

Binary Theta Series and Modular Forms with CM

1. Introduction

Let $r_n(q) = \#\{(x, y) \in \mathbb{Z}^2 : q(x, y) = n\}$ denote the **number** of representations of $n \in \mathbb{Z}$ by the **positive definite** binary quadratic form

$$q(x, y) = ax^2 + bxy + cy^2, \quad a, b, c \in \mathbb{Z}, a > 0.$$

Fermat, Euler, Lagrange, Gauss: When is $r_n(q) > 0$?

Dirichlet(1839), Weber(1882): If $\gcd(a, b, c) = 1$, i.e. if q is **primitive**, then \exists_∞ primes $p : r_p(q) > 0$. — Study:

$$Z_q(s) = \sum_{n \geq 1} r_n(q) n^{-s}.$$

Following **Jacobi, Hermite, Kronecker, Weber**, consider the closely related **binary theta series**

$$\vartheta_q(z) = \sum_{x, y \in \mathbb{Z}} e^{2\pi i q(x, y)z} = \sum_{n \geq 0} r_n(q) e^{2\pi i n z}.$$

Theorem 0 (a) Weber(1893): Let $D = \Delta(q) := b^2 - 4ac$ denote the discriminant of q and $\psi_D = \left(\frac{D}{\cdot}\right)$. Then

$$\vartheta_q\left(\frac{az + b}{cz + d}\right) = \psi_D(d)(cz + d)\vartheta_q(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(|D|).$$

(b) Hecke(1926), Schoeneberg(1939): ϑ_q is holomorphic at the cusps, so $\vartheta_q \in M_1(|D|, \psi_D)$.

Fix: a discriminant $D < 0$. Thus $D \equiv 0, 1 \pmod{4}$ and

$$D = f_D^2 d_K, \quad \text{where } K = \mathbb{Q}(\sqrt{D}), d_K = \text{disc}(K), \text{ and } f_D \geq 1$$

is some integer. Let

$$\Theta_D := \langle \vartheta_q : q \in Q_D \rangle_{\mathbb{C}} \quad \text{and} \quad \Theta(D) := \langle \vartheta_q : q \in Q(D) \rangle_{\mathbb{C}}$$

be the \mathbb{C} -subspaces **generated by the theta-series**, where

$$Q(D) = \{q = (a, b, c) \in \mathbb{Z}^2 : a > 0, \Delta(q) = D/t^2\},$$

$$Q_D = \{q = (a, b, c) \in Q(D) : \gcd(a, b, c) = 1, \Delta(q) = D\}.$$

Thus

$$\Theta_D \subset \Theta(D) \subset M_1(|D|, \psi_D).$$

Questions: 1) How large are the spaces Θ_D and $\Theta(D)$?

What is the dimension of the subspaces of cusp forms, i.e. of

$$\Theta_D^S = \Theta_D \cap S_1(|D|, \psi_D) \quad \text{and} \quad \Theta(D)^S = \Theta(D) \cap S_1(|D|, \psi_D)?$$

Hecke (1926): $\Theta(D) \neq M_1(|D|, \psi_D)$, for many D 's.

2) How can a binary theta series ϑ_q be expressed in terms of the (extended) **Atkin-Lehner basis** of $M_1(|D|, \psi_D)$?

3) How does the **Hecke algebra** $\mathbb{T}(D)$ act on these spaces? What are the L -functions of the Hecke eigenfunctions?

4) Is there an **intrinsic** characterization of these spaces?

2. Some Observations

1) The group $\mathrm{GL}_2(\mathbb{Z})$ acts on the sets Q_D and $Q(D)$, and

$$\vartheta_{q'} = \vartheta_q, \quad \text{for all } q' \in q \mathrm{GL}_2(\mathbb{Z}).$$

By using the **Dirichlet/Weber** result, one can show that the set $\{\vartheta_q : q \in Q_D / \mathrm{GL}_2(\mathbb{Z})\}$ is a **basis** of Θ_D . In particular,

$$\dim \Theta_D = \bar{h}_D := |Q_D / \mathrm{GL}_2(\mathbb{Z})|.$$

2) By **Gauss's** theory of composition of forms, the set

$$\mathrm{Cl}(D) = Q_D / \mathrm{SL}_2(\mathbb{Z})$$

has the structure of an abelian group. If $h_D := |\mathrm{Cl}(D)|$, then

$$\bar{h}_D = \frac{1}{2}(g_D + h_D), \quad \text{where } g_D = [\mathrm{Cl}(D) : \mathrm{Cl}(D)^2]$$

denotes the **number of genera** of forms of discriminant D .

3) For a **character** $\chi \in \mathrm{Cl}(D)^*$ on $\mathrm{Cl}(D)$, put

$$\vartheta_\chi(z) := \frac{1}{w_D} \sum_{q \in \mathrm{Cl}(D)} \chi(q) \vartheta_q(z) = \sum_{n \geq 0} a_n(\chi) e^{2\pi i n z} \in \Theta_D,$$

where $w_D = 2$ for $D < -4$ and $w_{-3} = 6, m_{-4} = 4$.

It is immediate that $\{\vartheta_\chi\}_{\chi \in \mathrm{Cl}(D)^*}$ generates Θ_D and hence by 1) forms a basis of Θ_D (subject to the identification $\vartheta_{\bar{\chi}} = \vartheta_\chi$).

Note: It turns out (cf. Theorem 1) that the coefficients $a_n(\chi)$ are **multiplicative** in n , and that hence ϑ_χ is a **Hecke eigenfunction** w.r.t. to the Hecke algebra $\mathbb{T}(D)$ generated by the Hecke operators T_p with $(p, D) = 1$.

4) The L -function associated to the form ϑ_χ is

$$L(s, \chi) := L(s, \vartheta_\chi) = \sum_{n \geq 1} a_n(\chi) n^{-s}.$$

This function is frequently found in the literature (e.g., in Lang, *Elliptic Functions*, 1st ed.), and was recently studied in detail by Z.-H. Sun and K. S. Williams (2006) (but without mentioning characters or modular forms).

5) If D is a fundamental discriminant, i.e., if $D = d_K$, then it is well-known that each ϑ_χ is a primitive form (newform) and hence in this case the ϑ_χ 's are part of the canonical Atkin-Lehner basis of $M_1(|D|, \psi_D)$.

However, in the general case this is no longer true for every $\chi \in \text{Cl}(D)^*$ because some of the characters $\chi \in \text{Cl}(D)^*$ are not primitive, i.e., they are lifts

$$\chi = \chi' \circ \pi \quad \text{of characters} \quad \chi' \in \text{Cl}(D')^*$$

of some “lower level” $D'|D$ (where $\frac{D}{D'} = t^2 > 1$) via the canonical surjection

$$\pi = \pi_{D, D'} : \text{Cl}(D) \rightarrow \text{Cl}(D').$$

3. Main Results

Theorem 1: The space Θ_D is a $\mathbb{T}(D)$ -submodule of $M_1(|D|, \psi_D)$ of **multiplicity one**, and has a canonical basis $\{\vartheta_\chi\}$ consisting of normalized $\mathbb{T}(D)$ -eigenforms. Furthermore, ϑ_χ is a cusp form if and only if χ is not a quadratic character.

Theorem 2: We have $\Theta_D = \Theta_D^E \oplus \Theta_D^S$, where

$\Theta_D^E = \Theta_D \cap E_1(|D|, \psi_D)$ denotes the **Eisenstein space part** and $\Theta_D^S = \Theta_D \cap S_1(|D|, \psi)$ denotes the **cusp space part** of Θ_D , and

$$(1) \quad \dim \Theta_D^E = g_D \quad \text{and} \quad \dim \Theta_D^S = \frac{1}{2}(h_D - g_D).$$

Remark: Thus $\Theta_D^S = 0 \Leftrightarrow h_D = g_D \stackrel{\text{def}}{\Leftrightarrow} D$ is an **idoneal discriminant**. (This implies a result of **Kitaoka (1971)**.)

Theorem 3: Let $\chi \in \text{Cl}(D)^*$, where $D = f_D^2 d_K$.

(a) $\exists!$ divisor $f_\chi | f_D$ and a unique **primitive character** $\chi_{pr} \in \text{Cl}(D_\chi)$, where $D_\chi = f_\chi^2 d_K$, such that $\chi = \chi_{pr} \circ \bar{\pi}_{D, D_\chi}$.

(b) The form $\vartheta_{\chi_{pr}} \in \Theta_{D_\chi}$ is a **primitive form (newform)** of level $|D_\chi|$. Moreover, there exist constants $c_n(\chi) \in \mathbb{R}$ such that

$$(2) \quad \vartheta_\chi(z) = \sum_{n|\bar{f}_\chi^2} c_n(\chi) \vartheta_{\chi_{pr}}(nz),$$

where $\bar{f}_\chi = f_D/f_\chi$. Furthermore, the function $n \mapsto c_n(\chi)$ is multiplicative and has the generating function

$$(3) \quad C(s, \chi) := \sum_{n|\bar{f}_\chi^2} c_n(\chi) n^{-s} = L(s, \vartheta_\chi) / L(s, \vartheta_{\chi_{pr}}).$$

Remark: While $L(s, \vartheta_{\chi_{pr}})$ is a classical Hecke L -function associated to a **Hecke character** and hence is well-understood, the L -function $L(s, \vartheta_\chi)$ is more complicated and is, in fact, **unknown in general**.

Thus, (3) does not help in determining the constants $c_n(\chi)$. However, $C(s, \chi)$ can be computed directly by using facts about ideals in **quadratic orders**.

As a consequence, we thus obtain an explicit expression for the L -function $L(s, \chi) = L(s, \vartheta_\chi)$:

Corollary: If $\chi \in \text{Cl}(D)^*$, then $L(s, \chi)$ has the Euler product

$$(4) \quad L(s, \chi) = \prod_p L_p(s, \chi)$$

where for $p \nmid \bar{f}_\chi$ the p -Euler factor $L_p(s, \chi)$ is given by

$$\begin{aligned} L_p(s, \chi) &= (1 - a_p(\chi)p^{-s} + \psi_D(p)p^{-2s})^{-1} \\ &= (1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s})^{-1}, \end{aligned}$$

whereas for $p \mid \bar{f}_\chi$ (and $p^{\bar{e}_p} \parallel \bar{f}_\chi$), it is given by

$$L_p(s, \chi) = \frac{1 - p^{(1-2s)\bar{e}_p}}{1 - p^{1-2s}} + \frac{\left(1 - \frac{1}{p}\psi_{D_\chi}(p)\right) p^{(1-2s)\bar{e}_p}}{1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}}.$$

Remark: This generalizes the work of **Sun and Williams (2006)** (for $D < 0$), who obtained a formula for the p -Euler factors of $L(s, \chi)$ in the case that the class group $\text{Cl}(D)$ is **cyclic**.

4. An example: $D = -144 = -4 \cdot 6^2$.

Put $q_0 = (1, 0, 36)$, $q_1 = (4, 0, 9)$, $q_2 = (5, 4, 8)$, $q_3 = (5, -4, 8) \in Q_D$.

Then $\text{Cl}(D) = \{cl(q_0), cl(q_1), cl(q_2), cl(q_3)\} = \langle cl(q_2) \rangle \simeq \mathbb{Z}/4\mathbb{Z}$,
 $\text{Cl}(D)^* = \langle \chi \rangle = \{1, \chi, \chi^2, \chi^3 = \bar{\chi}\}$, $\chi(q_2) = i$.

$\Rightarrow h_D = 4, g_D = 2$, so **Obs. 1,2** $\Rightarrow \dim \Theta_D = \bar{h}_D = \frac{1}{2}(4+2) = 3$.

Thus, Obs. 1, 3 $\Rightarrow \Theta_D = \langle \vartheta_{q_0}, \vartheta_{q_1}, \vartheta_{q_2} \rangle_{\mathbb{C}} = \langle \vartheta_1, \vartheta_{\chi}, \vartheta_{\chi^2} \rangle_{\mathbb{C}}$, with

$$\vartheta_1 = \frac{1}{2}(\vartheta_{q_0} + \vartheta_{q_1} + 2\vartheta_{q_2}), \quad \vartheta_{\chi} = \frac{1}{2}(\vartheta_{q_0} - \vartheta_{q_1}), \quad \vartheta_{\chi^2} = \frac{1}{2}(\vartheta_{q_0} + \vartheta_{q_1} - 2\vartheta_{q_2}).$$

Moreover, Theorems 1, 2 $\Rightarrow \Theta_D = \Theta_D^E \oplus \Theta_D^S$, where

$$\Theta_D^E = \langle \vartheta_1, \vartheta_{\chi^2} \rangle_{\mathbb{C}} \quad \text{and} \quad \Theta_D^S = \langle \vartheta_{\chi} \rangle_{\mathbb{C}}.$$

Since $h_{D/2^2} = h_{-36} = 2$ and $h_{D/3^2} = h_{-16} = 1$, we see that

χ is **primitive** (i.e., $f_{\chi} = 6$) and χ^2 has conductor $f_{\chi^2} = 3$.

Thus, Theorems 1, 3 $\Rightarrow \vartheta_{\chi}$ is a **newform** of level 144; in fact,

$\vartheta_{\chi}(z) = \eta(12z)^2$, where $\eta(z)$ is the **Dedekind eta-function**.

Recall: 1) (Genus theory) The **quadratic** characters in $\text{Cl}(D)^*$ are described (explicitly) by pairs of Dirichlet characters.

2) (Hecke) If χ_1, χ_2 are **Dirichlet characters** whose product $\chi := \chi_1\chi_2$ is **odd**, and if $N = \text{cond}(\chi_1)\text{cond}(\chi_2)$, then

$\exists! f_1(\cdot, \chi_1, \chi_2) \in M_1(N, \chi)$ with $L(s, f_1) = L(s, \chi_1)L(s, \chi_2)$.

Thus, the **primitive forms** associated to $1 = \chi^4$ and χ^2 are

$$\vartheta_K := f_1(\cdot; 1, \psi_{-4}) \quad \text{and} \quad \vartheta_{36} := f_1(\cdot; \psi_{-3}, \psi_{12}),$$

and hence by **Theorem 3** we have that

$$\vartheta_1(z) = \sum_{n|36} c_n \vartheta_K(nz) \quad \text{and} \quad \vartheta_{\chi^2}(z) = \sum_{n|4} c'_n \vartheta_{36}(nz).$$

By using **Theorem 3** and its **Corollary**, the coefficients c_n and c'_n can be determined explicitly:

$$C(s, 1) := \sum_{n|36} c_n n^{-s} = (1 - 2^{-s} + 2^{1-2s})(1 + 3^{1-2s}),$$

$$C(s, \chi^2) := \sum_{n|4} c'_n n^{-s} = (1 - 2^{-s} + 2^{1-2s});$$

here we used the fact that for all primes p ,

$$a_p((1)_{pr}) = 1 + \psi_{-4}(p) \quad \text{and} \quad a_p((\chi^2)_{pr}) = \psi_{-3}(p) + \psi_{12}(p),$$

which can be calculated explicitly. (We need only $p = 2, 3$.)

Comparing coefficients, we thus see that

$$c_1 = 1, \quad c_2 = -1, \quad c_4 = 2, \quad c_9 = 3, \quad c_{18} = -3, \quad c_{36} = 6,$$

and that $c_n = 0$ otherwise, and so

$$\vartheta_1(z) = \vartheta_K(z) - \vartheta_K(2z) + 2\vartheta_K(4z) + 3\vartheta_K(9z) - 3\vartheta_K(18z) + 6\vartheta_K(36z).$$

Similarly, $c'_1 = 1, c'_2 = -1, c'_4 = 2$ and so

$$\vartheta_{\chi^2} = \vartheta_{36}(z) - \vartheta_{36}(2z) + 2\vartheta_{36}(4z).$$

Note that the L -functions of ϑ_1 and ϑ_{χ^2} are

$$\begin{aligned} L(s, \vartheta_1) &= C(s, 1)L(s, (1)_{pr}) \\ &= (1 - 2^{-s} + 2^{1-2s})(1 + 3^{1-2s})\zeta_K(s), \\ L(s, \vartheta_{\chi^2}) &= C(s, \chi^2)L(s, (\chi^2)_{pr}) \\ &= (1 - 2^{-s} + 2^{1-2s})L(s, \psi_{-3})L(s, \psi_{12}), \end{aligned}$$

where $\zeta_K(s) = L(s, \vartheta_K)$ is the **Dedekind zeta-function** of the field $K := \mathbb{Q}(\sqrt{D}) = \mathbb{Q}(i)$.

5. Main Results II: CM-forms

Definition: Let $f \in M_k(N, \psi)$ be a $\mathbb{T}(N)$ -eigenfunction with eigencharacter $\lambda_f : \mathbb{T}(N) \rightarrow \mathbb{C}$. We say that f has **CM (complex multiplication)** by a Dirichlet character θ if

$$\lambda_f(T_p)\theta(p) = \lambda_f(T_p), \quad \text{for all } p \nmid N\text{cond}(\theta),$$

or, equivalently, if

$$\lambda_f(T_p) = 0, \quad \text{for all } p \nmid N\text{cond}(\theta) \text{ with } \theta(p) \neq 1.$$

Notation: We let $M_k^{CM}(N, \psi; \theta)$ denote the space generated by all $\mathbb{T}(N)$ -eigenfunctions $f \in M_k(N, \psi)$ which have CM by θ .

Theorem 4: For every discriminant $D < 0$ we have that

$$(5) \quad \Theta(D) = M_1^{CM}(|D|, \psi_D) := M_1^{CM}(|D|, \psi_D; \psi_D).$$

Corollary:

$$(6) \quad \dim \Theta(D) = \dim M_1^{CM}(|D|, \psi_D) = \sum_{f|f_D} 2^{\omega(f)} \bar{h}_{D/f^2},$$

where $\omega(f)$ denotes the number of distinct prime divisors of f . Moreover, the dimensions of the **Eisenstein part** and of the **cuspidal part** of $M_1^{CM}(|D|, \psi_D)$ are given by

$$\begin{aligned} \dim E_1^{CM}(|D|, \psi_D) &= \sum_{f|f_D} 2^{\omega(f)} g_{D/f^2}, \\ \dim S_1^{CM}(|D|, \psi_D) &= \sum_{f|f_D} 2^{\omega(f)} (f_{D/f^2} - g_{D/f^2}). \end{aligned}$$

Remark: There is **no** (known) formula for $\dim M_1(|D|, \psi_D)$.

6. Example: $D = -144$ (again)

Put $\mathcal{D} = \{-4, -16, -36, -144\}$. Then by definition

$$\Theta(D) := \langle \vartheta_q : \Delta(q) \in \mathcal{D} \rangle.$$

By **Theorem 4**:

$$\Theta(D) = M_1^{CM}(144, \psi_{-144}) = E_1^{CM}(144, \psi_{-144}) \oplus S_1^{CM}(144, \psi_{-144}).$$

Since $h_D = g_D$, for $D \in \mathcal{D}$, $D \neq -144$, the **Corollary** \Rightarrow

$$\dim S_1^{CM}(144, \psi_{-144}) = \frac{1}{2}(2^0(h_{-144} - g_{-144})) = \frac{1}{2}(4 - 2) = 1,$$

and so it follows that

$$S_1^{CM}(144, \psi_{-144}) = \mathbb{C}\vartheta_\chi(z) = \mathbb{C}\eta(12z)^2.$$

Moreover, from the **Corollary** we also have that

$$\begin{aligned} \dim E_1^{CM}(144, \psi_{-144}) &= 2^{\omega(6)}g_{-4} + 2^{\omega(3)}g_{-16} + 2^{\omega(2)}g_{-36} + g_{-144} \\ &= 4 \cdot 1 + 2 \cdot 1 + 2 \cdot 2 + 2 = 12. \end{aligned}$$

Now $E_1^{CM}(144, \psi_{-144})$ contains two $\mathbb{T}(144)$ -eigenspaces given by the eigenfunctions (**primitive forms**) ϑ_K and ϑ_{36} :

$$\begin{aligned} M_1(144, \psi_{-144})[\lambda_{\vartheta_K}] &= \langle \vartheta_K(nz) : n|36 \rangle, \\ M_1(144, \psi_{-144})[\lambda_{\vartheta_{36}}] &= \langle \vartheta_{36}(nz) : n|4 \rangle, \end{aligned}$$

which have dimension $d(36) = 9$ and $d(4) = 3$, respectively.

Thus

$$E_1^{CM}(144, \psi_{-144}) = M_1(144, \psi_{-144})[\lambda_{\vartheta_K}] \oplus M_1(144, \psi_{-144})[\lambda_{\vartheta_{36}}].$$

7. Ingredients

1) Dedekind's Isomorphism:

$$\lambda_D : \text{Cl}(D) \xrightarrow{\sim} \text{Pic}(\mathfrak{O}_D),$$

where $\mathfrak{O}_D = \mathbb{Z} + \mathbb{Z}\frac{D+\sqrt{D}}{2} \subset \mathfrak{O}_K$ is the **order** of discriminant D (and/or of conductor f_D in K).

2) A **classification** of the invertible ideals of \mathfrak{O}_D :

\Rightarrow the multiplicativity of $a_n(\chi)$,
the value of $c_n(\chi)$ for $n|D$, etc.

3) A study of the **conductor** of $\chi \in \text{Cl}(D)^*$: via the isomorphism

$$I_K(f_D\mathfrak{O}_K)/P_{K,\mathbb{Z}}(f_D) \xrightarrow{\sim} \text{Pic}(\mathfrak{O}_D),$$

one can identify each $\chi \in \text{Cl}(D)^*$ with a **Hecke character** $\tilde{\chi}$ on the group $I_K(f_D\mathfrak{O}_K)$ of fractional ideals prime to the ideal $f_D\mathfrak{O}_K$. A **key fact** is:

$$\chi \text{ is primitive on } \text{Cl}(D) \Leftrightarrow \tilde{\chi} \text{ is primitive mod } f_D\mathfrak{O}_K.$$

4) Genus theory (**Gauss/Kronecker/Weber**): identifies **quadratic** characters $\chi \in \text{Cl}(D)^*$ with **certain** Dirichlet characters.

5) Extended Atkin-Lehner (newform) theory: this describes:

1) the characters $\lambda \in \mathbb{T}(N)^* = \text{Hom}(\mathbb{T}(N), \mathbb{C})$ of the Hecke algebra $\mathbb{T}(N) \subset \text{End}(M_k(N, \psi))$ in terms of **primitive** eigenfunctions (newforms);

2) the structure of the $\mathbb{T}(N)$ -**eigenspace** associated to λ :

$$M_k(N, \psi)[\lambda] = \{f \in M_k(N, \psi) : f|_k T_n = \lambda(T_n)f, \forall (n, N) = 1\}$$

For **Theorem 4**, we also need:

6) (a) The **Deligne/Serre theory** of **Galois representations**

$$\rho_f : G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{C})$$

attached to $\mathbb{T}(N)$ -eigenfunctions $f \in M_1(N, \psi)$.

(b) A characterization of characters of ring class fields via **(strongly) dihedral** Galois representations of $G_{\mathbb{Q}}$ (= reinterpretation of a result of **Bruckner (1966)**).

(c) A characterization of **CM forms** via their associated Galois representations (\rightarrow **Theorem 5** below).

8. Galois representations

Deligne/Serre (1974): If $f \in M_1(N, \psi)$ is a normalized $\mathbb{T}(N)$ -eigenfunction, then $\exists!$ Galois representation

$$\rho_f : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{C})$$

such that for all primes $p \nmid N$

$$\begin{aligned} \mathrm{tr}(\rho_f(Fr_p)) &= \lambda_f(T_p) = a_p(f), \\ \mathrm{det}(\rho_f(Fr_p)) &= \psi(p). \end{aligned}$$

Furthermore, ρ_f is **irreducible** $\Leftrightarrow f$ is a cusp form.

Definition: An Galois representation $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{C})$ is called **strongly dihedral** if $\mathrm{Im}(\rho) \simeq D_n$ is a dihedral group ($n \geq 3$).

Moreover, ρ is said to be of **dihedral type** if $\mathrm{Im}(\rho)/Z(\mathrm{Im}(\rho)) \simeq D_n$ is a dihedral group ($n \geq 2$).

Theorem 5: Let $f \in S_1(N, \psi)$ be a newform.

- (a) f has CM by some character $\theta \Leftrightarrow \rho_f$ is of dihedral type.
- (b) f has CM by $\psi \Leftrightarrow \rho_f$ is strongly dihedral.

Theorem 6: Let $\rho : G \rightarrow \mathrm{GL}_2(\mathbb{C})$ be Galois representation.

- (a) **(Hecke)** If ρ is of dihedral type and is odd, then $\rho = \rho_f$ for some $f \in S_1(N, \psi)$.
- (b) **(Bruckner, 1966)** ρ is strongly dihedral if and only if the field $\mathrm{Fix}(\mathrm{Ker}(\rho))$ is contained in some **ring class field**.

Remark: Theorems 3, 5, 6 \Rightarrow Theorem 4.