Correspondence: Fix a convex $(n+2)$-gon and select a distinguished edge e of this polygon. Given a triangulation T of our convex $(n+2)$-gon, put a vertex in the interior of each triangle. Since T consists of n triangles, there are n vertices. The root is the vertex in the unique triangle containing the edge e. Draw an edge between any two vertices that are separated by a diagonal in T. If we arrive from the root to a vertex v by crossing a diagonal f of T, then we transverse the edges of the triangle containing v in a counterclockwise order beginning with the edge f. The first edge after f defines the left child of v and the second edge defines the right child of v; see Figure 4.3 for an example. Hence, we have constructed a binary tree B with n vertices.

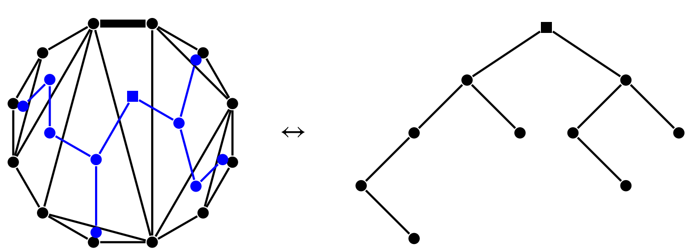

Figure 4.3: Bijection between a triangulation of a 12-gon and a binary tree

Conversely, given a binary tree B having n vertices, we build a triangulation our convex $(n+2)$-gon. As we traverse the binary tree B starting from the root, we insert a diagonal for each child of a vertex v. If the last edge on the path from the root to the vertex v corresponds to a diagonal f (or the distinguished edge e if v is the root), then we join the ends of the f to the vertex of the $(n+2)$-gon that is $|B_\ell|+1$ counterclockwise sides away from one end and $|B_r| + 1$ clockwise sides away from the other. Since B has $n - 1$ edges, we insert a total of $n - 1$ diagonals into the $(n + 2)$-gon and, therefore, obtain a triangulation T.

By construction, these maps compose, in either order, to the identity map. □

Bijection between (b) *and* (c).

Set 1: Consider all binary trees B with n vertices.

Set 2: Consider all planar trees P with $n + 1$ vertices.

Correspondence: Given a plane tree with $n + 1$ vertices, remove the root vertex and all incident edges. Next, remove every edge that is not the leftmost edge from a vertex. The remaining edges become the left edges in a binary tree B whose root is the leftmost child of the root in P. Draw edges from each child of a vertex in the plane tree to the next child. These horizontal edges are the right edges of the binary tree B. Hence, we have constructed a binary tree B with n vertices.

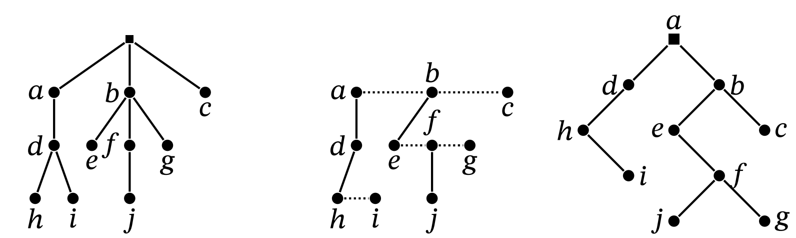

Figure 4.4: Bijection between a binary tree and a planar tree

This procedure is reversible, so we have a bijection. □

Exercises

Problem 4.1.2. A ***complete binary tree*** is a binary tree in which every vertex has either zero or two children. For any nonnegative integer n, show that the Catalan number C_n equals the number of complete binary trees with $2n + 1$ vertices.

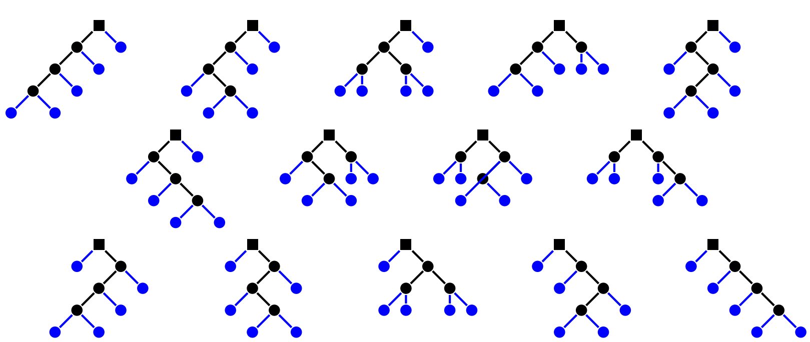

Figure 4.5: The 14 complete binary trees with 9 vertices

4.2 Lattice Paths

Because of their ubiquity in physical and probabilistic models, we introduce a new class of combinatorial objects. A ***lattice path*** in $\mathbb{Z}^d$ of length $\ell \in \mathbb{N}$ with steps belonging to a set $\mathcal{X}$ of integral vectors is a sequence $v_0, v_1, \dots, v_\ell \in \mathbb{Z}^d$ of lattice points such that, for all $1 \leqslant j \leqslant \ell$, the vector $v_j - v_{j-1}$ lies in the set $\mathcal{X}$.

We start with arguably the simplest enumeration problem for lattice paths. A ***north-east*** lattice path is a lattice path in $\mathbb{Z}^2$ with steps in the set $\{(0,1),(1,0)\}$.

Lemma 4.2.1. *For all nonnegative integers m and n, the number of north-east lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$.*

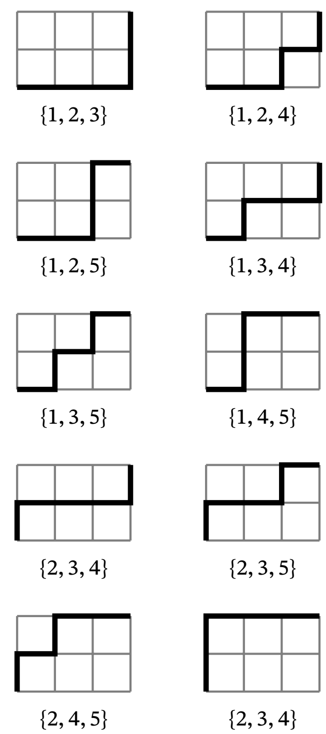

Figure 4.6: The 10 north-east lattice paths from $(0,0)$ to $(3,2)$.

Bijective proof. Since the possible steps form a basis for $\mathbb{Z}^2$, a north-east lattice path from $(0,0)$ to (m,n) has length $m+n$ and exactly m $(1,0)$-steps.

Set 1: Consider all subsets $\mathcal{A}$ of the set $[m+n]$ having cardinality m. The definition of the binomial coefficient implies that there are $\binom{m+n}{n}$ subsets having this form.

Set 2: Consider all north-east paths from $(0,0)$ to (m,n).

Correspondence: An subset $\mathcal{A}$ of $[m+n]$ having cardinality m maps to the lattice path $v_0, v_1, \dots, v_{m+n}$ where $v_0 = (0,0)$ and

$$v_j - v_{j-1} = \begin{cases} (1,0) & \text{if } j \in \mathcal{A}, \\ (0,1) & \text{if } j \notin \mathcal{A}. \end{cases}$$

Since $|\mathcal{A}| = m$, this path has m $(1,0)$-steps and n $(0,1)$-steps, so we deduce that $v_{m+n} = (m,n)$. Conversely, a north-east lattice path $v_0, v_1, \dots, v_{m+n}$ from $(0,0)$ to (m,n) maps to the subset

$$\{j \in [m+n] \mid v_j - v_{j-1} = (1,0)\}.$$

By construction, these maps compose, in either order, to the identity map.

Given these bijections, the number of north-east lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{n}$. □

To incorporate a linear boundary condition, we deploy the powerful reflection principle.

Theorem 4.2.2. *For any nonnegative integer n, the number of north-east lattice paths from $(0,0)$ to (n,n) that never go above the main diagonal is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Indirect bijective proof. To begin, we prove that there are $\binom{2n}{n-1}$ north-east lattice paths from $(0,0)$ to (n,n) that do go above the main diagonal (also known as is the line $y = x$).

Set 1: Consider all north-east paths from $(0,0)$ to $(n-1, n+1)$. Lemma 4.2.1 proves that this set has cardinality $\binom{2n}{n-1}$.

Set 2: Consider all north-east lattice paths from $(0,0)$ to (n,n) that go above the main diagonal.

Correspondence: Since the points $(0,0)$ and $(n-1,n+1)$ lie on opposite sides of the superdiagonal (also known as the line $y = x+1$), any piecewise linear curve joining these points must intersect the superdiagonal. Given a north-east lattice path $v_0, v_1, \dots, v_{2n}$ from $(0,0)$ to $(n-1,n+1)$, let v_{2j-1} be first point on the superdiagonal. Since $v_{2j-1} = (j-1,j)$ for some $1 \leqslant j < n$, the portion of this path coming after v_{2j-1} consists of $n-j$ $(1,0)$-steps and $n-j+1$ $(0,1)$-steps. We map this lattice path to a new lattice path obtained by reflecting, in the superdiagonal, the portion of the original path coming after v_{2j-1}. In other words, we interchange the last $2n-2j+1$ steps producing a lattice path from $(j-1,j)$ to $(j-1+n-j+1, j+n-j) = (n,n)$. On the other hand, given a north-east lattice path $w_0, w_1, \dots, w_{2n}$ from $(0,0)$ to (n,n) that goes above the main diagonal, let w_{2k-1} be the first point on the super diagonal. Since $w_{2k-1} = (k-1,k)$, for some $1 \leqslant k < n$, the portion of this path coming after w_{2k-1} consists of $n-k+1$ $(1,0)$-steps and $n-k$ $(0,1)$-steps. We map this lattice path to a new lattice path obtained by reflecting, in the superdiagonal, the portion of the original path coming after w_{2j-1}; we interchange the last $2n-2k+1$ steps giving a path from $(k-1,k)$ to $(k-1+n-k, k+n-k+1) = (n-1,n+1)$. Reflecting in the superdiagonal is an involution, so these maps are mutual inverses.

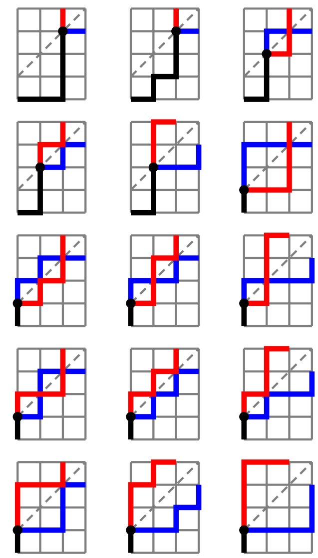

Figure 4.7: Bijection between the lattice paths from $(0,0)$ to $(2,4)$ and the lattice paths from $(0,0)$ to $(3,3)$ going above the diagonal.

Given these bijections, the number of north-east lattice paths from $(0,0)$ to (n,n) that go above the main diagonal is $\binom{2n}{n-1}$.

To finish the proof, we calculate that the number of north-east lattice paths from $(0,0)$ to (n,n) that never go above the main diagonal by subtract the number that do from the total number of lattice paths:

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \binom{2n}{n} - \frac{(2n)!}{(n-1)!(n+1)!} \\ &= \binom{2n}{n} - \left(\frac{n}{n+1}\right)\frac{(2n)!}{n!\,n!} \\ &= \left(1 - \frac{n}{n+1}\right)\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}. \qquad \square\end{aligned}$$

Remark 4.2.3. Rotating by $-\pi/4$ and reflecting in the horizontal axis, we obtain another collection of lattice paths is commonly associated with the Catalan numbers. A ***Dyck path*** is a lattice path on $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with steps in the set $\{(1,1),(1,-1)\}$ that never passes below the horizontal axis.

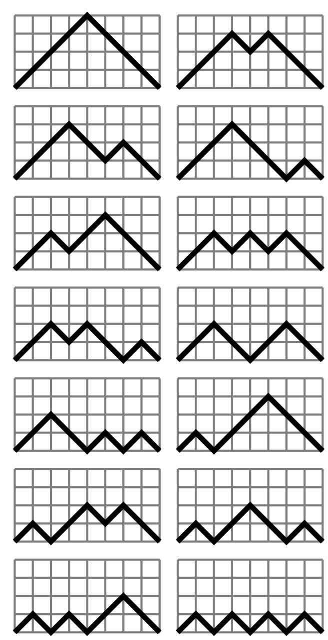

Figure 4.8: The 14 Dyck paths of length 8

Exercises

Problem 4.2.4. For any nonnegative integer n, prove that the Catalan number C_n counts the expressions containing n pairs of parentheses that are correctly matched.

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Figure 4.9: The 14 expressions containing 4 pairs of matched parentheses

Problem 4.2.5. A ***peak*** in a north-east lattice path is a point $(j, k) \in \mathbb{Z}^2$ on the path such that the points $(j, k-1)$ and $(j+1, k)$ are also on the path. Similarly, a ***valley*** is a point $(j, k) \in \mathbb{Z}^2$ on the path such that the points $(j-1, k)$ and $(j, k+1)$ are also on the path. For all nonnegative integers n and k, the ***Narayana number*** $N(n, k)$ counts the north-east lattice paths from $(0, 0)$ to (n, n) that never go above the main diagonal and have k peaks. Using a double-counting argument, show that $(k+1)\,N(n, k) = \binom{n}{k}\binom{n-1}{k}$.

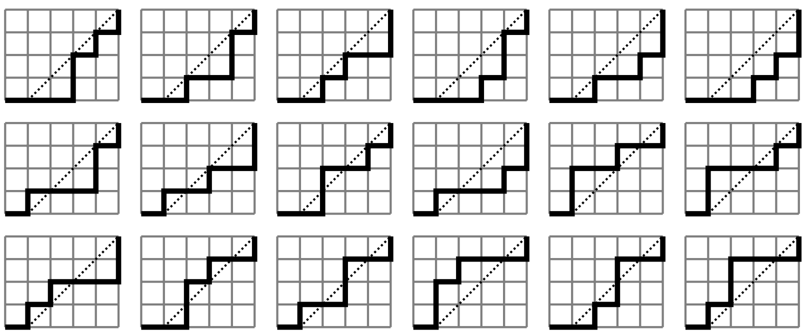

Figure 4.10: A special collection of 18 north-east lattice paths from $(-1, 0)$ to $(4, 4)$.

5
Involutions

Many valuable combinatorial formulas have both positive and negative signs. At first glance, it might seem that bijective proofs are not well-suited to deal with these formulas. We refine and extend our bijective techniques to address this apparent shortfall.

5.0 Signed Sets

To prove identities involving positive and negative signs, we focus on a special kind of bijection.

Definition 5.0.0. A ***signed set*** is a set $\mathcal{X}$ equipped with a function $\operatorname{sgn} : \mathcal{X} \to \{\pm 1\}$. This data is equivalent to partitioning the set $\mathcal{X}$ into its ***positive block*** $\mathcal{X}^+ := \{x \in \mathcal{X} \mid \operatorname{sgn}(x) = +1\}$ and its ***negative block*** $\mathcal{X}^- := \{x \in \mathcal{X} \mid \operatorname{sgn}(x) = -1\}$. A ***sign-reversing involution*** is a function $\varphi : \mathcal{X} \to \mathcal{X}$ such that $\varphi^2 = \varphi \circ \varphi = \operatorname{id}_{\mathcal{X}}$ and, for all $x \in \mathcal{X}$, either $\varphi(x) = x$ or $\operatorname{sgn}\big(\varphi(x)\big) = -\operatorname{sgn}(x)$. The ***fixed-point subset*** of the function φ is $\mathcal{X}^\varphi := \{x \in \mathcal{X} \mid \varphi(x) = x\}$.

This nomenclature leads to the next influential observation. The formula in the second part becomes very useful when the right sum has far fewer terms than the left one.

Proposition 5.0.1. *Let $\mathcal{X}$ be a finite signed set.*

i. *A function $\varphi : \mathcal{X} \to \mathcal{X}$ is an involution if and only if the set $\mathcal{X}$ is disjoint union of the fixed points and the 2-cycles of φ.*
ii. *For any sign-reversing involution $\varphi : \mathcal{X} \to \mathcal{X}$, we have*

$$\sum_{x \in \mathcal{X}^+} 1 - \sum_{x \in \mathcal{X}^-} 1 = \sum_{x \in \mathcal{X}} \operatorname{sgn}(x) = \sum_{x \in \mathcal{X}^\varphi} \operatorname{sgn}(x)\,.$$

Proof.

i. Since $\varphi : \mathcal{X} \to \mathcal{X}$ is a bijection, this function is a permutation of finite set $\mathcal{X}$, so φ can be decomposed into cycles [3.1.2]. By definition, the length of the cycle containing $x \in \mathcal{X}$ is the small positive integer m such that $\varphi^m(x) = x$. Thus, φ is an involution if and only if its cycle decomposition contains only 1-cycles (also known as fixed points) and 2-cycles.

ii. When the element $x \in \mathcal{X}$ belongs to a 2-cycle of φ, we have $\operatorname{sgn}(x) + \operatorname{sgn}\big(\varphi(x)\big) = 0$. It follows that all elements in 2-cycle cancel in the sum which leaves only the fixed points. □

As initial applications, we use sign-reversing involutions to establish some binomial identities.

Problem 5.0.2. For any nonnegative integer n, we have

$$\sum_{k\in\mathbb{Z}} (-1)^k \binom{n+1}{k} = 0\,.$$

Involutive solution. Consider all subsets of the set $[n+1]$.

Positive block: All subsets of $[n+1]$ having even cardinality. The definition of the binomial coefficients implies that there are $\sum_{k\in\mathbb{Z}} \binom{n+1}{2k}$ subsets having even cardinality.

Negative block: All subsets of $[n+1]$ having odd cardinality. The definition of the binomial coefficients implies that there are $\sum_{k\in\mathbb{Z}} \binom{n+1}{2k+1}$ subsets having odd cardinality.

Involution: For any subset $\mathcal{A} = \{a_1, a_2, \dots, a_k\}$ of the set $[n+1]$, the ***symmetric difference*** of $\mathcal{A}$ and the singleton $\{n+1\}$ is defined by

$$\mathcal{A} \ominus \{n+1\} := \begin{cases} \{a_1, a_2, \dots, a_k, n+1\} & \text{if } n+1 \notin \mathcal{A}; \\ \{a_1, a_2, \dots, a_{k-1}\} & \text{if } n+1 \in \mathcal{A}. \end{cases}$$

Since $(\mathcal{A} \ominus \{n+1\}) \ominus \{n+1\} = \mathcal{A}$, this operation is involutive. Moreover, the set $\mathcal{A} \ominus \{n+1\}$ has either one more or one fewer element than $\mathcal{A}$, so this involution is also sign-reversing.

More generally, the ***symmetric difference*** of two sets is the set of elements which are in either of the sets and not in their intersection. In other words, given two sets $\mathcal{X}$ and $\mathcal{Y}$, the symmetric difference is $\mathcal{X} \ominus \mathcal{Y} := (\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})$.

The sign-reversing involution has no fixed points, so we obtain

$$\sum_{k\in\mathbb{Z}} \binom{n+1}{2k} - \sum_{k\in\mathbb{Z}} \binom{n+1}{2k+1} = \sum_{k\in\mathbb{Z}} (-1)^k \binom{n+1}{k} = 0\,. \qquad \square$$

By tweaking our approach to the previous identity, we obtain a charming enhancement.

Problem 5.0.3. For all nonnegative integers m and n, we have

$$\sum_{k\leqslant m} (-1)^k \binom{n+1}{k} = (-1)^m \binom{n}{m}.$$

Involutive solution. Consider all subsets of the set $[n+1]$ having cardinality at most m.

Positive block: All subsets having even cardinality. The definition of the binomial coefficients implies that there are $\sum_{k\leqslant m/2} \binom{n+1}{2k}$ such subsets.

Negative block: All subsets having odd cardinality. The definition of the binomial coefficients implies that there are $\sum_{k\leqslant (m-1)/2} \binom{n+1}{2k+1}$ such subsets.

Involution: We have already have the sign-reversing involution $\mathcal{A} \mapsto \mathcal{A} \ominus \{n+1\}$ on all subset of $[n+1]$. However, it does not immediately restrict to subsets of cardinality at most m. To rectify this issue, we consider two cases.

- When m is even and the subset $\mathcal{A}$ has even cardinality, the image $\mathcal{A} \ominus \{n+1\}$ may have cardinality more m. This occurs in $\binom{n}{m}$ instances where $|\mathcal{A}| = m$ and $n+1 \notin \mathcal{A}$. Thus, we redefine the involution making these $\binom{n}{m}$ unmatched positive elements into fixed points.

- When m is odd and the subset $\mathcal{A}$ has even cardinality less than m, then image $\mathcal{A} \ominus \{n+1\}$ also has cardinality at most m. However, we miss some the subsets of odd cardinality at most m. This occurs in $\binom{n}{m}$ instances where the subset contains m elements from $[n]$. Once again, we redefine the involution making these $\binom{n}{m}$ unmatched negative elements into fixed points.

Combining the two cases, we obtain

$$\sum_{k \leqslant m/2} \binom{n+1}{2k} - \sum_{k \leqslant (m-1)/2} \binom{n+1}{2k+1} = \sum_{k \leqslant m} (-1)^k \binom{n+1}{k} = (-1)^m \binom{n}{m}. \square$$

Exercises

Problem 5.0.4. For any nonnegative n, use a sign-reversing involution to prove that

$$\sum_{k \in \mathbb{Z}} (-1)^k \left[{n+2 \atop k} \right] = 0 .$$

Problem 5.0.5. For all nonnegative integers m and n, use a sign-reversing involution to prove that

$$\sum_{k \in \mathbb{Z}} (-1)^k \binom{m+n}{m-k} \left(\!\!\binom{n}{k}\!\!\right) = 1 .$$