A matchings dual for graphs ☆

David A. Gregory

Dept. of Mathematics and Statistics, Queen's Univ., Kingston, Ont. Canada, K7L 3N6

Abstract

Let G be a graph with adjacency matrix A and let $\mathbb{F}$ be a field. An $\mathbb{F}$-matrix Q is a support matrix of G if $A = [Q \neq O]$, the zero-nonzero pattern of Q.

If G has an invertible skew-symmetric support $\mathbb{F}$-matrix S, the S-dual G^S of G is defined as the graph with adjacency matrix $[S^{-1} \neq O]$. An analogous adjacency matrix dual, G^+ has been examined in the literature for those bipartite graphs G with unique perfect matchings for which A^{-1} is sign-similar to an adjacency matrix. For such graphs G, the $+$-dual is a an example of an S-dual, that is, $G^+ \cong G^S$ for some choice of S.

If G is a graph with a perfect matching, the matchings dual of G is the graph G^* on the same vertex set but with vertices i, j adjacent in G^* if and only if $G - i - j$ has a perfect matching. Though G^S may depend on S, it is always a subgraph of G^* and is equal to G^* for some choice of S with integer entries.

For a large class of graphs, the corona graphs, it turns out that $G^* \cong G^+$ whenever G^+ is defined.

Keywords: Skew-adjacency matrices, pfaffians, matchings, graph duals
2000 MSC: 05C22, 05C31, 05C50, 97K30

☆These notes supplement seminar talks presented at the Royal Military College in Kingston. The talks were based on discussions of a workgroup consisting of M. Cavers, S. Cioabă, S. Fallat, D. Gregory, W. Haemers, S. Kirkland, J. McDonald and M. Tsatsomeros at the workshop on *Theory and Applications of Matrices Described by Patterns*, held at the Banff International Research Station, Alberta, Canada, January 31 - February 5, 2010. The results obtained there and some of the results given here appear in [4].

This revision, 8 May 2011. Updates at http://www.mast.queensu.ca/(tilde)gregoryd/

1. Introduction

Throughout, G denotes a simple graph on n vertices with edge set $E(G)$, vertex set $V(G)$ and adjacency matrix $A = A(G) = [a_{i,j}]$ where $a_{i,j} = 1$ if $ij \in E(G)$ and $a_{i,j} = 0$ otherwise. In most cases, $V(G) = [n] = \{1, 2, \ldots, n\}$.

A *support* $\mathbb{F}$*-matrix* of G is an $n \times n$ matrix S with entries from some field $\mathbb{F}$ such that entry $s_{i,j} \neq 0$ if and only if ij is an edge of G. Equivalently, an $\mathbb{F}$-matrix S is a support matrix of G if $A(G) = [S \neq O]$, the $\{0, 1\}$-matrix whose 1's mark the nonzero entries of S. A skew-symmetric support $\mathbb{F}$-matrix is referred to more briefly as a *skew-support* $\mathbb{F}$*-matrix*. A skew-support $\mathbb{F}$-matrix S of a graph G is often chosen to have entries from $\{0, 1, -1\}$. In that case, S is called a *skew-adjacency* $\mathbb{F}$*-matrix* of G.

If S is an invertible skew-support $\mathbb{F}$-matrix of G, define the S*-dual* of G to be the graph G^S on the same vertex set $[n]$ with i, j adjacent in G^S if $(S^{-1})_{i,j} \neq 0$. The following simple lemma justifies the use of the word *dual* in S-dual.

Lemma 1.1. *If S is an invertible skew-support $\mathbb{F}$-matrix of a simple graph G, then G^S is a simple graph, S^{-1} is an invertible skew-support $\mathbb{F}$-matrix of G^S, and $(G^S)^{S^{-1}} = G$.*

In some of the literature, the adjacency matrix $A = A(G)$ (with $\mathbb{F} = \mathbb{Q}$, the field of rationals) is used to define a dual of a graph G. There, when A is invertible, the A-dual G^A of G is defined as the graph with adjacency matrix $[A^{-1} \neq O]$. But A^{-1} may have nonzero diagonal entries so G^A need not be a simple graph. One way this difficulty with the adjacency matrix is avoided is by restricting G to be bipartite [9]. For, if G is a bipartite graph, the vertices of G may be ordered so that the adjacency matrix has the form

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}.$$

The matrix B is called the *biadjacency matrix* associated with A. If A is invertible, then B is invertible and

$$A^{-1} = \begin{bmatrix} O & (B^\top)^{-1} \\ B^{-1} & O \end{bmatrix}.$$

Thus, if an adjacency matrix A of a bipartite graph is invertible, then A^{-1} has zero diagonal and G^A is simple (and bipartite).

However, a special skew-adjacency $\mathbb{Q}$-matrix could just as well be used in place of A. For if G is bipartite with an invertible adjacency matrix A as above, and

$$S = \widetilde{A} = \begin{bmatrix} O & B \\ -B^\top & O \end{bmatrix}, \text{ then } S^{-1} = \begin{bmatrix} O & -(B^\top)^{-1} \\ B^{-1} & O \end{bmatrix}. \tag{1}$$

For this special choice $S = \widetilde{A}$ of a skew-adjacency matrix of a bipartite graph G with an invertible adjacency matrix A, we have $[A^{-1} \neq O] = [S^{-1} \neq O]$, so $G^A = G^S$. Thus we have the following lemma.

Lemma 1.2. *Suppose that G is a bipartite graph with an invertible adjacency matrix A. Then $G^A = G^S$ where $S = \widetilde{A}$ is the special skew-adjacency matrix obtained from A as in (1). Thus, in this case, the adjacency matrix dual G^A of G is an example of an S-dual of G.*

There are results in the literature for classes of bipartite graphs G for which A^{-1} exists and more closely resembles an adjacency matrix of G^A in some way. For example, if G is an h-graph, that is, if G is bipartite and has a unique perfect matching, then $\det A = \pm 1$, and A^{-1} is a symmetric support $\mathbb{Q}$-matrix of G^A with integral entries. Kirkland and Tifenbach [9] characterize the h-graphs G for which A^{-1} is sign-similar to the adjacency matrix of G^A; that is, for which $A^{-1} = D[A^{-1} \neq 0]D^\top$ for some diagonal $\{\pm 1\}$-matrix D. They first define the $+$-dual G^+ to be G^A but, later in their paper, they take G^+ to be the isomorphic copy of G^A obtained by interchanging the vertices on the unique perfect matching in G.

Let $\mathcal{S}_{\mathbb{F}}(G)$ denote the set of all invertible skew-support $\mathbb{F}$-matrices of a graph G. If G is a graph for which $\mathcal{S}_{\mathbb{F}}(G) \neq \emptyset$ and $S \in \mathcal{S}_{\mathbb{F}}(G)$, then S^{-1} is skew-symmetric and therefore, by definition, we *always* have $S^{-1} \in \mathcal{S}_{\mathbb{F}}(G^*)$ without any further restriction on G. In many of the classes of examples where G^+ is defined, we shall see that for all invertible skew-adjacency $\mathbb{F}$-matrices S of G, S^{-1} is an invertible skew-adjacency $\mathbb{F}$-matrix of G^S and G^S is isomorphic to G^+.

Because of the comments in the previous paragraph, skew-symmetric support matrices will be used exclusively here.

If G has a perfect matching and $\operatorname{char} \mathbb{F} \neq 2$ (so $1 \neq -1$ in $\mathbb{F}$), then $\mathcal{S}_{\mathbb{F}}(G) \neq \emptyset$, even if the adjacency matrix A is singular. In fact, $\mathcal{S}_{\mathbb{F}}(G)$ contains an invertible skew-adjacency $\mathbb{F}$-matrix (Lemma 2.3).

The *matchings dual* (or, more briefly, the $*$*-dual of* G) is determined combinatorially from G without using matrix inverses; vertices $i, j \in [n]$ are adjacent in G^* if and only if $G - i - j$ has a perfect matching (Definition 2.1, Theorem 2.4). In general, different $S \in \mathcal{S}_{\mathbb{F}}(G)$ may yield different S-duals, but all G^S are subgraphs of the matchings dual G^*. Thus, the zero-nonzero patterns $[S^{-1} \neq O]$ of the inverses of the matrices S in $\mathcal{S}_{\mathbb{F}}(G)$ are all subordinate to $A(G^*)$, the adjacency matrix of the $*$-dual of G.

If G has a perfect matching, and $\mathbb{F} \supseteq \mathbb{Q}$, the field of rationals, the matchings dual G^* turns out to be an example of an S-dual G^S; in fact, there is an $S \in \mathcal{S}_{\mathbb{F}}(G)$ with integer entries such that $G^S = G^*$ (Lemma 2.3). Often an S-dual G^S of G differs little from its matchings dual G^*, and may be determined quickly (See, e.g., 2.1). In fact, there are families of graphs G for which $G^S = G^*$ for all $\mathcal{S} \in \mathcal{S}(G)$ (Lemma 2.7, Corollary 2.8). For graphs G in such families, the zero-nonzero patterns $[S^{-1} \neq 0]$ of the inverses of the matrices S in $\mathcal{S}_{\mathbb{F}}(G) \neq \emptyset$ are identical. If G is a bipartite corona graph, then the adjacency dual G^+ is defined and is isomorphic to G^*.

The matchings dual is nonempty for every graph G with a perfect matching and is of interest in its own right. Its properties are developed in Section 3. In particular, it is shown there that if G has a perfect matching M, the idempotent bijection $\sigma_M : V(C(G)) \to V(C(G))$ that switches the end vertices of the edges of M is a homomorphism from G into G^*. It then follows that if G has a perfect matching, then G is a spanning subgraph of G^{**} and that G is $*$-self dual ($G \cong G^*$) if and only if $G = G^{**}$. Graphs G with perfect matchings that are $*$-self dual are of particular interest because they are precisely the graphs for which $(G^*)^{S^{-1}} = (G^*)^*$ whenever $G^S = G^*$. They are examined in Section 4.

2. Pfaffians of skew-support $\mathbb{F}$-matrices and the matchings dual

In this section, we employ a result for pfaffians used by Tutte [10] in his characterization of graphs with perfect matchings to determine when a graph G has an invertible skew-adjacency $\mathbb{F}$-matrix and to expose a connection between the S-dual G^S and the matchings in G.

A *matching* in G on k vertices is a set $M = \{i_1i_2,\ i_3i_4, \ldots, i_{k-1}i_k\}$ of vertex disjoint edges in G. A matching M in G is *perfect* if each vertex in G is in some edge of M. Let $\mathcal{M}(G)$ denote the set of all perfect matchings in G. Of course, $\mathcal{M}(G)$ is empty if G has odd order.

If G is a simple graph with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edge set $E(G)$, the *generic* skew-support matrix of G over $\mathbb{F}$ is the $n \times n$ skew-symmetric matrix $X(G) = X = [x_{i,j}]$ where the entries $x_{i,j}$ with $i < j$ and $ij \in E(G)$ are independent indeterminates over $\mathbb{F}$ and where $x_{i,j} = 0$ if $ij \notin E(G)$.

If X is a generic skew-support matrix of G over $\mathbb{F}$, then the *pfaffian* of X, $\operatorname{pf} X$, is defined by the rule

$$\operatorname{pf} X = \sum_{M \in \mathcal{M}(G)} \operatorname{wt}(X_M), \tag{2}$$

where $\mathcal{M}(G)$ denotes the set of all perfect matchings

$$M = \{i_1 i_2,\ i_3 i_4, \ldots, i_{n-1} i_n\}$$

in G and where $\operatorname{wt}(X_M)$ is equal to the product $\prod_{\{i_j, i_{j+1}\} \in M} x_{i_j, i_{j+1}}$ multiplied by the sign of the permutation that takes $(1, 2, \ldots, n)$ to $(i_1, i_2, \ldots, i_n)$. Because X is skew-symmetric, $\operatorname{wt}(X_M)$ is not affected by the order of the edges in M or the order chosen for the vertices of each edge. If n is odd, or if n is even and $\mathcal{M}(G)$ is empty, we take $\operatorname{pf} X = 0$.

It is well-known (see, e.g., [3, p.318]) that

$$\det X = (\operatorname{pf} X)^2. \tag{3}$$

Because the entries of X are independent indeterminates, $\det X = (\operatorname{pf} X)^2 \neq 0$ if and only if G has a perfect matching (see also [3, pp. 317-323]). Thus, X is invertible if and only if G has a perfect matching.

Of course, the *pfaffian*, $\operatorname{pf} S$, of a skew-support $\mathbb{F}$-matrix S of a graph G of order n is defined by setting $x_{i,j} = s_{i,j}$ in $\operatorname{pf} X$ where X is the generic skew-support matrix of G over $\mathbb{F}$. Thus, if S is an invertible skew-support $\mathbb{F}$-matrix of G, then $\det S$ is a nonzero square in $\mathbb{F}$. Also, if $\det S \neq 0$ then G must have a perfect matching. However, even if G has a perfect matching, it is possible that $\det S = 0$ because of cancellation in $\operatorname{pf} S$.

The following lemma is an immediate consequence of the definition of the pfaffian. Recall that a skew-adjacency $\mathbb{F}$-matrix of G is a skew-support $\mathbb{F}$-matrix S of G with entries from $\{0, 1, -1\}$, so $\operatorname{wt}(S_M) \in \{1, -1\}$ for each $M \in \mathcal{M}(G)$.

Lemma 2.1. *1. If some skew-support $\mathbb{F}$-matrix of G is invertible, then G has a perfect matching.*

2. If G has a unique perfect matching, then all skew-support $\mathbb{F}$-matrices of G are invertible.

3. If the number $m(G) = |\mathcal{M}(G)|$ of perfect matchings in G is odd, and $\operatorname{char} \mathbb{F} \in \{0, 2\}$ or $\operatorname{char} \mathbb{F} > m(G)$, then all skew-adjacency $\mathbb{F}$-matrices of G are invertible.

If M is an invertible matrix of order n with entries from $\mathbb{F}$, *Jacobi's identity* (see, e.g., [3, pp.301,320]) implies that

$$(\det M) \det M(i,j) = \det\left((\operatorname{adj} M)[i,j]\right) \tag{4}$$

where $M(i,j)$ denotes the principal $(n-2) \times (n-2)$ submatrix of M obtained by deleting rows i and j and columns i and j, and $(\operatorname{adj} M)[i,j]$ denotes the principal 2×2 submatrix of its adjugate obtained by selecting rows i and j and columns i and j.

Lemma 2.2. *Let G be a graph with an invertible skew-support $\mathbb{F}$-matrix S. Then vertices i, j are adjacent in G^S if and only if $\det S(i,j) \neq 0$.*

Proof. Jacobi's identity (4) implies that $C_{i,j}^2 = \det S(i,j) \det S$ where $C_{i,j} = (\det S)(S^{-1})_{j,i}$ is the i, j cofactor of S. Thus, $(S^{-1})_{i,j} \neq 0$ if and only if $\det S(i,j) \neq 0$.

Lemma 2.3. *Let G be a simple graph. Then:*

1. *If $\operatorname{char} \mathbb{F} \neq 2$, then some skew-adjacency $\mathbb{F}$-matrix of G is invertible ($\mathcal{S}_{\mathbb{F}}(G) \neq \emptyset$) if and only if G has a perfect matching.*
2. *If $\operatorname{char} \mathbb{F} = 2$, then G has a unique skew-adjacency matrix and it is invertible if and only if the number of perfect matchings in G is odd.*

Proof. 1. The necessity has already been observed in Lemma 2.1.

Suppose then that G has a perfect matching, M. Let m be the number of edges in G. Then $m \geq |M|$. We use induction on m to prove that $\mathcal{S}_{\mathbb{F}}(G) \neq \emptyset$. If $m = |M|$, then M is equal to $E(G)$ and every skew-adjacency matrix of G is invertible. Suppose then that $m > |M|$ and that the result is true for every subgraph of G that contains M and has $m - 1$ edges. Let kl be an edge of G not in M and let $\widehat{G} = G - kl$ be the subgraph of G obtained by deleting the edge kl. By the inductive assumption, $\widehat{G}$ has an invertible skew-adjacency matrix $\widehat{S}$. Let S be the skew-symmetric matrix with $s_{i,j} = \hat{s}_{i,j}$ when $ij \neq kl$

and with $s_{k,l} = x$, where x is an indeterminate. It is straightforward to verify that $\det S = x^2 \det \widehat{S}(k,l) - 2x\widehat{C}_{k,l} \det \widehat{S} + \det \widehat{S}$ where $\widehat{C}_{k,l}$ denotes the (k,l) cofactor of $\widehat{S}$. If $\widehat{C}_{k,l} \neq 0$, then $\det S \neq 0$ for some choice of $x = \pm 1$. If $\widehat{C}_{k,l} = 0$, then $\det S = x^2 \det \widehat{S}(k,l) + \det \widehat{S}$ where, by Jacobi's identity (4), $\det \widehat{S}(k,l) = 0$ since $(\operatorname{adj} \widehat{S})[k,l]$ is a zero matrix. Thus, in each case, x may be chosen so that S is an invertible skew-adjacency $\mathbb{F}$-matrix for G.

2. If $\operatorname{char} \mathbb{F} = 2$, then $1 = -1$ in $\mathbb{F}$, so the adjacency matrix A of G is the only skew-adjacency $\mathbb{F}$-matrix of G. By (2), A is nonsingular in $\mathbb{F}$ if and only if the number of perfect matchings in G is odd.

Remark 2.1. Suppose that $\operatorname{char} \mathbb{F} = 2$ and G is a graph of even order n. Then the adjacency matrix A of G is the only skew-adjacency $\mathbb{F}$-matrix of G and, by Lemma 2.3, it is invertible if and only if the number of perfect matchings in G is odd. (Such graphs must have some vertices of odd degree, otherwise $\det A \equiv 0 \pmod 2$.) Equivalently, A is invertible if and only if $A = QNQ^\top$ where Q and N are invertible matrices with entries in GF(2) and N is the direct sum of $n/2$ 2×2 matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (see, e.g., [6, p.183] or [7, p.128]). Thus, A is nonsingular in $\mathbb{F}$ if and only if the vertices of G may be labeled by vectors in a basis Ω of $\mathrm{GF}(2)^n$ (the rows of Q) so that vertices labeled with vectors $u, v \in \Omega$ are adjacent in G if and only if

$$u_1 v_2 + u_2 v_1 + \cdots + u_{n-1} v_n + u_n v_{n-1} \equiv 0 \pmod 2. \tag{5}$$

For example, if Ω is the standard basis of $\mathrm{GF}(2)^n$, then $E(G)$ is a perfect matching. Also, if Ω is the basis of $\mathrm{GF}(2)^6$ consisting of the vectors $(1,0;0,0;0,0)$, $(0,1;0,0;0,0)$, $(1,0;1,0;1,0)$, $(0,0;0,1;0,0)$, $(0,0;1,0;0,0)$, $(0,1;0,1;0,1)$, then G is the graph in Figure 1.

When Ω is a spanning subset of $\mathrm{GF}(2)^n$ that contains no zero vectors, the graph with vertex set Ω and adjacency defined by (5) is called a *symplectic* graph on Ω [6, p.183].

It follows from Remark 2.1 that a graph G of order n has an odd number of perfect matchings if and only if it is isomorphic to a symplectic graph on a set Ω that is a basis of $\mathrm{GF}(2)^n$.

Definition 2.1. Let G be a graph of order n. If $n \geq 3$, let G^* (respectively, $G^{\#}$) be the graph on the same vertex set, but with vertices i, j adjacent if $G - i - j$ has a perfect matching (respectively, a unique perfect matching). If $n \leq 2$, let $G^* = G^{\#} = G$.

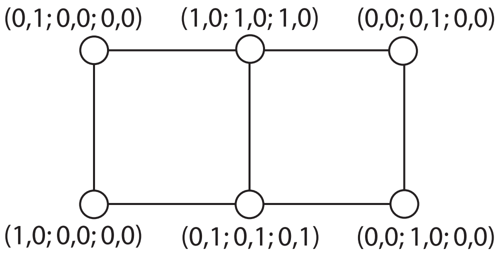

Figure 1: *A symplectic graph on a basis of* $\mathrm{GF}(2)^6$.

We call G^* the *matchings dual* (or, more briefly, the $*$-*dual*) of G. Clearly, G^* is empty (has no edges) if n is odd or if G has 3 or more connected components with no perfect matching. Also, the skew-support matrices S of G are all singular in these cases, so $\mathcal{S}(G) = \emptyset$.

If G and H are graphs, we write $H \subseteq G$ if H is a subgraph of G; that is, if $V(H) \subseteq V(G)$ and $E(G) \subseteq E(H)$. If H is a subgraph of G obtained by deleting a proper subset U of $V(G)$, we say that H is an *induced* subgraph of G and write $H = G - U$. A subgraph H of G is said to be a *spanning* subgraph if $V(H) = V(G)$.

The next theorem traps the skew-duals G^S between $G^\#$ and G^*. Recall that $\mathcal{S}_{\mathbb{F}}(G)$ denotes the set of all *invertible* skew-support $\mathbb{F}$-matrices of G. Thus, if $S \in \mathcal{S}_{\mathbb{F}}(G)$, then G must have a perfect matching.

Theorem 2.4. *If $S \in \mathcal{S}_{\mathbb{F}}(G)$, then $G^\# \subseteq G^S \subseteq G^*$.*

Proof. Let $S \in \mathcal{S}_{\mathbb{F}}(G)$ and let $i, j \in [n]$. Then $S(i,j)$ is a skew-support matrix of $G - i - j$, the induced subgraph of G obtained by deleting vertices i and j.

If $ij \in E(G^S)$, then $(\operatorname{pf} S(i,j))^2 = \det S(i,j) \neq 0$ by Lemma 2.2, so $G - i - j$ has a perfect matching and $ij \in E(G^*)$. Thus, $G^S \subseteq G^*$.

If $ij \in E(G^\#)$ then $G-i-j$ has a unique perfect matching, so $\det S(i,j) = (\operatorname{pf} S(i,j))^2 \neq 0$ and $ij \in E(G^S)$ by Lemma 2.2. Thus, $G^\# \subseteq G^S$.

As the following example shows, sometimes G^* and $G^\#$ differ very little and the skew-duals G^S can be determined quickly.

Example 2.1. *(A graph G and its S-duals.)*

Figure 2 shows two different S-duals for a graph G. The skew-support matrices used are the skew-adjacency $\mathbb{Q}$-matrices S determined by the orientations shown on the two copies of G. (Take $s_{i,j} = 1$ if $i \to j$ in G.) Since G has a unique perfect matching, both matrices are invertible. Note that $G - i - j$ has two perfect matchings when $\{i, j\} = \{1, 6\}$, and at most one perfect matching for all other vertex pairs $\{i, j\}$. It is then straightforward to check that $G^{\#}$ is the fourth graph in the diagram and that G^* is the second. Using the definition of pfaffian, we see that $\det S(1, 6) = 4$ for the first orientation and $\det S(1, 6) = 0$ for the second. Thus, by Lemma 2.2, $\{1, 6\}$ is an edge of G^S in the first case, but not in the second. Thus, $G^S = G^*$ in the first case while $G^S = G^{\#}$ in the second. Because G^* and $G^{\#}$ differ by only one edge, Theorem 2.4 implies that they are the *only* possible S-duals of G for *any* field $\mathbb{F}$ for which $\mathcal{S}_{\mathbb{F}}(G) \neq \emptyset$.

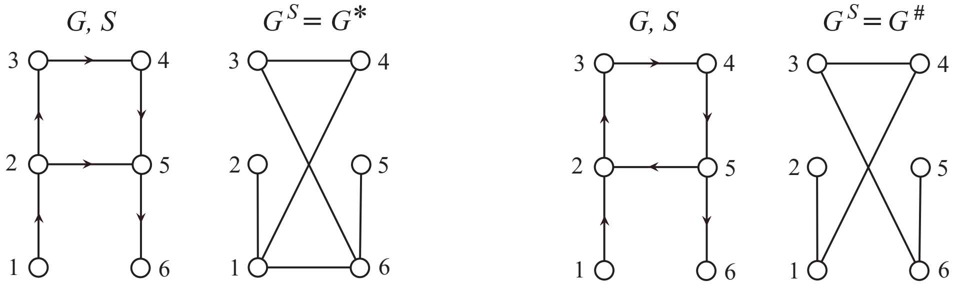

Figure 2: *A graph G and its S-duals G^S.*

Theorem 2.4 motivates the following definition.

Definition 2.2. A skew-support $\mathbb{F}$-matrix S of a graph G is a $*$-*support* $\mathbb{F}$-*matrix* of G if S is invertible and $G^S = G^*$.

The advantage of a $*$-support matrix S of G is that G^S has a simple graphical interpretation: $G^S = G^*$.

Let $\mathcal{S}^*_{\mathbb{F}}(G)$ denote the set of all $*$-support $\mathbb{F}$-matrices of G. Thus, $\mathcal{S}^*_{\mathbb{F}}(G) = \{S \in \mathcal{S}(G) \mid G^S = G^*\}$. By Lemma 2.2, if $S \in \mathcal{S}_{\mathbb{F}}(G)$ then $\mathcal{S} \in \mathcal{S}^*_{\mathbb{F}}(G)$ if and only if $S(i, j) \neq 0$ whenever $G - i - j$ has a perfect matching.

Lemma 2.5. *If G has a perfect matching and X is the generic skew-support matrix of G over $\mathbb{F}$, then $G^X = G^*$.*

PROOF. If $i, j \in [n]$, then $X(i,j)$ is a generic skew-support matrix of $G-i-j$. Also, because $X(i,j)$ is generic, $\det X(i,j) = (\operatorname{pf} X(i,j))^2 \neq 0$ if and only if $G-i-j$ has a perfect matching [3, p.317-323]. Thus, $G^X = G^*$ by Lemma 2.2 and Definition 2.1.

The following corollary implies that if G has a perfect matching and $\mathbb{F} \supseteq \mathbb{Q}$, then $\mathcal{S}_{\mathbb{F}}^*(G) \neq \emptyset$. This implies that G^* is itself an S-dual if G has a perfect matching.

Corollary 2.6. *If G has a perfect matching and $\operatorname{char} \mathbb{F} = 0$, then G has a $*$-support $\mathbb{F}$-matrix with integer entries.*

PROOF. It follows from Lemma 2.5 that if G is a graph with a perfect matching, then $X = X(G)$ is invertible and the zero-nonzero pattern of X^{-1} is equal to the adjacency matrix of G^*:

$$[X^{-1} \neq O] = A(G^*). \tag{6}$$

Since the polynomial $(\det X) \prod_{ij \in E(G^*)} C_{i,j}(X)$ is nonzero for some choice of rational entries (see, e.g., [2, Lem. 2.1]), it follows that the system of conditions (6) has a solution $X = S$ with rational entries and therefore a solution with integer entries.

Recall that a skew-adjacency matrix has entries from $\{0, 1, -1\}$.

Problem 1. Which graphs with perfect matchings have a skew-adjacency $\mathbb{Q}$-matrix that is a $*$-support matrix?

The following lemma provides graphs for which $\mathcal{S}_{\mathbb{F}}(G) = \mathcal{S}_{\mathbb{F}}^*(G)$. The proof is an immediate consequence of Theorem 2.4.

Lemma 2.7. *Suppose $\mathcal{S}_{\mathbb{F}}(G) \neq \emptyset$, and $G-i-j$ has at most one perfect matching for each pair i, j of vertices in G. Then $G^* = G^S$ for all $S \in \mathcal{S}_{\mathbb{F}}(G)$; that is, $\mathcal{S}_{\mathbb{F}}(G) = \mathcal{S}_{\mathbb{F}}^*(G)$.*

An *odd-cycle graph* is a graph with no even cycles. Of course, a graph with no odd cycles (an *even-cycle graph*) is a bipartite graph. A graph is a *forest* if it is both an odd-cycle and an even-cycle graph.

Corollary 2.8. *If G is an odd-cycle graph with a perfect matching and $\mathbb{F}$ is any field, then all skew-support $\mathbb{F}$-matrices of G are invertible and $G^* = G^S$ for all $S \in \mathcal{S}(G)$.*

Proof. If G is an odd-cycle graph, then no induced subgraph of G could have two perfect matchings because their symmetric difference would contain an even cycle. Thus, G has only one perfect matching, so every skew-support matrix is invertible. Also $G - i - j$ has at most one perfect matching for all i, j, and Lemma 2.7 applies.

The matchings dual G^* of an odd-cycle graph G with a perfect matching need not be an odd-cycle graph. For example, P_6 is a tree, and therefore an odd-cycle graph, but P_6^* (the first graph in Figure 2) has an even cycle.

The following lemma characterizes the graphs G with $*$-support matrices whose inverses are also $*$-support matrices.

Lemma 2.9. *If S is a $*$-support matrix for a graph G, then S^{-1} is a $*$-support matrix for G^* if and only if $G = G^{**}$.*

Note that the graph G in Figure 2 is isomorphic to its matchings dual, $G \cong G^*$. We say that a graph G is *$*$-self-dual* if $G \cong G^*$. In Section 3, we shall see that if G has a perfect matching, then $G \subseteq G^{**}$ and that equality holds if and only if G is $*$-self-dual.

It then follows from Lemma 2.9 that the $*$-self-dual graphs are precisely those graphs whose $*$-support matrices have inverses that are also $*$-support matrices. The important class of $*$-self-dual graphs is examined in Section 4.

3. Properties of the matchings dual

Every graph G has a matchings dual G^*. However, if G has odd order, then G^* has no edges. So, when determining G^*, we only consider graphs G of even order. If we restrict our attention to connected graphs with connected $*$-duals, we shall see that the graphs and their $*$-duals must have perfect matchings.

Example 3.1. *(The matchings duals of some common graphs)*

1. The nonempty graphs of order 4 and their $*$-duals are shown in Figure 3. All but two of the graphs there are $*$-self-dual. The remaining two graphs satisfy $G^{**} = G$.

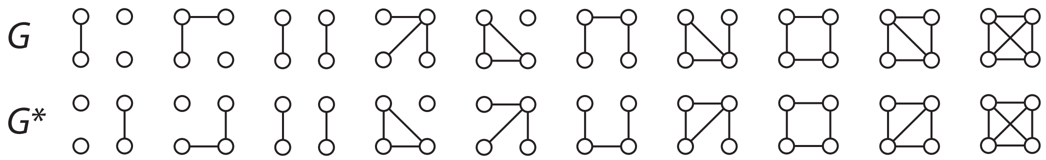

Figure 3: *The 10 nonempty graphs G on 4 vertices and their matchings-duals, G^*.*

2. The complete bipartite graph $K_{m,n}$ is $*$-self-dual if and only if $m = n$. The complete graph K_n is $*$-self-dual if and only if n is even. Also, the join $K_n \vee \overline{K}_n$, of a complete graph with an empty graph of the same order, is $*$-self-dual.

3. If P_n is a path with $n = 2m$ vertices $u_1 - v_1 - u_2 - v_2 - \cdots - u_m - v_m$, then it follows from Def. 2.1 that u_i is adjacent to v_j in P_n^* if and only if $i \leq j$ (see also [1, 9]). In particular, P_2 and P_4 are $*$-self-dual, but P_6^* consists of a 4-cycle $u_1 - v_2 - u_2 - v_3 - u_1$ with pendant edges u_1v_1 and v_3u_3 (see Figure 4).

4. If C_n is an even cycle, $n = 2k$ say, then $C_n^* \cong K_{k,k}$, the complete bipartite graph with k vertices in each vertex part. If n and m are odd, then the vertex disjoint sums $C_n \dot{+} C_m$ and $K_n \dot{+} K_m$ both have $*$-dual $K_{n,m}$. If $n = 2k$ and $m = 2l$ are even, then $(C_n \dot{+} C_m)^* = K_{k,k} \dot{+} K_{l,l}$ while $(K_n \dot{+} K_m)^* = K_n \dot{+} K_m$.

A subgraph H of G is said to be *nice* [8, p.125] if $G - V(H)$ (the induced subgraph of G obtained by deleting the vertices of H) has a perfect matching.

Let $M(G)$ be the set of all those edges in G that are in perfect matchings of G. Let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G. The following properties of the matchings dual follow directly from Definition 2.1 and hold for all graphs, whether or not they have perfect matchings.

Lemma 3.1. *The following properties hold for all graphs G of even order.*

1. *If G is bipartite, then G^* is bipartite and the two graphs share a common vertex bipartition.*
2. *If H is a spanning subgraph of G or a nice subgraph of G, then $G^* \supseteq H^*$.*
3. $E(G) \cap E(G^*) = M(G)$.
4. *G is $*$-self-dual if and only if there is a bijection $\sigma : V(G) \to V(G)$ such that $G - \sigma(i) - \sigma(j)$ contains a perfect matching of G if and only if ij is an edge of G.*

5. $\mathrm{Aut}(G) \subseteq \mathrm{Aut}(G^*)$.

Lemma 3.1(3) implies that if G has a perfect matching, then G^* has a perfect matching as well. The following lemma implies a partial converse: in the special case that G is connected, if G^* has a perfect matching then G must have a perfect matching.

Lemma 3.2. *Let G be a connected graph of order $n \geq 2$. If G^* has no isolated vertices, then G has a perfect matching.*

PROOF. Let $D(G)$ denote the set of all vertices in G which are not covered by at least one maximum matching. Because G^* has no isolated vertices, for each vertex i in G there is a vertex i' in G such that $G - i - i'$ has a perfect matching, $M_{i,i'}$ say. Suppose that G has no perfect matching. Then the matchings $M_{i,i'}$ are maximum. Thus n is even and $D(G) = V(G)$. But, by the Gallai-Edmonds Structure Theorem [8, p.94], the components of the subgraph of G induced by $D(G)$ all have odd order. Thus the components of G have odd order. But G is connected, so G has odd order. This is a contradiction so G must have a perfect matching.

Let M be a matching in G. An path in G is *M-alternating* if it edges are alternately in M and $E(G)\backslash M$. An *M-augmenting path* is an M-alternating path whose end vertices are not incident to edges in M. A theorem of Berge asserts that a matching M in G has the maximum possible number of edges if and only if G has no M-augmenting path (see, e.g., [11]).

Lemma 3.3. *Let G be a graph with a perfect matching.*

1. If either $ij \in M(G)$ or if there is some *perfect matching M in G for which there is an M-alternating (i,j)-path in G whose end edges are in M, then $ij \in E(G^*)$.*

2. If $ij \in E(G^*)$, then either $ij \in M(G)$ or for each *perfect matching M in G there is an M-alternating (i,j)-path in G whose end edges are in M.*

PROOF. Let $G' = G - i - j$.

1. Suppose that M is a perfect matching in G and P is an (i,j)-path in G whose end edges are in M. Then $P' = P - i - j$ is an M-alternating path in G whose end edges are not in M. The edges of M in G' that are not on P' together with the edges of P' that are not in M form a perfect matching in G'. Thus $ij \in G^*$.

2. Suppose that $ij \in G^*$. Then $G' = G - i - j$ has a perfect matching. Since G has a perfect matching, each component of G has even order, so i and j must be in the same component of G. If ij is in some perfect matching of G, then $ij \in E(G^*)$ by Lemma 3.1(3). Suppose then that $G - i - j$ has a perfect matching and that ij is in no perfect matching of G. Let M be a perfect matching in G. Since ij is not in M, there are edges ii' and jj' in G that are in M. Let M' be the restriction of M to G'. Then M' covers all of the vertices of $G - i - j$ except for i' and j'. Because G' has a perfect matching, M' is not a maximum matching in G'. Thus G' must contain an M'-augmenting path P'. Because the end edges of P' are not in M' and i' and j' are the only vertices of G' that are not covered by M', the path P' must be an (i', j')-path in G'. Appending the edges ii' and jj' to the ends of P' yields an M-alternating path P in G whose end edges are in M.

Note that if a graph G has a perfect matching M, and H is a subgraph of G that has a perfect matching contained in M, then H is nice and so $G^* \supseteq H^*$.

If G has a perfect matching M, let $\sigma_M : V(G) \to V(G)$ be the bijection that switches the ends of the edges in M. Thus, $\sigma_M(x) = x'$ where xx' is the edge of M that is incident to x. Property 1 in the following lemma asserts that σ_M is a *homomorphism* from G to G^*. Because σ_M is a bijection, property 1 also implies that G is isomorphic to a subgraph of G^*.

Lemma 3.4. *The following properties hold for a graph G with a perfect matching M and associated bijection σ_M on $V(G)$.*

1. *If $xy \in E(G)$, then $\sigma_M(x)\sigma_M(y) \in E(G^*)$. Thus $|E(G)| \leq |E(G^*)|$.*
2. *G is $*$-self-dual if and only if σ_M is an isomorphism from G to G^*.*
3. *$G \subseteq G^{**}$. Moreover, $G = G^{**}$ if and only if G is $*$-self-dual, $G \cong G^*$.*
4. *If G is connected, then G has a spanning tree containing M.*
5. *G^* is connected if and only if G is connected.*

Proof. If $n = 2$, then $G = K_2$ and properties 1-5 follow immediately since $K_2^* = K_2$ by Definition 2.1. Suppose then that $n \geq 4$.

1. If $xy \in M$ then $\sigma_M(x)\sigma_M(y) = x'y' = xy \in M \subseteq E(G^*)$. If $xy \in E(G)$ and $xy \notin M$, then $x' - x - y - y'$ is a nice P_4 in G, so $G^* \supseteq P_4^*$. But $x'y'$ is an edge of P_4^* so $x'y' = \sigma_M(x)\sigma_M(y)$ is an edge of G^*.

If xy is a 2-subset of $V(G)$, let $\sigma_M(xy) := \sigma_M(x)\sigma_M(y)$. Then property 1 asserts that $\sigma_M(E(G)) \subseteq E(G^*)$.

2. The necessity is immediate. Suppose then that $G \cong G^*$. Then $|E(G)| = |E(G^*)|$. Because σ_M determines a 1-1 correspondence on the 2-subsets of $V(G)$, it follows from (1) that $\sigma_M(E(G)) = E(G^*)$ and that σ is an isomorphism from G to G^*.

3. Because M is also a perfect matching in G^*, $\sigma_M(E(G^*)) \subseteq E(G^{**})$ by property 1. Thus $E(G) = \sigma_M^2(E(G)) \subseteq \sigma_M(E(G^*)) \subseteq E(G^{**})$, so G is a subgraph of G^{**}. Moreover, if $G = G^{**}$, then equality holds throughout and $\sigma_M(E(G)) = E(G^*)$. Since σ_M is injective on $E(G)$, it follows that σ is an isomorphism from G to G^*. Conversely, if G is isomorphic to G^*, then G^* must be isomorphic to G^{**}, so G must be isomorphic to (and therefore equal to) G^{**}.

4. Of all the connected spanning subgraphs of G that contain M, let T be one with the fewest edges. Then T can contain no cycles because a cycle in T would contain an edge not in M and deleting that cycle edge would yield a spanning connected subgraph containing M but with one less edge. Thus T is a tree.

5. Suppose that G^* is connected. If G is not connected, each of its components must have an even number of vertices since G has a perfect matching. But then vertices i, j in distinct components G_k, G_l of G could never be adjacent in G^*, since $G_k - i$ and $G_l - j$ do not have perfect matchings. But then G is not connected, a contradiction. Thus, if G^* is connected, then G is connected.

Suppose G is connected. By property 2, G has a spanning tree T with the same perfect matching M. By Lemma 3.1(2), $G^* \supseteq T^*$ so it is sufficient to show that T^* is connected. We argue by induction on the order n of T, n even. If $n = 2$, $T^* = T$. Suppose now that $n \geq 4$ and that the result holds for trees of even order less than n that have perfect matchings. Let uv be an edge in T that is not in M, let $T - uv$ be the subgraph obtained by deleting the edge uv (but not the vertices u, v) and let T_u and T_v be the two components of $T - uv$ containing u and v respectively. Then T_u and T_v have perfect matchings contained in M and, because $uv \notin M$, there are vertices u' in T_u and v' in T_v with edges uu' and vv' in M. The path P_4 with $u' - u - v - v'$ is also a tree with a perfect matching in M. By Lemma 3.1(2) and the strong inductive assumption, the duals of these three nice trees are connected subgraphs of T^*. Thus T^* is connected.

Corollary 3.5. *If G is a connected graph with a perfect matching, then G^{*k} is $*$-self-dual for some k.*

Proof. By Lemma 3.4(1), $|E(G^{*k})| = |E(G^{*(k+1)})|$ for some (smallest k), and so G^{*k} is $*$-self-dual by Lemma 3.4(3).

4. $*$-self-dual graphs

If G has isolated vertices, then it is straightforward to check that G is $*$-self-dual only if it is empty or if it is one of $\overline{K}_2\dot{+}K_2$, $P_3\dot{+}K_1$ (the first two graphs in Figure 3). Thus, in searching for $*$-self-dual graphs, it is sufficient to restrict attention to those with no isolated vertices. The following lemma shows that attention may be further restricted to connected graphs with perfect matchings.

Lemma 4.1. *If G has no isolated vertices, then G is $*$-self-dual if and only if each of its connected components is $*$-self-dual and has a perfect matching.*

Proof. Let $G = G_1\dot{+}\cdots\dot{+}G_k$ where the G_i, $i \in [k]$ are the connected components of G. Since G has no isolated vertices, each component G_i of G has two or more vertices.

Suppose that each component G_i is $*$-self-dual and has a perfect matching, M_i say. Then each G_i has even order and is a nice subgraph of G, so $G^* = G_1^*\dot{+}\cdots\dot{+}G_k^*$. Because each G_i is $*$-self-dual, each $G_i^* = \sigma_{Mi}(G_i)$ by Lemma 3.4(2). Thus $G^* = \sigma_{M_1}(G_1)\dot{+}\cdots\dot{+}\sigma_{M_k}(G_k) = \sigma_M(G)$, where $M = \cup_{i=1}^k M_i$ is a perfect matching in G. Therefore $G^* \cong G$.

Suppose now that G has no isolated vertices and $G^* \cong G$. Then G^* has no isolated vertices. If $k = 1$, then $G_1 = G$ is $*$-self-dual and has a perfect matching by Lemma 3.2. So we may assume that $k \geq 2$.

Let $H = G_1\dot{+}\cdots\dot{+}G_{k-1}$. Suppose that G_k has no perfect matching. Then no two vertices in $V(H)$ are adjacent in G^*. But G^* has no isolated vertices, so each vertex u in $V(H)$ is adjacent in G^* to some vertex in $V(G_k)$. Thus, $H - u$ has a perfect matching for each $u \in V(H)$. Therefore H has only one component, $H = G_1$, and $G = G_1\dot{+}G_2$ where G_1 and G_2 have odd order, and $G_1 - u$ has a perfect matching for all vertices u in G_1. Since G_1 has odd order, it has no perfect matching, and the argument above implies that $G_2 - v$ has a perfect matching for all vertices v in G_2. Thus, $G = G_1\dot{+}G_2 \cong (G_1\dot{+}G_2)^*$ is complete bipartite, a contradiction. Thus G_k has a perfect matching. By reordering the components of G, it follows that each component G_i of G has a perfect matching, M_i say. But then, as seen earlier, $G^* = G_1^*\dot{+}\cdots\dot{+}G_k^*$ and $M = \cup_{i=1}^k M_i$ is a perfect matching in G. Then $G^* = \sigma_M(G)$ by Lemma

3.4(2) and so $G_1^* \dot{+} \cdots \dot{+} G_k^* = G^* = \sigma_M(G) = \sigma_{M_1}(G_1) \dot{+} \cdots \dot{+} \sigma_{M_k}(G_k)$, where $\sigma_{M_i}(G_i) \subseteq G_i^*$ for each $i \in [k]$. Therefore $G_i^* = \sigma(G_i)$ for each $i \in [k]$. Thus each component of G is $*$-self-dual.

Example 4.1. *(The $*$-self-dual connected graphs of order $n \leq 6$.)*

The single vertex graph K_1 is the only $*$-self-dual connected graph of odd order.

The only connected graph of order 2 is K_2 and it is $*$-self-dual.

From Figure 3, we see that there are precisely five connected $*$-self-dual graphs on 4 vertices (the last five graphs in the figure).

Figure 4 shows the connected $*$-self-dual graphs G on 6 vertices. To obtain these graphs, we first note that Lemma 3.2 implies that each such graph G must have a perfect matching. By Corollary 3.5, iterating the dual operation will eventually yield a $*$-self-dual graph containing any desired connected graph on 6 vertices that has a perfect matching. The first 3 graphs in the figure contain C_6 (and so contain $C_6^* = K_{3,3}$). The next 7 contain C_5 (and so contain G_6^* where G_6 is the graph obtained by attaching a pendant edge to C_5) but not C_6. Those in row 4 contain P_6 (and so P_6^*) but not C_5 or C_6. This accounts for all cases containing the tree P_6. The last 4 contain T_6, the only other spanning tree of order 6 that has a perfect matching.

The *corona* graph of a graph G of order n is the graph $C(G)$ of order $2n$ obtained by attaching a vertex i' to each vertex i of G. It has been observed (refs) that if $C(G)$ is bipartite then it is an h-graph and $C(G)^+$ exists and is isomorphic to $C(G)$. The following lemma implies that $C(G)$ is $*$-self-dual for *every* graph G and that its skew-support $\mathbb{F}$-matrices are all $*$-support matrices.

Lemma 4.2. *If $C(G)$ is the corona graph of a graph G, then $C(G) \cong C(G)^* = C(G)^S$ for all skew-support matrices S of $C(G)$.*

Proof. Let $V = [n]$ be the vertices of G and for each $i \in V$, let $i' = i + n$ be the degree 1 vertex attached to i in $C(G)$. Note that the set $V' = V + n$ of attached vertices all have degree 1 in $C(G)$ and the set M of pendant edges $ii', i \in V, i' \in V'$ forms the only perfect matching in $C(G)$.

If i, j are in V, then i, j are not adjacent in $C(G)$ because $C(G) - i - j$ has two isolated vertices (i' and j') and so could not have a perfect matching.

If $i \in V$, then i, i' are adjacent in $C(G)$ since $M \backslash \{ii'\}$ is a perfect matching in $G - i - i'$. Also, i' is the only vertex in V' adjacent to i in $C(G)$ because, if

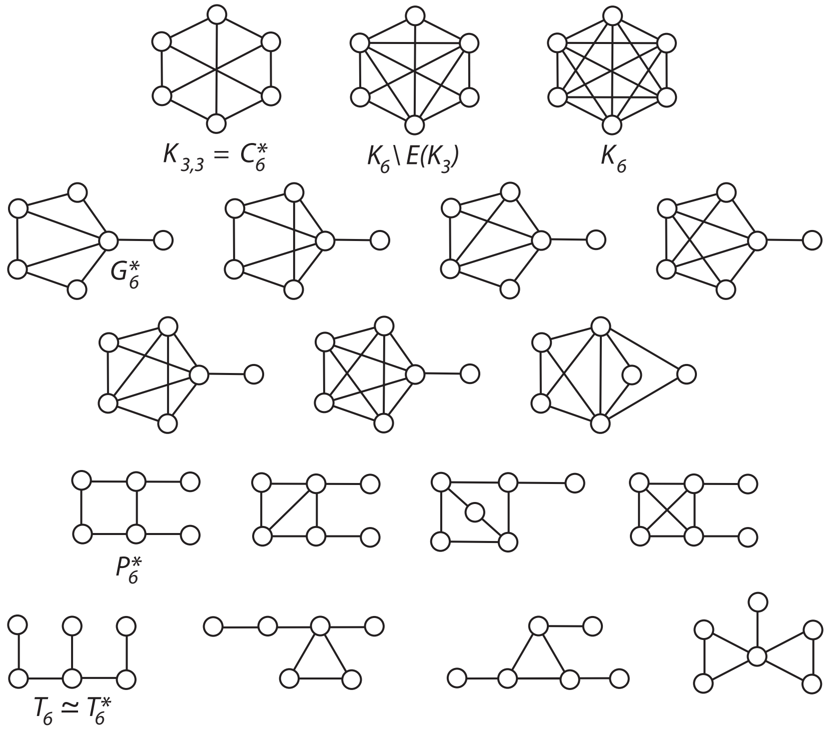

Figure 4: *The connected $*$-self-dual graphs on 6 vertices.*

$j' \neq i'$ then $C(G) - i - j'$ has an isolated vertex, i', and so could not contain a perfect matching.

Finally, if i' and j' are both in V' then i', j' are adjacent in $C(G)$ if and only if i, j are adjacent in G. This follows because a perfect matching in $G - i' - j'$ must cover all the degree 1 vertices in $V' \backslash \{i', j'\}$ and so all the vertices in $V \backslash \{i, j\}$. This leaves the vertices i, j, and they may be covered by an edge if and only if i, j are adjacent in G.

Thus, the correspondence that interchanges each vertex $i \in V$ with $i' \in V'$ establishes an isomorphism between $C(G)$ and $C(G)^*$.

Because $C(G)$ has a unique perfect matching, every skew-support matrix of G is invertible. Also, it was shown above that $C(G) - u - v$ has at most one perfect matching for all vertices u, v of $C(G)$. Thus, by Lemma 2.7, $C(G)^* = C(G)^S$ for all skew-support matrices S of $C(G)$.

Lemma 4.3. *A tree of order $n \geq 2$ is $*$-self-dual if and only if it contains a* *perfect matching and is the corona of a tree.*

Proof. The sufficiency follows from Lemma 4.2.

For the necessity, suppose that T is a tree and that $T \cong T^*$. The result is clearly true when $n = 2$, so we may assume that $n \geq 3$. By Lemma 3.2, T has a perfect matching, M say. Let ij be an edge in M. Then at least one of i, j must have degree 1 in T, otherwise, by extending ij from both ends, it would follow that T would contain a path P_6 on 6-vertices with a perfect matching consisting of edges of M. But then $T^* \supseteq P_6^*$ would contain a 4-cycle and so T^* could not be a tree. Thus $T = C(T_0)$ where T_0 is the subtree of T obtained by deleting the degree 1 vertices of T.

If G is an odd-cycle graph, then $C(G)$ is an odd-cycle graph and Lemma 4.2 implies that $C(G)$ is $*$-self-dual. But, unlike the result in Lemma 4.3 for trees, there are other $*$-self-dual odd-cycle graphs. Let H_n be the odd-cycle graph of order n obtained from the star $K_{1,n-1}$ on n vertices by joining pairs of vertices of degree 1 by edges. When n is odd, H_n will consist of $(n-1)/2$ triangles attached at a common vertex. When n is even, H_n will consist of $(n-2)/$ triangles and a single pendant edge all attached at a common vertex. When n is even, H_n has a perfect matching and one of the matching edges will be attached to the unique vertex of degree 1.

Lemma 4.4. *The graphs H_n of even order n are the only connected $*$-self-dual odd-cycle graphs with perfect matchings and one pendant edge.*

Proof. Using the definition of $*$-dual, it is straightforward to check that $H_n^* \cong H_n$ when n is even.

Suppose now that G is an odd-cycle graph with a perfect matching M and that $G \cong G^*$. If $n = 2$, then $G = K_2$ is $*$-self-dual, so we may assume that $n \geq 4$. Let uv be an edge in M. If each end of uv has degree greater than 1, then u has a neighbor u' other than v and v has a neighbor v' other that u. It is necessary that $u' = v'$, otherwise u' and v' would each be incident to an edge of M and G so G would contain a nice path P_6 on 6 vertices and so $G^* \supseteq P_6^*$ would contain a 4-cycle. Thus, each edge of M is either a pendant edge of G or is on a triangle in G.

Since $n \geq 4$, some edge uv of M is an a triangle $u - v - w$. Then w must be incident to an edge wx of M. The matching edge wx must be a pendant edge, otherwise it would be in a triangle wxy and y would be incident to another edge yz of M and then $u-v-w-x-y-x$ would be a nice 6-path in G, leading to a contradiction as before. There is only one pendant edge so all but one edge of M is on a triangle attached to the remaining (pendant) edge in M.

5. h-graphs, +-duals, and ∗-duals

The first matrix that comes to mind in defining a matrix dual for a simple graph G is the adjacency matrix A of G. As long as A^{-1} exists and has all diagonal entries equal to 0, $[A^{-1} \neq O]$ is an adjacency matrix and we let G^A be the associated graph. Then A^{-1} is a support matrix of G^A and we have $(G^A)^{A^{-1}} = G$. As observed in Lemma 1.2, if G is an h-graph then $G^A = G^{\widetilde{A}}$ where $\widetilde{A}$ is the special skew-adjacency matrix in (1), so we may use a skew-adjacency matrix in place of the adjacency matrix.

Recall from Section 1 that an h-graph G with adjacency matrix A is said to have a +-dual if A^{-1} is sign-similar to an adjacency matrix (necessarily that of G^A). If (as in the last half of the paper [9]) the +-dual G^+ is taken to be the isomorphic copy of G^A obtained by interchanging the vertices on the unique perfect matching, it follows from Lemma 3.4 that G is a subgraph of G^+. Moreover, if G has a +-dual, then so does G^+ and $G^{++} = G$ since $(G^A)^{A^{-1}} = G$.

The ∗-dual and the +-dual agree for many h-graphs for which the +-dual is known to exist.

Lemma 5.1. *If G is an h-graph with adjacency matrix A and G^+ is defined, then $G^+ \cong G^*$ if and only if $\widetilde{A}$ is a ∗-support matrix for G.*

Proof. $G^+ \cong G^A$ by definition and $G^A = G^{\widetilde{A}}$ by Lemma 1.2, so $G^+ \cong G^*$ if and only if $G^{\widetilde{A}} = G^*$, that is, if and only if $\widetilde{A}$ is a ∗-support matrix of G.

Corollary 5.2. *Let G be an h-graph such that $G-i-j$ has at most one perfect matching for all vertices i, j in G. If G^+ is defined, then $G^+ \cong G^*$.*

Proof. For then, *every* invertible skew support matrix for G will be a skew ∗-support matrix by Lemma 2.7.

Example 5.1. 1. It is observed in [9] that if G is bipartite, then $C(G)$ is an h-graph and $C(G)^+$ exists. Also, it was observed in Lemma 4.2 that $C(G)-i-j$ has at most one perfect matching for all vertices i, j in $C(G)$

and that $C(G) \cong C(G)^*$. Thus, by Corollary 5.2, $C(G) \cong C(G)^* \cong C(G)^+$ for all bipartite graphs G. Thus, $C(G) \cong C(G)^+$, as observed in [9].

2. It is also observed in [9] that if T is a tree with a perfect matching, then T^+ is defined. Thus, by Corollary 5.2, $T^+ \cong T^*$ whenever T is a tree with a perfect matching. In particular, $P_6^* \cong P_6^+$.

3. There are h-graphs whose $+$-duals exist and differ from the $*$-dual. For example, if G is the $*$-self-dual graph $P_6^* \cong P_6^+$ in Figure 4, then $G^+ \ncong G^*$ because $G^+ = P_6^{++} = P_6$ while $G^* \cong G$.

Question 1. Suppose that G is an h-graph satisfying the conditions of the corollary and that A is the adjacency matrix of G. Then A^{-1} exists and its entries are all in $\{0, 1, -1\}$. Must G^+ be defined? Equivalently, must A^{-1} be sign-similar to a nonnegative matrix?

References

[1] S. Akbari and S. J. Kirkland, On unimodular graphs, *Linear Algebra and its Applications*, 421 (2007) 3-15.

[2] N. Alon, Combinatorial Nullstellensatz, *Combin. Probab. Comput.* 8 (1999) 7-29.

[3] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge 1991.

[4] M. Cavers, S. M. Cioabă, S. Fallat, D. A. Gregory, W. H. Haemers, S. J. Kirkland, J. J. McDonald and M. Tsatsomeros, Skew-adjacency matrices of graphs. Submitted to Linear Algebra Appl. 30 December 2010.

[5] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.

[6] C. D. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2000.

[7] D. A. Gregory, K. N. Vander Meulen and B. L. Shader, Skew rank decompositions, *Linear Algebra Appl.* 244 (1996) 123-154.

[8] L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland, New York, 1986.

[9] R. M. Tifenbach and S. J. Kirkland, Directed intervals and the dual of a graph, *Linear Algebra and its Applications* 431 (2009) 792-807.

[10] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.,* 22 (1947) 107-111.

[11] D. B. West, *Introduction to Graph Theory*, Second Edition, Prentice-Hall, Upper Saddle River, New Jersey, 2001.