

9.2 The Gosper Algorithm

Recall the following algorithm.

Algorithm 9.2.1 (Gosper)**.**

input: a hypergeometric term t_n

output: a hypergeometric term s_n such that $s_{n+1} - s_n = t_n$
if one exists, otherwise `null`.

Write $\frac{t_{n+1}}{t_n} = \frac{f(n)}{g(n)} \frac{h(n+1)}{h(n)}$ where $f, g, h \in \mathbb{C}[x]$ and $\gcd\big(f(n), g(n+j)\big) = 1$ for all nonnegative integers j.
If there exists a nonzero polynomial $p(n)$ such that $f(n)\,p(n+1) - g(n-1)\,p(n) = h(n)$
then return $\frac{g(n-1)\,p(n)}{h(n)}\, t_n$
else return `null`.

Problem 9.2.2. Can $\sum_{k=0}^{n-1} k!$ be expressed in closed form?

Solution. Following the Gosper algorithm, we have

$$\frac{t_{n+1}}{t_n} = \frac{(n+1)!}{n!} = n+1$$

so $f(n) = n+1$, $g(n) = 1$, and $h(n) = 1$. The functional equation is $(n+1)p(n+1) - p(n) = 1$ which has no solution. Therefore, this indefinite sum is not hypergeometric. □

Correctness of the Gosper algorithm. Given Lemmas 9.1.5–9.1.7, it remains to show that a rational function $p(n)$ that satisfies

$$f(n)\,p(n+1) - g(n-1)\,p(n) = h(n)$$

is a polynomial.

Let $p(n) = a(n)/b(n)$ where $\gcd\big(a(n), b(n)\big) = 1$. To prove that $b(n) = 1$, we show that $\gcd\big(b(n), b(n+k)\big) = 1$ for all nonnegative integers k. The case $k = 0$ establishes that $b(n) = 1$.

Suppose otherwise and let j be maximal nonnegative integer such that $q(n) := \gcd\big(b(n), b(n+j)\big) \neq 1$. If $b(n) \neq 1$, then such an index j exists. Indeed, if ξ is a root of $q(n)$, then ξ and $\zeta := \xi + j$ are roots of $b(n)$. When $j > \max\{\zeta - \xi \mid \xi \text{ and } \zeta \text{ roots of } b(n)\}$, the defining equation of $q(n)$ cannot be satisfied. Hence, the definition of j implies that $\gcd\big(b(n), b(n+j+1)\big) = 1$.

Since $p(n) = a(n)/b(n)$, the functional equation becomes

$$f(n)\,a(n+1)\,b(n) - g(n-1)\,a(n)\,b(n+1) = h(n)\,b(n)\,b(n+1)\,.$$

We derive the desired contradiction by showing that
(a) $q(n+1)$ divides $f(n)$, and
(b) $q(n+1)$ divides $g(n+j)$.
As a consequence, the maximality of the degree $h(n)$ implies that $\gcd\big(f(n), g(n+j)\big) = 1$, for all nonnegative integers j, and we deduce that $q(n+1) = 1$.

Proof of a. Set $\varphi(n) := \gcd\bigl(q(n+1), b(n)\bigr)$. It follows that $\varphi(n)$ divides $q(n+1)$ which divides $b(n+j+1)$. We also see that $\varphi(n)$ divides $b(n)$. Because $\gcd\bigl(b(n), b(n+j+1)\bigr) = 1$, we deduce that $\varphi(n) = 1$. Since $q(n+1)$ divides $b(n+1)$, and $q(n+1)$ is relatively prime to $a(n+1)$ and $b(n)$, the functional equation implies that $q(n+1)$ divides $f(n)$.

Proof of b. Set $\psi(n) := \gcd\bigl(q(n-j), b(n+1)\bigr)$. It follows that $\psi(n+j)$ divides $q(n)$ which divides $b(n)$. We also see that $\psi(n+j)$ divides $b(n+j+1)$. Again because

$$\gcd\bigl(b(n), b(n+j+1)\bigr) = 1\,,$$

we deduce that $\psi(n) = 1$. Since $q(n-j)$ divides $b(n)$ and $q(n-j)$ is relatively prime to both $a(n)$ and $b(n+1)$, the functional equations implies that $q(n-j)$ divides $g(n-1)$ or $q(n+1)$ divides $g(n+j)$. □

Problem 9.2.3. For any nonnegative integer m, can the sums

$$\sum_{k=0}^{n-1} (-1)^k \binom{m}{k} \qquad \text{and} \qquad \sum_{k=0}^{n-1} \binom{m}{k}$$

be expressed in closed form?

Solution. Following the Gosper algorithm, we have

$$\frac{t_{n+1}}{t_n} = \frac{(-1)^{n+1}\binom{m}{n+1}}{(-1)^n\binom{m}{n}} = (-1)\frac{m!}{(n+1)!(m-n-1)!}\,\frac{n!(m-n)!}{m!} = \frac{n-m}{n+1}\,,$$

so $f(n) = n-m$, $g(n) = n+1$, and $h(n) = 1$. Hence, we have $\gcd(n-m, n+1+j) = 1$ for all nonnegative integers j. The functional equation $(n-m)\,p(n+1) - (n)\,p(n) = 1$ has the constant polynomial $p(n) = -1/m$ as a solution. We conclude that $s_n := -\frac{n}{m}(-1)^n\binom{m}{n} = (-1)^{n+1}\binom{m-1}{n-1}$ satisfies

$$\begin{aligned} s_{n+1} - s_n &= (-1)^{n+2}\binom{m-1}{n} - (-1)^{n+1}\binom{m-1}{n-1} \\ &= (-1)^n\left[\binom{m-1}{n} + \binom{m-1}{n-1}\right] = (-1)^n\binom{m}{n}. \end{aligned}$$

Thus, we have $\sum_{k=0}^{n-1}(-1)^k\binom{m}{k} = (-1)^{n+1}\binom{m-1}{n-1}$ for all nonnegative integers n.

For the second sum, the Gosper algorithm gives

$$\frac{t_{n+1}}{t_n} = \frac{\binom{m}{n+1}}{\binom{m}{n}} = \frac{m!}{(n+1)!\,(m-n-1)!}\,\frac{n!\,(m-n)!}{m!} = \frac{m-n}{n+1}$$

so $f(n) = m-n$, $g(n) = n+1$, and $h(n) = 1$. Observe that $\gcd(m-n, n+1+j) = 1$ for all nonnegative integers j. However, the functional equation $(m-n)\,p(n+1) - (n)\,p(n) = 1$ has no solution because the highest coefficients do not cancel. Therefore, this indefinite sum is not hypergeometric. □

9.3 The Zeilberger Algorithm

Although the infinite sum $\sum_{k\in\mathbb{Z}} \binom{m}{k} = 2^m$ has a simple form, the indefinite sum $\sum_{k=0}^{n-1} \binom{m}{k}$ is not hypergeometric. Extending our analogy, the function $\exp(-x^2)$ is not the derivative of an elementary function, so the infinite integral $\int \exp(-x^2)\,dx$ cannot be expressed as an elementary function. Nevertheless, the definite improper integral is $\int_{-\infty}^{\infty} \exp(-x^2)\,dx = \sqrt{\pi}$.

Consider a sum $\sum_k F(n,k)$ where $F(n,k)$ is a hypergeometric term in both arguments: $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions of n and k. Can we find a recurrence for the sum? Even though we cannot expect, in general, to find a term $G(n,k)$ such that $F(n,k) = G(n,k+1) - G(n,k)$, we often get lucky and find a $G(n,k)$ for which

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)\,.$$

When this happens, we can prove that the definite sum is a constant. By extending the Gosper algorithm, Zeilberger provides a method for determining if such a recurrence of a given order exists. Rather than discuss all of the details, we simply illustrate the basic idea in a two examples.

Problem 9.3.1. For any nonnegative integer n, does the sum

$$s_n := \sum_{k\in\mathbb{Z}} (-1)^k \binom{n}{k}\binom{n+k}{k}$$

satisfy a first-order recurrence?

Solution. For any two polynomials $\alpha, \beta \in \mathbb{C}[n]$, consider the first-order recurrence operator:

$$\begin{aligned} H(n,k) &= \alpha(n)\,F(n,k) + \beta(n)\,F(n+1,k) \\ &= \alpha(n)\,(-1)^k\binom{n}{k}\binom{n+k}{k} + \beta(n)\,(-1)^k\binom{n+1}{k}\binom{n+1+k}{k} \\ &= \frac{(n+k)!(-1)^k}{(k!)^2(n+1-k)!}\Big(k\big(\alpha(n)-\beta(n)\big) + (n+1)\big(\alpha(n)+\beta(n)\big)\Big). \end{aligned}$$

Set $h(k) := k\big(\alpha(n) - \beta(n)\big) + (n+1)\big(\alpha(n) + \beta(n)\big) \in (\mathbb{Q}[n])[k]$. Following the Gosper algorithm, we have

$$\begin{aligned} \frac{H(n,k+1)}{H(n,k)} &= \frac{(n+k+1)!\,(-1)^{k+1}}{\big((k+1)!\big)^2(n-k)!}\,\frac{(k!)^2(n+1-k)!}{(n+k)!\,(-1)^k}\,\frac{h(k+1)}{h(k)} \\ &= \frac{\big(k+(n+1)\big)\big((k-(n+1)\big)}{(k+1)^2}\,\frac{h(k+1)}{h(k)}\,, \end{aligned}$$

so $f(k) = \big(k+(n+1)\big)\big((k-(n+1)\big)$, $g(k) = (k+1)^2$, and $\gcd\big(f(k), g(k+j)\big) = 1$ for all nonnegative integers j. The functional equation becomes

$$\big(k+(n+1)\big)\big((k-(n+1)\big)\,p(k+1) - k^2\,p(k) = k\big(\alpha(n)-\beta(n)\big) + (n+1)\big(\alpha(n)+\beta(n)\big)$$

which has the solution $\alpha(n) = \beta(n) = n + 1$ and $p(k) = -2$. We conclude that

$$G(n,k) = \frac{g(k-1)\,p(k)}{h(k)}\,H(n,k) = \frac{(-1)^{k+1}\,2\,k^2\,(n+k)!}{(k!)^2\,(n+1-k)!}$$

satisfies

$$\begin{aligned} G(n,k+1) - G(n,k) &= \frac{(-1)^{k+2}\,2\,(k+1)^2\,(n+k+1)!}{\big((k+1)!\big)^2\,(n-k)!} - \frac{(-1)^{k+1}\,2\,k^2\,(n+k)!}{(k!)^2\,(n+1-k)!} \\ &= \frac{(-1)^{k+1}\,2\,(n+k)!}{(k!)^2(n-k)!}\left[\frac{(k+1)^2\,(n+k+1)}{(k+1)^2} + \frac{k^2}{n+1-k}\right] \\ &= -2(n+1)^2 H(n,k)\,. \end{aligned}$$

Hence, we have

$$\sum_{k\in\mathbb{Z}} H(n,k) = -2(n+1)^2 \sum_{k\in\mathbb{Z}} G(n,k+1) - G(n,k) = 0\,,$$

so $(n+1)s_n + (n+1)s_{n+1} = 0$, $s_{n+1} = -s_n$, and $s_n = (-1)^n$. □

Problem 9.3.2. For any nonnegative integer n, does the sum

$$s_n := \sum_{k\in\mathbb{Z}} \binom{n}{k}\binom{n+k}{k}$$

satisfy a second-order recurrence?

Solution. For any polynomials $\alpha, \beta, \gamma \in \mathbb{C}[n]$, consider the second-order recurrence operator:

$$\begin{aligned} H(n,k) &= \alpha(n)\,F(n,k) + \beta(n)\,F(n+1,k) + \gamma(n)\,F(n+2,k) \\ &= \alpha(n)\binom{n}{k}\binom{n+k}{k} + \beta(n)\binom{n+1}{k}\binom{n+1+k}{k} + \gamma(n)\binom{n+2}{k}\binom{n+2+k}{k} \\ &= \frac{(n+k)!}{(k!)^2(n-k+2)!}h(k)\,. \end{aligned}$$

For brevity, set

$$h(k) := (n-k+2)(n-k+1)\,\alpha(n) + (n+k+1)(n-k+2)\,\beta(n) + (n+k+2)(n+k+1)\,\gamma(n)\,.$$

Following the Gosper algorithm, we have

$$\begin{aligned} \frac{H(n,k+1)}{H(n,k)} &= \frac{(n+k+1)!}{\big((k+1)!\big)^2(n-k+1)!}\,\frac{(k!)^2\,(n-k+2)!}{(n+k)!}\,\frac{h(k+1)}{h(k)} \\ &= \frac{(n+k+1)(n-k+2)}{(k+1)^2}\,\frac{h(k+1)}{h(k)} \end{aligned}$$

so we obtain $f(k) = (n+k+1)(n-k+2)$, $g(k) = (k+1)^2$, and $\gcd\big(f(k), g(k+j)\big) = 1$ for all nonnegative integers j. The functional equation is $(n+k+1)(n-k+2)\,p(k+1) - k^2 p(k) = h(k)$. It follows that

$$\begin{aligned} k^0 &: \quad (n+1)(n+1)p = (n+1)(n+2)\big(\alpha(n) + \beta(n) + \gamma(n)\big) \\ k^1 &: \quad p = (-2n-3)\alpha(n) + \beta(n) + (2n+3)\gamma(n) \\ k^2 &: \quad -2p = \alpha(n) - \beta(n) + \gamma(n) \end{aligned}$$

which has $\alpha(n) = n + 1$, $\beta(n) = -6n - 9$, $\gamma(n) = n + 2$, and $p(n) = -4n - 6$ as a solution. We conclude that

$$(n+1)s_n - (6n+9)s_{n+1} + (n+2)s_{n+2} = 0\,. \qquad \square$$

Remark 9.3.3. The numbers satisfying this last recurrence are called the ***central Delannoy numbers***. Their ordinary generating function is $(1 - 6x + x^2)^{-1/2}$.