Problems 6

Due: Friday, 5 November 2021 before 17:00 EDT

Students registered in MATH 402 should submit solutions to 4 of the following problems. Students in MATH 802 should submit solutions to all 5.

1. A ***bridge hand*** is a subset of cardinality 13 selected from a standard deck of 52 cards. A ***face card*** is a jack, queen, king, or ace. How many bridge hands have at least one of each kind of face card?

2. For any nonnegative integer, let D_n denote the number of derangements of the set $[n]$.

(i) For any nonnegative integer n, verify via double counting that

$$D_{n+2} = (n+1)\big(D_{n+1} + D_n\big)\,.$$

(ii) For any nonnegative integer n, show that $D_{n+1} = (n+1)\,D_n + (-1)^{n+1}$ using part (i) and induction.

3. Let R denote a commutative domain of characteristic zero. Consider two formal power series $f(x) := \sum_{j \in \mathbb{N}} a_j\, x^j \in R[[x]]$ and $g(x) := \sum_{j \in \mathbb{N}} b_j\, x^j \in R[[x]]$. When $f(x) \in R[x]$ or $[x^0]\big(g(x)\big) = b_0 = 0$, show that the composition

$$f\big(g(x)\big) := \sum_{j \in \mathbb{N}} a_j\, \big(g(x)\big)^j$$

is a well-defined element of $R[[x]]$.

4. Fix a nonnegative integer n.

(i) Use the power conversion identity for Stirling subset numbers to show that

$$(x+1)^n = \sum_{m \in \mathbb{Z}} \left\{ {n+1 \atop m+1} \right\} x^{\underline{m}}\,.$$

(ii) By comparing appropriate coefficients in a polynomial equation, prove that

$$\left\{ {n+1 \atop m+1} \right\} = \sum_{k \in \mathbb{Z}} \binom{n}{k} \left\{ {k \atop m} \right\}.$$

(iii) By comparing appropriate coefficients in a polynomial equation, prove that

$$\left[{n+1 \atop m+1} \right] = \sum_{k \in \mathbb{Z}} \left[{n \atop k} \right] \binom{k}{m}.$$

5. Let K be a field of characteristic zero. Consider the K-algebra $K((x))$ of formal Laurent series. The ***formal residue*** map $\operatorname{Res} : K((x)) \to K$ is defined by $\operatorname{Res}(f) := [x^{-1}](f)$. For any two $f, g \in K((x))$, prove the following:

(i) $\operatorname{Res}\!\left(\frac{df}{dx}\right) = 0$; *Hint*: Differentiation is defined term-by-term.

(ii) $\operatorname{Res}\!\left(\frac{df}{dx}\, g\right) = -\operatorname{Res}\!\left(f\, \frac{dg}{dx}\right)$; *Hint*: Assume the product rule holds.

(iii) $\operatorname{Res}\!\left(\frac{1}{f}\, \frac{df}{dx}\right) = \operatorname{ord}(f)$ for all $f \neq 0$.