7.3 Partition Identities from Series

Generating series also provide new insights into identities involving Stirling subset numbers and integer partitions. For completeness, we start by reproving Theorem 3.0.4.

Proposition 7.3.1 (Stirling series)**.** *For any nonnegative integer k, we have*
$$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{n\in\mathbb{N}} \left\{ {n \atop k} \right\} x^n .$$

Inductive series proof. For any nonnegative integer k, set
$$S_k(x) := \sum_{n\in\mathbb{N}} \left\{ {n \atop k} \right\} x^n .$$
When $k = 0$, we have $S_0(x) = 1$ because $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {n+1 \atop 0} \right\} = 0$ for all nonnegative integers n. The addition formula [3.0.3] for Stirling subset numbers gives
$$\begin{aligned} S_{k+1}(x) &= \sum_{n\in\mathbb{N}} \left\{ {n \atop k+1} \right\} x^n = \sum_{n\in\mathbb{N}} \left(\left\{ {n \atop k} \right\} + (k+1) \left\{ {n \atop k+1} \right\} \right) x^n \\ &= \left(\sum_{n\in\mathbb{N}} \left\{ {n-1 \atop k} \right\} x^n \right) + (k+1) \left(\sum_{n\in\mathbb{N}} \left\{ {n-1 \atop k+1} \right\} x^n \right) \\ &= x\, S_k(x) + (k+1)x\, S_{k+1}(x) \end{aligned}$$
We deduce that $\big(1-(k+1)x\big)S_{k+1}(x) = x\, S_k(x)$. The induction hypothesis states that $S_k(x) = \prod_{j=1}^{k} \frac{x}{1-jx}$, so we conclude that $S_{k+1}(x) = \prod_{j=1}^{k+1} \frac{x}{1-jx}$. □

Problem 7.3.2. For all nonnegative integers n and k,
$$\left\{ {n+1 \atop k+1} \right\} = \sum_{j=0}^{n} \left\{ {j \atop k} \right\} (k+1)^{n-j} .$$

Series solution. The Stirling series is
$$S_k(x) := \sum_{n\in\mathbb{N}} \left\{ {n \atop k} \right\} x^n = \prod_{j=1}^{k} \frac{x}{1-j\,x} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} .$$
Since $\left\{ {0 \atop k+1} \right\} = 0$ for all nonnegative integers k, it follows that
$$\begin{aligned} \sum_{n\in\mathbb{N}} \left\{ {n+1 \atop k+1} \right\} x^n &= \frac{1}{x}\left(S_{k+1}(x) - \left\{ {0 \atop k+1} \right\} \right) \\ &= \left(\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \right) \left(\frac{1}{1-(k+1)x} \right) \\ &= \left(\sum_{n\in\mathbb{N}} \left\{ {n \atop k} \right\} x^n \right) \left(\sum_{j\in\mathbb{N}} (k+1)^j\, x^j \right) \\ &= \sum_{n\in\mathbb{N}} \left(\sum_{j=0}^{n} \left\{ {j \atop k} \right\} (k+1)^{n-j} \right) x^n . \end{aligned}$$
Extracting the coefficient of x^n from both sides of the equation establishes the identity. □

Problem 7.3.3. For all nonnegative integers n and k, demonstrate that
$$k! \left\{ {n \atop k} \right\} = \sum_{j \in \mathbb{N}} (-1)^{k-j} \binom{k}{j} j^n .$$

Series solution. Consider the partial fraction expansion
$$\frac{1}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{j=1}^{k} \frac{\alpha_j}{1-jx} .$$
To solve for $\alpha_1, \alpha_2, \dots, \alpha_k$, multiply both sides of this equation by $1 - ix$ for some $1 \leqslant i \leqslant k$ and set $x = i^{-1}$ to obtain
$$\begin{aligned} \alpha_i &= \frac{1}{(1-\frac{1}{i})(1-\frac{2}{i})\cdots(1-\frac{i-1}{i})(1-\frac{i+1}{i})\cdots(1-\frac{k}{i})} \\ &= \frac{i^{k-1}}{(i-1)(i-2)\cdots(1)(-1)(-2)\cdots(k-i)} = \frac{(-1)^{k-i}\, i^{k-1}}{(i-1)!(k-i)!} . \end{aligned}$$
Extracting the coefficient of x^n from the generating series gives
$$\begin{aligned} \left\{ {n \atop k} \right\} &= [x^n]\left(\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \right) = [x^{n-k}]\left(\frac{1}{(1-x)(1-2x)\cdots(1-kx)} \right) \\ &= [x^{n-k}]\left(\sum_{j=1}^{k} \frac{\alpha_j}{1-jx} \right) = \sum_{j=1}^{k} \alpha_j [x^{n-k}]\left(\frac{1}{1-jx} \right) = \sum_{j=1}^{k} \alpha_j\, j^{n-k} \\ &= \sum_{j=1}^{k} (-1)^{k-j} \frac{j^{k-1}}{(j-1)!(k-j)!} j^{n-k} = \frac{1}{k!} \sum_{j \in \mathbb{N}} (-1)^{k-j} \binom{k}{j} j^n . \end{aligned}$$
□

For integer partitions with k parts, there is a very similar expression for the generating series. As a consequence, the generating series for all integer partitions is expressed as an infinite product.

Proposition 7.3.4 (Integer partition series)**.** *For all nonnegative integers k, we have*
$$\sum_{n \in \mathbb{N}} p_k(n)\, x^n = \prod_{j=1}^{k} \frac{x}{1-x^j} \quad \textit{and} \quad \sum_{n \in \mathbb{N}} p(n)\, x^n = \prod_{j \in \mathbb{N}} \frac{1}{1-x^{j+1}} .$$

Inductive proof of first identity. For all nonnegative integer k, set
$$G_k(x) := \sum_{n \in \mathbb{N}} p_k(n)\, x^n .$$
When $k = 0$, we see that $G_0(x) = 1$ because we have $p_0(0) = 0$ and $p_0(n+1) = 0$ for all nonnegative integers n. The addition formula [4.0.3] for integer partitions gives
$$\begin{aligned} G_{k+1}(x) &= \sum_{n \in \mathbb{N}} p_{k+1}(n)\, x^n = \sum_{n \in \mathbb{N}} \big(p_k(n-1) + p_{k+1}(n-k-1) \big)\, x^n \\ &= x\, G_k(x) + x^{k+1}\, G_{k+1}(x) , \end{aligned}$$
and we deduce that $(1 - x^{k+1})\, G_{k+1}(x) = x\, G_k(x)$. The induction hypothesis states that $G_k(x) = \prod_{j=1}^{k} x\,(1-x^j)^{-1}$, so we conclude that $G_{k+1}(x) = \prod_{j=1}^{k+1} x\,(1-x^j)^{-1}$. □

Counting proof of the second identity. For all nonnegative integer j, we have

$$\operatorname{ord}\left(\frac{1}{1-x^{j+1}} - 1\right) = \operatorname{ord}\left(\sum_{n \in \mathbb{N}} x^{(j+1)(n+1)}\right) = j+1\,.$$

It follows that

$$\begin{aligned}[x^n]\left(\prod_{j\in\mathbb{N}} \frac{1}{1-x^{j+1}}\right) &= [x^n]\left(\prod_{j=0}^{n-1} \frac{1}{1-x^{j+1}}\right) \\ &= [x^n]\big((1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)\cdots(1+x^n+x^{2n}+x^{3n}+\cdots)\big),\end{aligned}$$

and this infinite product is a well-defined element in the formal power series ring $\mathbb{Z}[[x]]$. Moreover, we see that the coefficient of x^n is the number of solutions $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{N}^n$ to the equation $n = a_0 + 2a_1 + 3a_2 + \cdots + n a_{n-1}$. Thus, this solution set consists of the partitions $(n^{a_{n-1}}, \dots, 2^{a_1}, 1^{a_0})$ of the integer n. □

Problem 7.3.5. For any nonnegative integers n, Let $q(n)$ be the number of partitions of n into distinct parts and let $p_{\text{odd}}(n)$ be the number of partitions of n into odd parts. Demonstrate that

$$q(n) = p_{\text{odd}}(n)\,.$$

Sketch of proof. Since the respective generatoring series are

$$\begin{aligned}\sum_{n\in\mathbb{N}} q(n)\,x^n &= (1+x)(1+x^2)(1+x^3)\cdots = \prod_{j\in\mathbb{N}}(1+x^{j+1}) \\ \sum_{n\in\mathbb{N}} p_{\text{odd}}(n)\,x^n &= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots = \prod_{j\in\mathbb{N}} \frac{1}{1-x^{2j+1}}\,,\end{aligned}$$

we have

$$\begin{aligned}\prod_{j\in\mathbb{N}}(1+x^{j+1}) &= \prod_{j\in\mathbb{N}}\left(\frac{1-x^{2(j+1)}}{1-x^{j+1}}\right) \\ &= \frac{\prod_{j\in\mathbb{N}}(1-x^{2(j+1)})}{\prod_{j\in\mathbb{N}}(1-x^{j+1})} = \prod_{j\in\mathbb{N}} \frac{1}{1-x^{2j+1}}\,.\end{aligned}$$ □

Exercises

Problem 7.3.6. For any nonnegative integer n, use the power conversion identity to show that

$$(x+1)^n = \sum_{m\in\mathbb{Z}} \left\{ {n+1 \atop m+1} \right\} x^{\underline{m}}\,.$$

Problem 7.3.7. For any nonnegative integer n, prove that

$$\left\{ {n+1 \atop m+1} \right\} = \sum_{k\in\mathbb{Z}} \binom{n}{k} \left\{ {k \atop m} \right\}.$$

Problem 7.3.8. For any nonnegative integer n, prove that

$$\left[{n+1 \atop m+1} \right] = \sum_{k\in\mathbb{Z}} \left[{n \atop k} \right] \binom{k}{m}.$$

8

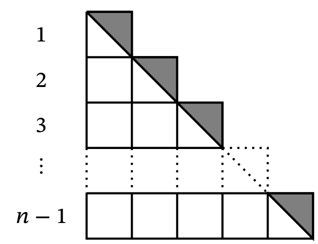
Recurrence Relations

8.0 Bernoulli Numbers

A famous sequence of *rational* numbers leads to a formula for the sum of m-th powers of the first n positive integers. We begin by examining the first three cases.

Problem 8.0.1. For any nonnegative integer n, verify that

$$\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2} = \binom{n}{2}.$$



area = $\frac{1}{2}(n-1)^2 + \frac{1}{2}(n-1)$

Algebraic solution. By summing twice, we obtain

$$\begin{aligned} 2\left(\sum_{k=0}^{n-1} k\right) &= \begin{pmatrix} 0 & + & 1 & + \cdots + & (n-1) \\ +\,(n-1) & + & (n-2) & + \cdots + & 0 \end{pmatrix} \\ &= \big((n-1) + (n-1) + \cdots + (n-1) \big) \\ &= \sum_{k=0}^{n} (n-1) = (n-1)(n)\,. \end{aligned}$$

□

Problem 8.0.2. For any nonnegative integer n, prove that

$$\sum_{k=0}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6}.$$

Algebraic solution. Since $k^2 = k\left(\sum_{j=0}^{k-1} 1\right) = \sum_{j=0}^{k-1} k$, we have

$$\begin{aligned} 3\sum_{k=0}^{n-1} k^2 &= 3\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} k \\ &= \begin{pmatrix} 1 & + \big(& 2 & + & 2 & \big) + \cdots + \big(& (n-1) + & (n-1) + \cdots + & (n-1) \big) \\ +\,(n-1) & + \big(& (n-1) & + & (n-2) & \big) + \cdots + \big(& (n-1) + & (n-2) + \cdots + & 1 \big) \\ +\,(n-1) & + \big(& (n-2) & + & (n-1) & \big) + \cdots + \big(& 1 + & 2 + \cdots + & (n-1) \big) \end{pmatrix} \\ &= \Big((2n-1) + \big((2n-1) + (2n-1) \big) + \cdots + \big((2n-1) + (2n-1) + \cdots + (2n-1) \big) \Big) \\ &= \sum_{k=0}^{n-1}\sum_{j=0}^{k-1} (2n-1) = (2n-1)\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} 1 \\ &= (2n-1)\sum_{k=0}^{n-1} k = (2n-1)\frac{n(n-1)}{2}\,. \end{aligned}$$

□

Problem 8.0.3. For any nonnegative integer n, show that

$$\sum_{k=0}^{n-1} k^3 = \binom{n}{2}^2 = \left(\sum_{k=0}^{n-1} k\right)^2.$$

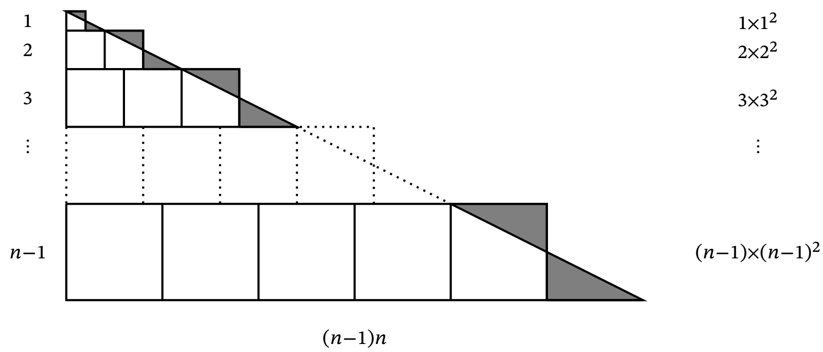


Figure 8.1: Triangular region used to compute the sum $\sum_{k=0}^{n-1} k^3$

Geometric solution. We show that $\sum_{k=0}^{n-1} k^2 = \frac{1}{2}\binom{n}{2}(n-1)(n)$ by computing the area of the triangular region in Figure 8.1. □

We now consider the general case. For all nonnegative integers m and n, set $\mathfrak{B}_m(n) := \sum_{k=0}^{n-1} k^m$. To understand this finite sum, we create a telescoping sum, use the binomial theorem, and reorder the sums to obtain

$$n^m = \sum_{k=0}^{n-1}\Big((k+1)^m - k^m\Big) = \sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\binom{m}{j}k^j = \sum_{j=0}^{m-1}\binom{m}{j}\mathfrak{B}_j(n)\,.$$

Hence, we have the following matrix equation

$$\begin{bmatrix} n \\ n^2 \\ n^3 \\ \vdots \\ n^{m+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 3 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{m+1}{0} & \binom{m+1}{1} & \binom{m+1}{2} & \cdots & \binom{m+1}{m} \end{bmatrix} \begin{bmatrix} \mathfrak{B}_0(n) \\ \mathfrak{B}_1(n) \\ \mathfrak{B}_2(n) \\ \vdots \\ \mathfrak{B}_m(n) \end{bmatrix}$$

To invert the $\big((m+1)\times(m+1)\big)$-matrix, suppose that

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 3 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{m+1}{0} & \binom{m+1}{1} & \binom{m+1}{2} & \cdots & \binom{m+1}{m} \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or, equivalently, that $\sum_{j=0}^{m}\binom{m+1}{j}B_j = \delta_{m,0}$ for all nonnegative integers m. For any nonnegative integer j, the ***Bernoulli number*** B_j is defined by this implicit recurrence relation. The first few values are

j	0	1	2	3	4	5	6	7	8	9	10	11	12
B_j	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

Table 8.1: Bernoulli numbers

Proposition 8.0.4. *For any nonnegative integers m and n, we have*

$$\mathfrak{B}_m(n) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j\, n^{n+1-j}\,.$$

Algebraic proof. It is enough to demonstrate that the inverse of the $\big((m+1)\times(m+1)\big)$-matrix whose (j,k)-entry is $\binom{j+1}{k}$ is the matrix whose (j,k)-entry is $\frac{1}{j+1}\binom{j+1}{j-k}B_{j-k}$. By reindexing the sum, using the absorption, symmetry, and trinomial revision identities on binomial coefficients, and recognizing the defining Bernoulli recurrence, we see that the (i,k)-entry in the matrix product is

$$\begin{aligned}\sum_{j=k}^{i}\binom{i+1}{j}\frac{1}{j+1}\binom{j+1}{j-k}B_{j-k} &= \sum_{\ell=0}^{i-k}\frac{1}{\ell+k+1}\binom{i+1}{\ell+k}\binom{\ell+k+1}{\ell}B_\ell \\ &= \sum_{\ell=0}^{i-k}\frac{1}{i+2}\binom{i+2}{\ell+k+1}\binom{\ell+k+1}{k+1}B_\ell \\ &= \frac{1}{i+2}\binom{i+2}{k+1}\sum_{\ell=0}^{i-k}\binom{i-k+1}{\ell}B_\ell \\ &= \frac{1}{i+2}\binom{i+2}{i-k+1}\delta_{i-k,0} = \delta_{i,k}\,. \qquad \square\end{aligned}$$

Exercises

Problem 8.0.5. For any nonnegative integer n, the ***Bernoulli number*** B_n is defined the recurrence

$$(n+1)B_n = -\sum_{k=0}^{n-1}\binom{n+1}{k}B_k$$

and the initial condition $B_0 = 1$.

(i) Prove that

$$\frac{x}{\exp(x)-1} = \sum_{j\in\mathbb{N}} B_j\,\frac{x^j}{j!}\,.$$

(ii) Use the part (i) to demonstrate that $B_{2j+1} = 0$ for all $j \geqslant 1$.

8.1 Solving Recurrences

Generating series are made so solve recurrences. We illustrate this feature with two examples.

Definition 8.1.1. For any nonnegative integer n, the ***Chebyshev polynomials*** (of the first kind) are defined by the recurrence $T_{n+2}(x) = 2x\,T_{n+1}(x) - T_n(x)$ and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. The next few Chebyshev polynomials are

$$\begin{aligned} T_2(x) &= 2x^2 - 1\,, & T_4(x) &= 8x^4 - 8x^2 + 1\,, \\ T_3(x) &= 4x^2 - 3x\,, & T_5(x) &= 16x^5 - 20x^3 + 5x\,. \end{aligned}$$

Problem 8.1.2. For any nonnegative integer n, verify that

$$T_n\big(\cos(\theta)\big) = \cos(n\theta)\,.$$

Inductive proof. When $n = 0$ and $n = 1$, we have

$$T_0\big(\cos(\theta)\big) = 1 = \cos\big((0)\theta\big) \quad T_1\big(\cos(\theta)\big) = \cos(\theta) = \cos\big((1)\theta\big),$$

so the base cases hold. Assume that the formula holds for all nonnegative integers less than $n + 2$. Using the trigonometric identity $2\cos(\varphi)\cos(\psi) = \cos(\varphi + \psi) + \cos(\varphi - \psi)$, the defining recurrence, and the induction hypothesis gives

$$\begin{aligned} T_{n+2}\big(\cos(\theta)\big) &= 2\big(\cos(\theta)\big)\, T_{n+1}\big(\cos(\theta)\big) - T_n\big(\cos(\theta)\big) \\ &= 2\cos(\theta)\cos\big((n+1)\theta\big) - \cos(n\theta) \\ &= \cos\big((n+1)\theta + \theta\big) + \cos\big((n+1)\theta - \theta\big) - \cos(n\theta) \\ &= \cos\big((n+2)\theta\big). \end{aligned}$$ □

Remark 8.1.3. The same trigonometric identity also implies that, for all nonnegative integers m and n satisfying $m \geqslant n$, we have

$$\begin{aligned} 2\, T_m(x)\, T_n(x) &= 2T_m\big(\cos(\theta)\big)\, T_n\big(\cos(\theta)\big) \\ &= 2\cos(m\theta)\, \cos(n\theta) \\ &= \cos(m\theta + n\theta) + \cos(m\theta - n\theta) \\ &= T_{m+n}(x) + T_{m-n}(x). \end{aligned}$$

Remark 8.1.4. Since $\cos\big((2k-1)\frac{\pi}{2}\big)$ for all integers k, we see that the zeros of $T_n(x) \in \mathbb{Z}[x]$ are $\cos\big(\frac{2k-1}{2n}\pi\big)$ for all $1 \leqslant k \leqslant n$.

Proposition 8.1.5 (Chebyshev series)**.** *Setting*

$$\Psi(t) := \sum_{n \in \mathbb{N}} T_n(x)\, t^n \in \mathbb{Q}[x][[t]],$$

we have

$$\Psi(t) = \frac{1 - xt}{1 - 2xt + t^2}.$$

Series proof. The basic maneuvers with generating series give

$$\begin{aligned} \frac{\Psi(t) - T_0(x) - T_1(x)t}{t^2} &= \sum_{n \in \mathbb{N}} T_{n+2}(x)\, t^n \\ &= 2x \sum_{n \in \mathbb{N}} T_{n+1}(x)\, t^n - \sum_{n \in \mathbb{N}} T_n(x)\, t^n \\ &= 2x\left(\frac{\Psi(t) - T_0(x)}{t}\right) - \Psi(t) \end{aligned}$$

so we obtain $(1 - 2xt + t^2)\,\Psi(t) = 1 + xt - 2xt = 1 - xt$. □

Problem 8.1.6. Find a closed-form for $T_n(x)$.

Series solution. Viewing $1 - 2xt + t^2$ as a polynomial in t, we see that $t = \frac{1}{2}\big(2x \pm \sqrt{4x^2 - 4}\big) = x \pm \sqrt{x^2 - 1}$ and we obtain

$$1 - 2xt + t^2 = \big(1 - (x - \sqrt{x^2 - 1})\, t\big)\big(1 - (x + \sqrt{x^2 - 1})\, t\big).$$

The partial fraction decomposition of the Chebyshev series is

$$\Psi(t) = \frac{1 - xt}{1 - 2xt + t^2} = \frac{\alpha}{1 - (x - \sqrt{x^2 - 1})t} + \frac{\beta}{1 - (x + \sqrt{x^2 - 1})t}.$$

As $\alpha\big(1-(x+\sqrt{x^2-1})t\big)+\beta\big(1-(x-\sqrt{x^2-1})t\big)=1-xt$, we have

$$t=\tfrac{1}{x-\sqrt{x^2-1}}:\quad \alpha\Big(1-\tfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\Big)=1-\tfrac{x}{x-\sqrt{x^2-1}}\quad\Rightarrow\quad \alpha=\Big(\tfrac{-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\Big)\Big(\tfrac{x-\sqrt{x^2-1}}{-2\sqrt{x^2-1}}\Big)=\tfrac{1}{2}$$
$$t=\tfrac{1}{x+\sqrt{x^2-1}}:\quad \beta\Big(1-\tfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\Big)=1-\tfrac{x}{x+\sqrt{x^2-1}}\quad\Rightarrow\quad \beta=\Big(\tfrac{\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\Big)\Big(\tfrac{x+\sqrt{x^2-1}}{2\sqrt{x^2-1}}\Big)=\tfrac{1}{2}\,.$$

Hence, expanding $\Psi(t)$ as a power series produces

$$\begin{aligned}\Psi(x)&=\frac{1}{2}\left(\frac{1}{1-(x-\sqrt{x^2-1})\,t}+\frac{1}{1-(x+\sqrt{x^2-1})\,t}\right)\\&=\sum_{n\in\mathbb{N}}\left(\frac{(x-\sqrt{x^2-1})^n+(x+\sqrt{x^2-1})^n}{2}\right)t^n\end{aligned}$$

which yields $T_n(x)=\frac{1}{2}\big[(x-\sqrt{x^2-1})^n+(x+\sqrt{x^2-1})^n\big]$. Using the binomial theorem, we also obtain

$$\begin{aligned}T_n(x)&=\frac{1}{2}\Big((x-\sqrt{x^2-1})^n+(x+\sqrt{x^2-1})^n\Big)\\&=\frac{x^n}{2}\left[\left(1-\frac{\sqrt{x^2-1}}{x}\right)^n+\left(1+\frac{\sqrt{x^2-1}}{x}\right)^n\right]\\&=\frac{x^n}{2}\left[\sum_{k\in\mathbb{Z}}\binom{n}{k}(-1)^k\left(\frac{\sqrt{x^2-1}}{x}\right)^k+\sum_{k\in\mathbb{Z}}\binom{n}{k}\left(\frac{\sqrt{x^2-1}}{x}\right)^k\right]\\&=x^n\sum_{k\in\mathbb{Z}}\binom{n}{2k}\left(\frac{\sqrt{x^2-1}}{x}\right)^{2k}\\&=\sum_{k\in\mathbb{Z}}\binom{n}{2k}(x^2-1)^k\,x^{n-2k}\,.\end{aligned}$$

which the desired closed-form expression. □

Remark 8.1.7. From the closed-form expression, we see that $T_n(-x)=(-1)^n\,T_n(x)$ for all nonnegative integers n.

Problem 8.1.8. For any nonnegative integer n, solve the simultaneous recurrences $a_{n+1}=5a_n+12b_n$ and $b_{n+1}=2a_n+5b_n$ satisfying the initial conditions $a_0=1$ and $b_0=0$.

Series solution. Let $f:=\sum_{n\in\mathbb{N}}a_n\,x^n$ and $g:=\sum_{n\in\mathbb{N}}b_n\,x^n$. The basic maneuvers with generating functions give

$$\begin{aligned}\frac{f-a_0}{x}&=\sum_{n\in\mathbb{N}}a_{n+1}\,x^n=5\sum_{n\in\mathbb{N}}a_n\,x^n+12\sum_{n\in\mathbb{N}}b_n\,x^n=5\,f+12\,g\\\frac{g-b_0}{x}&=\sum_{n\in\mathbb{N}}b_{n+1}\,x^n=2\sum_{n\in\mathbb{N}}a_n\,x^n+5\sum_{n\in\mathbb{N}}b_n\,x^n=2\,f+5\,g\,,\end{aligned}$$

so $(1-5x)\,f-12\,g=1$ and $2x\,f+(5x-1)\,g=0$. Hence, we have $g=\frac{2x}{1-5x}f$ and $f=\frac{1-5x}{(1-5x)^2-12(2x)}=\frac{1-5x}{1-10x+x^2}$. Consider the partial fraction decompositions

$$\begin{aligned}f&=\frac{1-5x}{1-10x+x^2}=\frac{\alpha}{1-(5-2\sqrt{6})x}+\frac{\beta}{1-(5+2\sqrt{6})x}\,,\\g&=\frac{2x}{1-10x+x^2}=\frac{\gamma}{1-(5-2\sqrt{6})x}+\frac{\delta}{1-(5+2\sqrt{6})x}\,.\end{aligned}$$

Since $\alpha\big(1-(5+2\sqrt{6})x\big)+\beta\big(1-(5-2\sqrt{6})x\big) = 1-5x$ and $\gamma\big(1-(5+2\sqrt{6})x\big)+\delta\big(1-(5-2\sqrt{6})x\big) = 2x$, we have

$$\begin{aligned}
x = \tfrac{1}{5-2\sqrt{6}} \ :\quad & \alpha\left(1-\tfrac{5+2\sqrt{6}}{5-2\sqrt{6}}\right) = 1-\tfrac{5}{5-2\sqrt{6}} && \alpha = \left(\tfrac{-2\sqrt{6}}{5-2\sqrt{6}}\right)\left(\tfrac{5-2\sqrt{6}}{-4\sqrt{6}}\right) = \tfrac{1}{2} \\
& \gamma\left(1-\tfrac{5+2\sqrt{6}}{5-2\sqrt{6}}\right) = \tfrac{2}{5-2\sqrt{6}} && \gamma = \left(\tfrac{2}{5-2\sqrt{6}}\right)\left(\tfrac{5-2\sqrt{6}}{-4\sqrt{6}}\right) = -\tfrac{\sqrt{6}}{12} \\
x = \tfrac{1}{5+2\sqrt{6}} \ :\quad & \beta\left(1-\tfrac{5-2\sqrt{6}}{5+2\sqrt{6}}\right) = 1-\tfrac{5}{5+2\sqrt{6}} && \beta = \left(\tfrac{2\sqrt{6}}{5+2\sqrt{6}}\right)\left(\tfrac{5+2\sqrt{6}}{4\sqrt{6}}\right) = \tfrac{1}{2} \\
& \delta\left(1-\tfrac{5-2\sqrt{6}}{5+2\sqrt{6}}\right) = \tfrac{2}{5+2\sqrt{6}} && \delta = \left(\tfrac{2}{5+2\sqrt{6}}\right)\left(\tfrac{5+2\sqrt{6}}{4\sqrt{6}}\right) = \tfrac{\sqrt{6}}{12}\,.
\end{aligned}$$

Expanding f and g as power series produces

$$\begin{aligned}
f &= \frac{1}{2}\left(\frac{1}{1-(5-2\sqrt{6})x}+\frac{1}{1-(5+2\sqrt{6})x}\right) = \sum_{n\in\mathbb{N}}\left[\frac{(5-2\sqrt{6})^n+(5+2\sqrt{6})^n}{2}\right]x^n\,, \\
g &= \frac{\sqrt{6}}{12}\left(\frac{1}{1-(5+2\sqrt{6})x}-\frac{1}{1-(5-2\sqrt{6})x}\right) = \sum_{n\in\mathbb{N}}\left[\frac{\sqrt{6}\big((5+2\sqrt{6})^n-(5-2\sqrt{6})^n\big)}{12}\right]x^n\,.
\end{aligned}$$

Therefore, we deduce that

$$a_n = \sum_{k\in\mathbb{N}}\binom{n}{k}5^{n-2k}\,(24)^k \qquad b_n = 2\sum_{k\in\mathbb{N}}\binom{n}{k}5^{n-2k-1}\,(24)^k\,. \qquad \square$$

Exercises

Problem 8.1.9. For all nonnegative integer n, the ***Laguerre polynomials*** are defined by the recurrence

$$(n+2)\,L_{n+2}(x) = \big(2(n+1)+(1-x)\big)\,L_{n+1}(x)-(n+1)\,L_n(x)\,,$$

and the initial conditions $L_0(x)=1$ and $L_1(x)=1-x$.

(i) Show that the Laguerre series is

$$\Phi(t) := \sum_{n\in\mathbb{N}} L_n(x)\,t^n = \frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right).$$

(ii) Find a closed formula for $L_n(x)$.