

8.2 Rational Generating Functions

The numerical sequences whose generating series are rational functions have an appealing characterization.

Theorem 8.2.1. *Fix a positive integer d. Given complex numbers $c_1, c_2, \dots, c_d$ such that $c_d \neq 0$, consider the polynomial*

$$\begin{aligned} q(x) &:= 1 + c_1\,x + c_2\,x^2 + \cdots + c_d\,x^d \\ &= (1-\lambda_1\,x)^{m_1}(1-\lambda_2\,x)^{m_2}\cdots(1-\lambda_k\,x)^{m_k} \end{aligned}$$

where the nonzero complex numbers $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_k^{-1}$ are the distinct roots and m_j denotes the multiplicity of λ_j^{-1} for all $1 \leqslant j \leqslant k$. For any sequence $(a_0, a_1, a_2, \dots)$ of complex numbers, the following conditions are equivalent:

(R1) *The generating series is a rational function such that*

$$\sum_{n \in \mathbb{N}} a_n\,x^n = \frac{p(x)}{q(x)}$$

where $p(x)$ is a polynomial in $\mathbb{C}[x]$ of degree less than d.

(R2) *For any nonnegative integer n, we have the recurrence*

$$a_{n+d} + c_1\,a_{n+d-1} + c_2\,a_{n+d-2} + \cdots + c_d\,a_n = 0\,.$$

(R3) *For all $1 \leqslant i \leqslant k$, there exists polynomials $g_i(x)$ of degree less than m_i such that, for all nonnegative integers n, we have the closed-formula $a_n = \sum_{i=1}^{k} g_i(n)\,\lambda_i^n$.*

Algebraic proof. For all $1 \leqslant \ell \leqslant 3$, consider the $\mathbb{C}$-vector space V_ℓ of all sequences satisfying condition (Rℓ); each set is clearly closed under taking linear combinations. Moreover, each $\mathbb{C}$-vector space V_ℓ has dimension d:

- in (R1), the d coefficients of $p(x) \in \mathbb{C}[x]$ are arbitrary;
- in (R2), the initial values $a_0, a_1, \dots, a_{d-1}$ are arbitrary;
- in (R3), the m_i coefficients of the polynomial $g_i \in \mathbb{C}[x]$ are arbitrary and we have $m_1 + m_2 + \cdots + m_k = d$.

To prove $V_i = V_j$, it suffices to show that $V_i \subseteq V_j$. Hence, it is enough to consider two cases. Suppose that the sequence $(a_0, a_1, a_2, \dots)$ lies in V_1.

$V_1 \subseteq V_2$: Extracting the coefficient of x^{n+d} from both sides of the equation

$$q(x)\left(\sum_{n \in \mathbb{N}} a_n\,x^n\right) = p(x)$$

produces the recurrence in (R2), so $V_1 \subseteq V_2$.

$V_1 \subseteq V_3$: Consider the partial fraction decomposition

$$\sum_{n \in \mathbb{N}} a_n\,x^n = \sum_{i=1}^{k} \frac{g_i(x)}{(1-\lambda_i x)^{m_i}} = \sum_{i=1}^{k} g_i(x)\left(\sum_{n \in \mathbb{N}} \binom{m_i + n - 1}{m_i - 1} \lambda_i^n\,x^n\right).$$

Writing $g_i(x) := g_{i,0} + g_{i,1}x + \cdots + g_{i,m_i-1}x^{m_i-1}$, we obtain

$$\sum_{n \in \mathbb{N}} a_n\, x^n = \sum_{n \in \mathbb{N}} \left[\sum_{i=1}^{k} \left(\sum_{j=1}^{m_i-1} g_{i,j} \binom{m_i + n - 1}{m_i - 1} \lambda_i^{-j} \right) \lambda_i^n \right] x^n$$

so we deduce that $V_1 \subseteq V_3$. □

Eulerian numbers arise as the coefficients in the numerator for a simple rational generating series.

Problem 8.2.2 (Carlitz identity)**.** For any nonnegative integer m, show that

$$\sum_{n \in \mathbb{N}} n^m\, x^n = \frac{\sum_{k \in \mathbb{Z}} \left\langle {m \atop k} \right\rangle x^{k+1}}{(1-x)^{m+1}} \,.$$

Inductive solution. Since $\sum_{n \in \mathbb{N}} n^m\, x^m = \left(x \frac{d}{dx}\right)^m \big((1-x)^{-1}\big)$, it suffices to show that

$$\left(x \frac{d}{dx}\right)^m \left(\frac{1}{1-x}\right) = \frac{\sum_{k \in \mathbb{Z}} \left\langle {m \atop k} \right\rangle x^{k+1}}{(1-x)^{m+1}} \,.$$

When $m = 0$, we have $1 = \sum_{k \in \mathbb{Z}} \left\langle {0 \atop k} \right\rangle x^k$ because $\left\langle {0 \atop 0} \right\rangle = 1$ and $\left\langle {n \atop 0} \right\rangle = 0$ for all nonzero integers n. Hence, the base case holds. Assume that the identity holds for some nonnegative integer m. The induction hypothesis and the addition formula [3.3.4] for Eulerian numbers give

$$\begin{aligned}
&\left(x \frac{d}{dx}\right)^{m+1} \left[\frac{1}{1-x}\right] \\
&= \left(x \frac{d}{dx}\right) \left[\left(x \frac{d}{dx}\right)^m \left(\frac{1}{1-x}\right) \right] \\
&= \left(x \frac{d}{dx}\right) \left[\frac{\sum_{k \in \mathbb{Z}} \left\langle {m \atop k} \right\rangle x^{k+1}}{(1-x)^{m+1}} \right] \\
&= \frac{x}{(1-x)^{m+1}} \left[\sum_{k \in \mathbb{Z}} \left\langle {m \atop k} \right\rangle (k+1)\, x^k \right] + \frac{(m+1)\,x}{(1-x)^{m+2}} \left[\sum_{k \in \mathbb{Z}} \left\langle {m \atop k} \right\rangle x^{k+1} \right] \\
&= \frac{1}{(1-x)^{m+2}} \left[\sum_{k \in \mathbb{Z}} (k+1) \left\langle {m \atop k} \right\rangle x^{k+1} + \sum_{k \in \mathbb{Z}} (m-k) \left\langle {m \atop k} \right\rangle x^{k+2} \right] \\
&= \frac{1}{(1-x)^{m+2}} \left[\sum_{k \in \mathbb{Z}} \left((k+1) \left\langle {m \atop k} \right\rangle + (m-k+1) \left\langle {m \atop k-1} \right\rangle \right) x^{k+1} \right] \\
&= \frac{\sum_{k \in \mathbb{Z}} \left\langle {m+1 \atop k} \right\rangle x^{k+1}}{(1-x)^{m+2}} \,.
\end{aligned}$$

□

Using a recurrence relation to create a sequence of complex numbers for all integer subscripts has a useful interpretation in terms of rational functions.

Corollary 8.2.3. *Fix a positive integer* d *and consider complex numbers* $c_1, c_2, \dots, c_d$ *such that* $c_d \neq 0$*. Given the doubly-infinite sequence* $(\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$ *of complex numbers satisfying*

$$a_{n+d} + c_1\, a_{n+d-1} + c_2\, a_{n+d-2} + \cdots + c_d\, a_n = 0$$

for all integers n, the generating series

$$f(x) := \sum_{n \in \mathbb{N}} a_n \, x^n \qquad \text{and} \qquad g(x) := \sum_{n \in \mathbb{N}} a_{-n-1} \, x^{n+1}$$

are both rational functions. Moreover, we have $g(x) = -f(1/x)$ as rational functions.

Sketch of proof. Theorem 8.2.1 establishes that $f(x) = p(x)/q(x)$ where $q(x) = 1 + c_1 \, x + c_2 \, x^2 + \cdots + c_d \, x^d$. The hypothesis on the doubly-infinity sequence implies that $q(x) \big(\sum_{n \in \mathbb{Z}} a_n \, x^n \big) = 0$. Since multiplication by the polynomial $q(x)$ is linear, we obtain

$$q(x) \left(\sum_{n \in \mathbb{N}} a_{-n-1} \, x^{-n-1} \right) = -q(x) \left(\sum_{n \in \mathbb{N}} a_n \, x^n \right) = -p(x) \, .$$

Hence, the substitution $x \mapsto 1/x$ gives

$$\sum_{n \in \mathbb{N}} a_{-n-1} \, x^{n+1} = -\frac{p(1/x)}{q(1/x)} = -f(1/x) \, . \qquad \square$$

Corollary 8.2.3 is a statement about the equality of rational functions. For example, when $a_j = 1$ for all integers j, we have $f(x) := \sum_{n \in \mathbb{N}} x^n = (1-x)^{-1}$ and $g(x) := \sum_{n \in \mathbb{N}} x^{n+1} = x(1-x)^{-1}$, so

$$-f(1/x) = \frac{1}{1 - 1/x} = -\frac{x}{x-1} = \frac{x}{1-x} = g(x) \, .$$

9

Hypergeometric Functions

9.0 Hypergeometric Series

The ratio of consecutive terms in a geometric series $\sum_{k \in \mathbb{N}} a_k$ is constant: for any nonnegative integer k, we have $a_{k+1} / a_k = r$ for some fixed complex number r. It follows that, for all nonnegative integers k, we have $a_k = a_0 \, r^k$. Generalizing this observation, we introduce a new class of series.

Definition 9.0.1. In a ***hypergeometric series*** $\sum_{k \in \mathbb{N}} t_k$, the ratio of consecutive terms is a fixed rational function in the summation index: for any nonnegative integer k, we have

$$\frac{t_{k+1}}{t_k} = \frac{p(k)}{q(k)} \, ,$$

where $p(x)$ and $q(x)$ are polynomials in $\mathbb{C}[x]$.

Problem 9.0.2. Verify that these series are hypergeometric:

$$\sum_{n \in \mathbb{N}} x^n \, , \qquad \sum_{k \in \mathbb{N}} k! \, , \qquad \sum_{j \in \mathbb{N}} \frac{(2j+7)!}{(j-3)!} \, .$$

Solution. Since

$$\frac{x^{n+1}}{x^n} = x \, , \quad \frac{(k+1)!}{k!} = k+1 \, , \quad \frac{(2j+9)!}{(j-2)!} \frac{(j-3)!}{(2j+7)!} = \frac{(2j+9)(2j+9)}{j-2} \, ,$$

we see that all three series are hypergeometric. □

When we normalize the series by assuming that $t_0 = 1$, there is an accepted notation for a hypergeometric function. In the ratio of consecutive terms, factor the numerator and denominator completely as

$$\frac{t_{k+1}}{t_k} = \frac{p(k)}{q(k)} = \frac{(k+a_1)(k+a_2) \cdots (k+a_m)}{(k+b_1)(k+b_2) \cdots (k+b_n)} \frac{x}{k+1}$$

where $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m, x \in \mathbb{C}$. The hypergeometric series with the terms t_k is denoted by

$$\mathrm{F}\left(\begin{matrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_n \end{matrix} ; x \right) = {}_m\mathrm{F}_n \left(\begin{matrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_n \end{matrix} ; x \right) = \sum_{k \in \mathbb{N}} t_k \, .$$

The distinguished factor $(k+1)$ in the denominator is a historical tradition. If there were no factor of $(k+1)$ in the denominator of your ratio of consecutive terms, then put it in and compensate by putting extra factor in the numerator.

Problem 9.0.3. Describe $\exp(x)$ and $\sum_{k \in \mathbb{N}} 2^k / (k!)^2$ in terms of standard notation for a hypergeometric series.

Proof. Since

$$\exp(x) = \sum_{k \in \mathbb{N}} \frac{x^k}{k!} \qquad \text{and} \qquad \frac{x^{k+1}}{(k+1)!} \frac{k!}{x^k} = \frac{x}{k+1}$$

we see that $\exp(x) = \mathrm{F}\left(\genfrac{}{}{0pt}{}{-}{-} ; x \right) = \mathrm{F}\left(\genfrac{}{}{0pt}{}{1}{1} ; x \right)$. Similarly, we have

$$\frac{2^{k+1}}{\big((k+1)!\big)^2} \frac{(k!)^2}{2^k} = \frac{2}{(k+1)^2} \, ,$$

so it follows that $\displaystyle\sum_{k \in \mathbb{N}} \frac{2^k}{(k!)^2} = \mathrm{F}\left(\genfrac{}{}{0pt}{}{-}{1} ; 2 \right) = \mathrm{F}\left(\genfrac{}{}{0pt}{}{\;1\;}{1 \quad 1} ; 2 \right)$. □

Remark 9.0.4. We do not change a hypergeometric function if we cancel a parameter that occurs in both the numerator and denominator or conversely if we insert two identical parameters.

Problem 9.0.5. Demonstrate that

$$\mathrm{F}\left(\genfrac{}{}{0pt}{}{r \;\; 1}{1} ; x \right) = \frac{1}{(1-x)^r} \qquad \text{and} \qquad \mathrm{F}\left(\genfrac{}{}{0pt}{}{-r \;\; 1}{1} ; -x \right) = (1+x)^r \, .$$

Proof. The generalized binomial theorem states that

$$(1-x)^{-r} = \sum_{k \in \mathbb{N}} \left(\!\! \binom{r}{k} \!\!\right) x^k \qquad \text{and} \qquad (1+x)^r = \sum_{k \in \mathbb{N}} \binom{r}{k} x^k \, ,$$

so the ratio of consecutive terms are

$$\frac{r^{\overline{k+1}} \, x^{k+1}}{(k+1)!} \frac{k!}{r^{\overline{k}} \, x^k} = \frac{r(r+1)(r+2) \cdots (r+k) \, x^{k+1}}{(k+1)\,(r)(r+1)(r+2) \cdots (r+k-1) \, x^k} = \frac{(k+r)\,x}{(k+1)} \, ,$$

$$\frac{r^{\underline{k+1}} \, x^{k+1}}{(k+1)!} \frac{k!}{r^{\underline{k}} \, x^k} = \frac{r(r-1)(r-2) \cdots (r-k) \, x^{k+1}}{(k+1)\,(r)(r-1)(r-2) \cdots (r-k+1) \, x^k} = \frac{(r-k)\,x}{(k+1)} = \frac{\big(k + (-r)\big)(-x)}{(k+1)} \, .$$

Since the initial terms are 1, we deduce the given formula. □

Problem 9.0.6. Is the Bessel function

$$J_p(x) := \sum_{k \in \mathbb{N}} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+p}}{k! \, (k+p)!}$$

a hypergeometric function?

Proof. The ratio of consecutive terms is

$$\frac{(-1)^{k+1} \left(\frac{x}{2}\right)^{2k+2+p}}{(k+1)! \, (k+p+1)!} \frac{k! \, (k+p)!}{(-1)^k \left(\frac{x}{2}\right)^{2k+p}} = \frac{-\frac{x^2}{4}}{(k+1)(k+p+1)}$$

and initial term is $\frac{1}{p!}\left(\frac{x}{2}\right)^p$, so $J_p(x) = \frac{1}{p!}\left(\frac{x}{2}\right)^p \; \mathrm{F}\left(\genfrac{}{}{0pt}{}{-}{p+1} ; -\frac{x^2}{4} \right)$. □

Proposition 9.0.7. *The general hypergeometric series is*

$$\mathrm{F}\left(\genfrac{}{}{0pt}{}{a_1 \;\; a_2 \;\; \cdots \;\; a_m}{b_1 \;\; b_2 \;\; \cdots \;\; b_n} ; x \right) = \sum_{k \in \mathbb{N}} \frac{a_1^{\overline{k}} \, a_2^{\overline{k}} \, \cdots \, a_m^{\overline{k}}}{b_1^{\overline{k}} \, b_2^{\overline{k}} \, \cdots \, b_n^{\overline{k}}} \frac{x^k}{k!} \, .$$

Proof. Since $a^{\overline{k+1}} = (a)(a+1)\cdots(a+k-1)(a+k) = a^{\overline{k}}(a+k)$, the ratio of consecutive terms is

$$\left(\frac{a_1^{\overline{k+1}}\, a_2^{\overline{k+1}} \cdots a_m^{\overline{k+1}}}{b_1^{\overline{k+1}}\, b_2^{\overline{k+1}} \cdots b_n^{\overline{k+1}}} \frac{x^{k+1}}{(k+1)!}\right)\left(\frac{b_1^{\overline{k}}\, b_2^{\overline{k}} \cdots b_n^{\overline{k}}}{a_1^{\overline{k}}\, a_2^{\overline{k}} \cdots a_m^{\overline{k}}} \frac{k!}{x^k}\right) = \frac{(a_1+k)(a_2+k)\cdots(a_m+k)}{(b_1+k)(b_2+k)\cdots(b_n+k)} \frac{x}{k+1},$$

and the initial term is 1. Therefore, the right side is the specified hypergeometric function. □

Remark 9.0.8. If any of the upper parameters $a_1, a_2, \dots, a_m$ is a nonpositive integer, then the general hypergeometric series is a polynomial, otherwise is a power series.

Remark 9.0.9. There are some surprisingly simple identities for differentiating hypergeometric functions:

$$\begin{aligned}\left(x\frac{d}{dx} + a_1\right) \mathrm{F}\left(\begin{matrix}a_1 & a_2 & \cdots & a_m\\ b_1 & b_2 & \cdots & b_n\end{matrix}; x\right) &= \sum_{k \in \mathbb{N}} \frac{(k+a_1)a_1^{\overline{k}}\, a_2^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}}\, b_2^{\overline{k}} \cdots b_n^{\overline{k}}} \frac{x^k}{k!} \\ &= \sum_{k \in \mathbb{N}} \frac{a_1(a_1+1)^{\overline{k}}\, a_2^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}}\, b_2^{\overline{k}} \cdots b_n^{\overline{k}}} \frac{x^k}{k!} = a_1\ \mathrm{F}\left(\begin{matrix}a_1{+}1 & a_2 & \cdots & a_m\\ b_1 & b_2 & \cdots & b_n\end{matrix}; x\right)\end{aligned}$$

$$\left(x\frac{d}{dx} + b_1 - 1\right) \mathrm{F}\left(\begin{matrix}a_1 & a_2 & \cdots & a_m\\ b_1 & b_2 & \cdots & b_n\end{matrix}; x\right) = (b_1-1)\ \mathrm{F}\left(\begin{matrix}a_1 & a_2 & \cdots & a_m\\ b_1{-}1 & b_2 & \cdots & b_n\end{matrix}; x\right)$$

Exercises

Problem 9.0.10. A ***Gaussian*** hypergeometric series is given by

$$\mathrm{F}\left(\begin{matrix}a\ b\\ c\end{matrix}; x\right) := \sum_{k \in \mathbb{N}} \frac{a^{\overline{k}}\, b^{\overline{k}}}{c^{\overline{k}}} \frac{x^k}{k!}.$$

(i) Show that the Gaussian hypergeometric function is a solution to the differential equation

$$x(1-x)\frac{d^2y}{dx^2} + \big(c - (a+b+1)x\big)\frac{dy}{dx} - ab\, y = 0.$$

(ii) Establish the reflection identity

$$\frac{1}{(1-x)^a} \mathrm{F}\left(\begin{matrix}a\ b\\ c\end{matrix}; \frac{-x}{1-x}\right) = \mathrm{F}\left(\begin{matrix}a & c-b\\ c & \end{matrix}; x\right).$$

9.1 Indefinite Sums

The indefinite summation problem asks when $s_n := \sum_{k=0}^{n-1} t_n$ has a closed form that does not involve the summation sign. For any nonnegative integer n, we regard the indefinite sum s_n as the discrete analogue of an antiderivative. Instead of its derivative being the integrand, its difference is the summand: $s_{n+1} - s_n = t_n$. This equation implies that

$$\frac{t_{n+1}}{t_n} = \frac{s_{n+2} - s_{n+1}}{s_{n+1} - s_n} = \frac{s_{n+2}/s_{n+1} - 1}{1 - s_n/s_{n+1}}.$$

It follows that, when s_n is hypergeometric, t_n is also hypergeometric. In 1970's, Bill Gosper discovered a procedure for finding sums of hypergeometric terms that are hypergeometric.

Algorithm 9.1.1 (Gosper)**.**
input: a hypergeometric term t_n
output: a hypergeometric term s_n such that $s_{n+1} - s_n = t_n$
if one exists, otherwise `null`.

Write $\frac{t_{n+1}}{t_n} = \frac{f(n)}{g(n)} \frac{h(n+1)}{h(n)}$ where $f, g, h \in \mathbb{C}[x]$ and $\gcd\big(f(n), g(n+j)\big) = 1$ for all nonnegative integers j.
If there exists a nonzero polynomial $p(n)$ such that $f(n)\, p(n+1) - g(n-1)\, p(n) = h(n)$
then return $\frac{g(n-1)\, p(n)}{h(n)}\, t_n$
else return `null`.

Remark 9.1.2. The Gosper algorithm determines the indefinite sum up to a constant: $s_n - \sum_{k=0}^{n-1} t_k \in \mathbb{C}$.

Before analyzing the correctness of this algorithm, we first illustrate it with a few examples.

Problem 9.1.3. Can $\sum\limits_{k=0}^{n-1} k(k!)$ be expressed in closed form?

Solution. Following the Gosper algorithm, we have

$$\frac{t_{n+1}}{t_n} = \frac{(n+1)(n+1)!}{(n)(n!)} = \frac{(n+1)(n+1)}{n} = \left(\frac{n+1}{1}\right)\left(\frac{n+1}{n}\right) = \frac{f(n)}{g(n)} \frac{h(n+1)}{h(n)}$$

and $\gcd(n+1, 1) = 1$. The constant polynomial $p(n) = 1$ satisfies

$$f(n)p(n+1) - p(n) = (n+1)(1) = (1) = n = h(n)\,,$$

so we conclude that $s_n := \big(n\, n!\big)/n = n!$ satisfies

$$s_{n+1} - s_n = (n+1)! - n! = (n+1-1)(n!) = (n)(n!)\,.$$

Thus, we have $\sum\limits_{k=0}^{n-1} k(k!) = n! - 1$ for all nonnegative integers n. □

Problem 9.1.4. Can the sum

$$\sum_{k=0}^{n-1} (k^2 + 3k + 1)(k!)$$

be expressed in closed form?

Solution. Following the Gosper algorithm, we have

$$\frac{t_{n+1}}{t_n} = \frac{\big((n+1)^2 + 3(n+1) + 1\big)(n+1)!}{(n^2 + 3n + 1)(n!)} = \left(\frac{n+1}{1}\right)\left(\frac{(n+1)^2 + 3(n+1) + 1}{n^2 + 3n + 1}\right) = \frac{f(n)}{g(n)} \frac{h(n+1)}{h(n)}$$

and $\gcd(n+1, 1) = 1$. If $p(n) = \alpha\, n + \beta$ and

$$\begin{aligned} f(n)p(n+1) - p(n) &= (n+1)\big(\alpha(n+1) + \beta\big) - (\alpha n + \beta) \\ &= \alpha n^2 + (\alpha + \beta)n + \alpha = n^2 + 3n + 1 = h(n)\,, \end{aligned}$$

we see that $p(n) = n + 2$. Hence, the expression

$$\begin{aligned} s_n &:= \frac{g(n-1)\,p(n)}{h(n)}\,t_n \\ &= \frac{n+2}{n^2+3n+1}(n^2+3n+1)(n!) = (n+2)n! \end{aligned}$$

satisfies

$$\begin{aligned} s_{n+1} - s_n &= (n+3)\big((n+1)!\big) - (n+2)(n!) \\ &= \big((n+3)(n+1) - (n+2)\big)(n!) \\ &= (n^2+3n+1)(n!) = t_n \end{aligned}$$

and $\sum_{k=0}^{n-1}(k^2+3k+1)(k!) = (n+2)(n!) - 2$ for all nonnegative integers n. □

To establish the correctness of the Gosper algorithm, we first collect a few preliminary results.

Lemma 9.1.5. *The maximality of the degree of the polynomial h implies that $\gcd\big(f(n), g(n+j)\big) = 1$ for all nonnegative integers j.*

Proof by contradiction. For some positive integer j, suppose that $q(n) = \gcd\big(f(n), g(n+j)\big) \neq 1$. It follows that $q(n)$ divides $f(n)$ and $q(n-j)$ divides $g(n)$. Hence, by setting $f(n) = q(n)\,f^*(n)$ and $g(n) = q(n-j)\,g^*(n)$, we obtain

$$\frac{f(n)}{g(n)} = \frac{q(n)}{q(n-1)}\,\frac{q(n-1)}{q(n-2)} \cdots \frac{q(n-j+1)}{q(n-j)}\,\frac{f^*(n)}{g^*(n)}\,,$$

and moving the product $q(n)\,q(n-1)\cdots q(n-j+1)$ into $h(n)$ contradicts the maximality of the degree. □

Lemma 9.1.6. *The output $s_n = \frac{g(n-1)\,t_n}{h(n)}\,p(n)$ is hypergeometric if and only if $p(n)$ is a rational function.*

Proof.
⇒: Suppose that $s_n = \frac{g(n-1)\,t_n}{h(n)}\,p(n)$ is hypergeometric. It follows that

$$\begin{aligned} p(n) &= \frac{h(n)\,s_n}{g(n-1)\,t_n} \\ &= \frac{h(n)\,s_n}{g(n-1)\,(s_{n+1}-s_n)} = \frac{h(n)}{g(n-1)\big(1-\frac{s_n}{s_{n+1}}\big)} \end{aligned}$$

so $p(n)$ is a rational function.
⇐: Suppose that $p(n)$ is a rational function. The sequence s_n is hypergeometric because

$$\frac{s_{n+1}}{s_n} = \frac{g(n)\,t_{n+1}\,p(n+1)}{h(n+1)}\,\frac{h(n)}{g(n-1)\,t_n\,p(n)} = \frac{g(n)}{g(n-1)}\,\frac{h(n)}{h(n+1)}\,\frac{p(n+1)}{p(n)}\,\frac{t_{n+1}}{t_n}$$

and t_n is also hypergeometric. □

Lemma 9.1.7. *If the rational function $p(n)$ satisfies*

$$f(n)\,p(n+1) - g(n-1)\,p(n) = h(n)\,,$$

then the output $s_n = \frac{g(n-1)\,t_n}{h(n)} p(n)$ satisfies $s_{n+1} - s_n = t_n$.

Proof. Since $\frac{t_{n+1}}{t_n} = \frac{f(n)}{g(n)} \frac{h(n+1)}{h(n)}$, we have

$$\begin{aligned} s_{n+1} - s_n &= \frac{g(n)\,t_{n+1}\,p(n+1)}{h(n+1)} - \frac{g(n-1)\,t_n\,p(n)}{h(n)} \\ &= t_n \left(\frac{f(n)\,p(n+1)}{h(n)} - \frac{g(n-1)\,p(n)}{h(n)} \right) \\ &= \frac{t_n}{h(n)} \big(f(n)\,p(n+1) - g(n-1)\,p(n) \big) = t_n\,. \end{aligned}$$ □

It remains to show that $p(n)$ must be a polynomial.