Binary Theta Series and
Modular Forms with CM

1. Introduction

Let \( r_n(q) = \# \{(x, y) \in \mathbb{Z}^2 : q(x, y) = n\} \) denote the number of representations of \( n \in \mathbb{Z} \) by the positive definite binary quadratic form
\[
q(x, y) = ax^2 + bxy + cy^2, \quad a, b, c \in \mathbb{Z}, a > 0.
\]

**Fermat, Euler, Lagrange, Gauss:** When is \( r_n(q) > 0 \)?

**Dirichlet(1839), Weber(1882):** If \( \gcd(a, b, c) = 1 \), i.e. if \( q \) is primitive, then \( \exists \infty \) primes \( p : r_p(q) > 0 \). — Study:

\[
Z_q(s) = \sum_{n \geq 1} r_n(q)n^{-s}.
\]

Following Jacobi, Hermite, Kronecker, Weber, consider the closely related binary theta series

\[
\vartheta_q(z) = \sum_{x,y \in \mathbb{Z}} e^{2\pi i q(x,y)z} = \sum_{n \geq 0} r_n(q)e^{2\pi inz}.
\]

**Theorem 0 (a) Weber(1893):** Let \( D = \Delta(q) := b^2 - 4ac \) denote the discriminant of \( q \) and \( \psi_D = \left( \frac{D}{.} \right) \). Then

\[
\vartheta_q \left( \frac{az + b}{cz + d} \right) = \psi_D(d)(cz + d)\vartheta_q(z), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(|D|).
\]

**(b) Hecke(1926), Schoeneberg(1939):** \( \vartheta_q \) is holomorphic at the cusps, so \( \vartheta_q \in M_1(|D|, \psi_D) \).
Fix: a discriminant $D < 0$. Thus $D \equiv 0, 1 \pmod{4}$ and

$$D = f_D^2 d_K,$$

where $K = \mathbb{Q}(\sqrt{D})$, $d_K = \text{disc}(K)$, and $f_D \geq 1$ is some integer. Let

$$\Theta_D := \langle \vartheta_q : q \in Q_D \rangle_{\mathbb{C}} \quad \text{and} \quad \Theta(D) := \langle \vartheta_q : q \in Q(D) \rangle_{\mathbb{C}}$$

be the $\mathbb{C}$-subspaces generated by the theta-series, where

$$Q(D) = \{ q = (a, b, c) \in \mathbb{Z}^2 : a > 0, \Delta(q) = D/t^2 \},$$

$$Q_D = \{ q = (a, b, c) \in Q(D) : \gcd(a, b, c) = 1, \Delta(q) = D \}.$$

Thus

$$\Theta_D \subset \Theta(D) \subset M_1(|D|, \psi_D).$$

Questions: 1) How large are the spaces $\Theta_D$ and $\Theta(D)$? What is the dimension of the subspaces of cusp forms, i.e. of

$$\Theta_D^S = \Theta_D \cap S_1(|D|, \psi_D) \quad \text{and} \quad \Theta(D)^S = \Theta(D) \cap S_1(|D|, \psi_D)?$$

Hecke (1926): $\Theta(D) \neq M_1(|D|, \psi_D)$, for many $D$’s.

2) How can a binary theta series $\vartheta_q$ be expressed in terms of the (extended) Atkin-Lehner basis of $M_1(|D|, \psi_D)$?

3) How does the Hecke algebra $\mathbb{T}(D)$ act on these spaces? What are the $L$-functions of the Hecke eigenfunctions?

4) Is there an intrinsic characterization of these spaces?
2. Some Observations

1) The group $\text{GL}_2(\mathbb{Z})$ acts on the sets $Q_D$ and $Q(D)$, and
\[ \vartheta_{q'} = \vartheta_q, \quad \text{for all } q' \in q \text{ GL}_2(\mathbb{Z}). \]

By using the Dirichlet/Weber result, one can show that the set $\{ \vartheta_q : q \in Q_D/\text{GL}_2(\mathbb{Z}) \}$ is a basis of $\Theta_D$. In particular,
\[ \dim \Theta_D = \overline{h}_D := |Q_D/\text{GL}_2(\mathbb{Z})|. \]

2) By Gauss’s theory of composition of forms, the set
\[ \text{Cl}(D) = Q_D/\text{SL}_2(\mathbb{Z}) \]
has the structure of an abelian group. If $h_D := |\text{Cl}(D)|$, then
\[ \overline{h}_D = \frac{1}{2}(g_D + h_D), \quad \text{where } g_D = [\text{Cl}(D) : \text{Cl}(D)^2] \]
denotes the number of genera of forms of discriminant $D$.

3) For a character $\chi \in \text{Cl}(D)^*$ on $\text{Cl}(D)$, put
\[ \vartheta_\chi(z) := \frac{1}{w_D} \sum_{q \in \text{Cl}(D)} \chi(q) \vartheta_q(z) = \sum_{n \geq 0} a_n(\chi)e^{2\pi inz} \in \Theta_D, \]
where $w_D = 2$ for $D < -4$ and $w_{-3} = 6$, $m_{-4} = 4$.

It is immediate that $\{ \vartheta_\chi \}_{\chi \in \text{Cl}(D)^*}$ generates $\Theta_D$ and hence by 1) forms a basis of $\Theta_D$ (subject to the identification $\vartheta_\chi = \vartheta_\chi$).

Note: It turns out (cf. Theorem 1) that the coefficients $a_n(\chi)$ are multiplicative in $n$, and that hence $\vartheta_\chi$ is a Hecke eigenfunction w.r.t. to the Hecke algebra $\mathcal{T}(D)$ generated by the Hecke operators $T_p$ with $(p, D) = 1$. 
4) The $L$-function associated to the form $\vartheta_\chi$ is

$$L(s, \chi) := L(s, \vartheta_\chi) = \sum_{n \geq 1} a_n(\chi) n^{-s}.$$ 

This function is frequently found in the literature (e.g., in Lang, *Elliptic Functions*, 1st ed.), and was recently studied in detail by Z.-H. Sun and K. S. Williams (2006) (but without mentioning characters or modular forms).

5) If $D$ is a fundamental discriminant, i.e., if $D = d_K$, then it is well-known that each $\vartheta_\chi$ is a primitive form (newform) and hence in this case the $\vartheta_\chi$'s are part of the canonical Atkin-Lehner basis of $M_1(|D|, \psi_D)$.

However, in the general case this is no longer true for every $\chi \in \text{Cl}(D)^*$ because some of the characters $\chi \in \text{Cl}(D)^*$ are not primitive, i.e., they are lifts

$$\chi = \chi' \circ \pi$$

of characters $\chi' \in \text{Cl}(D')^*$ of some “lower level” $D'|D$ (where $\frac{D}{D'} = t^2 > 1$) via the canonical surjection

$$\pi = \pi_{D,D'} : \text{Cl}(D) \to \text{Cl}(D').$$
3. Main Results

**Theorem 1:** The space $\Theta_D$ is a $\mathbb{T}(D)$-submodule of $M_1(|D|, \psi_D)$ of multiplicity one, and has a canonical basis $\{\vartheta_\chi\}$ consisting of normalized $\mathbb{T}(D)$-eigenforms. Furthermore, $\vartheta_\chi$ is a cusp form if and only if $\chi$ is not a quadratic character.

**Theorem 2:** We have $\Theta_D = \Theta^E_D \oplus \Theta^S_D$, where $\Theta^E_D = \Theta_D \cap E_1(|D|, \psi_D)$ denotes the Eisenstein space part and $\Theta^S_D = \Theta_D \cap S_1(|D|, \psi)$ denotes the cusp space part of $\Theta_D$, and 

$$\dim \Theta^E_D = g_D \quad \text{and} \quad \dim \Theta^S_D = \frac{1}{2}(h_D - g_D).$$

**Remark:** Thus $\Theta^S_D = 0 \iff h_D = g_D$ \textit{def} $D$ is an idoneal discriminant. (This implies a result of Kitaoka (1971).)

**Theorem 3:** Let $\chi \in \text{Cl}(D)^*$, where $D = f_D^2 d_K$.

(a) $\exists!$ divisor $f_\chi | f_D$ and a unique primitive character $\chi_{pr} \in \text{Cl}(D_\chi)$, where $D_\chi = f_\chi^2 d_K$, such that $\chi = \chi_{pr} \circ \bar{\pi}_{D,D_\chi}$.

(b) The form $\vartheta_{\chi_{pr}} \in \Theta_{D_\chi}$ is a primitive form (newform) of level $|D_\chi|$. Moreover, there exist constants $c_n(\chi) \in \mathbb{R}$ such that

$$\vartheta_\chi(z) = \sum_{n \mid \bar{f}_\chi^2} c_n(\chi) \vartheta_{\chi_{pr}}(nz),$$

where $\bar{f}_\chi = f_D/f_\chi$. Furthermore, the function $n \mapsto c_n(\chi)$ is multiplicative and has the generating function

$$C(s, \chi) := \sum_{n \mid \bar{f}_\chi^2} c_n(\chi)n^{-s} = L(s, \vartheta_\chi)/L(s, \vartheta_{\chi_{pr}}).$$
Remark: While \( L(s, \vartheta_{\chi_{pr}}) \) is a classical Hecke \( L \)-function associated to a Hecke character and hence is well-understood, the \( L \)-function \( L(s, \vartheta_{\chi}) \) is more complicated and is, in fact, unknown in general.

Thus, (3) does not help in determining the constants \( c_n(\chi) \). However, \( C(s, \chi) \) can be computed directly by using facts about ideals in quadratic orders.

As a consequence, we thus obtain an explicit expression for the \( L \)-function \( L(s, \chi) = L(s, \vartheta_{\chi}) \):

**Corollary:** If \( \chi \in \text{Cl}(D)^* \), then \( L(s, \chi) \) has the Euler product

\[
L(s, \chi) = \prod_p L_p(s, \chi)
\]

where for \( p \nmid \bar{f}_\chi \) the \( p \)-Euler factor \( L_p(s, \chi) \) is given by

\[
L_p(s, \chi) = \left( 1 - a_p(\chi)p^{-s} + \psi_D(p)p^{-2s} \right)^{-1}
\]

\[
= \left( 1 - a_p(\chi_{pr})p^{-s} + \psi_D(\chi_{pr})(p)p^{-2s} \right)^{-1},
\]

whereas for \( p \mid \bar{f}_\chi \) (and \( p^{\bar{e}_p} \mid \mid \bar{f}_\chi \)), it is given by

\[
L_p(s, \chi) = \frac{1 - p^{(1-2s)\bar{e}_p}}{1 - p^{1-2s}} + \left( 1 - \frac{1}{p}\psi_D(\chi)(p) \right) p^{(1-2s)\bar{e}_p}
\]

\[
\frac{1 - a_p(\chi_{pr})p^{-s} + \psi_D(\chi_{pr})(p)p^{-2s}}{1 - a_p(\chi_{pr})p^{-s} + \psi_D(\chi_{pr})(p)p^{-2s}}.
\]

Remark: This generalizes the work of Sun and Williams (2006) (for \( D < 0 \)), who obtained a formula for the \( p \)-Euler factors of \( L(s, \chi) \) in the case that the class group \( \text{Cl}(D) \) is cyclic.
4. An example: $D = -144 = -4 \cdot 6^2$.

Put $q_0 = (1, 0, 36), q_1 = (4, 0, 9), q_2 = (5, 4, 8), q_3 = (5, -4, 8) \in \mathbb{Q}_D$.

Then $\text{Cl}(D) = \{\text{cl}(q_0), \text{cl}(q_1), \text{cl}(q_2), \text{cl}(q_3)\} = \langle \text{cl}(q_2) \rangle \simeq \mathbb{Z}/4\mathbb{Z},$

$\text{Cl}(D)^* = \langle \chi \rangle = \{1, \chi, \chi^2, \chi^3 = \overline{\chi}\}$, $\chi(q_2) = i.$

$\Rightarrow h_D = 4, g_D = 2,$ so Obs. 1, 2 $\Rightarrow \dim \Theta_D = \frac{1}{2}(4 + 2) = 3.$

Thus, Obs. 1, 3 $\Rightarrow \Theta_D = \langle \vartheta_{q_0}, \vartheta_{q_1}, \vartheta_{q_2} \rangle_{\mathbb{C}} = \langle \vartheta_1, \vartheta_\chi, \vartheta_\chi^2 \rangle_{\mathbb{C}}$, with

$\vartheta_1 = \frac{1}{2}(\vartheta_{q_0} + \vartheta_{q_1} + 2\vartheta_{q_2}),$ $\vartheta_\chi = \frac{1}{2}(\vartheta_{q_0} - \vartheta_{q_1}),$ $\vartheta_\chi^2 = \frac{1}{2}(\vartheta_{q_0} + \vartheta_{q_1} - 2\vartheta_{q_2}).$

Moreover, Theorems 1, 2 $\Rightarrow \Theta_D = \Theta^E_D \oplus \Theta^S_D$, where

$\Theta^E_D = \langle \vartheta_1, \vartheta_\chi^2 \rangle_{\mathbb{C}}$ and $\Theta^S_D = \langle \vartheta_\chi \rangle_{\mathbb{C}}.$

Since $h_{D/2^2} = h_{-36} = 2$ and $h_{D/3^2} = h_{-16} = 1,$ we see that

$\chi$ is primitive (i.e., $f_\chi = 6$) and $\chi^2$ has conductor $f_{\chi^2} = 3.$

Thus, Theorems 1, 3 $\Rightarrow \vartheta_\chi$ is a newform of level 144; in fact, $\vartheta_\chi(z) = \eta(12z)^2,$ where $\eta(z)$ is the Dedekind eta-function.

Recall: 1) (Genus theory) The quadratic characters in $\text{Cl}(D)^*$ are described (explicitly) by pairs of Dirichlet characters.

2) (Hecke) If $\chi_1, \chi_2$ are Dirichlet characters whose product $\chi := \chi_1\chi_2$ is odd, and if $N = \text{cond}(\chi_1)\text{cond}(\chi_2),$ then

$\exists! f_1(\cdot; \chi_1, \chi_2) \in M_1(N, \chi)$ with $L(s, f_1) = L(s, \chi_1)L(s, \chi_2).$

Thus, the primitive forms associated to $1 = \chi^4$ and $\chi^2$ are

$\vartheta_K := f_1(\cdot; 1, \psi_{-4})$ and $\vartheta_{36} := f_1(\cdot; \psi_{-3}, \psi_{12}),$

and hence by Theorem 3 we have that

$\vartheta_1(z) = \sum_{n|36} c_n \vartheta_K(nz)$ and $\vartheta_\chi^2(z) = \sum_{n|4} c'_n \vartheta_{36}(nz).$
By using Theorem 3 and its Corollary, the coefficients $c_n$ and $c'_n$ can be determined explicitly:

$$C(s, 1) := \sum_{n|36} c_n n^{-s} = (1 - 2^{-s} + 2^{1-2s})(1 + 3^{1-2s}),$$

$$C(s, \chi^2) := \sum_{n|4} c'_n n^{-s} = (1 - 2^{-s} + 2^{1-2s});$$

here we used the fact that for all primes $p,$

$$a_p((1)_{pr}) = 1 + \psi_{-4}(p) \quad \text{and} \quad a_p((\chi^2)_{pr}) = \psi_{-3}(p) + \psi_{12}(p),$$

which can be calculated explicitly. (We need only $p = 2, 3.$)

Comparing coefficients, we thus see that

$$c_1 = 1, \ c_2 = -1, \ c_4 = 2, \ c_9 = 3, \ c_{18} = -3, \ c_{36} = 6,$$

and that $c_n = 0$ otherwise, and so

$$\vartheta_1(z) = \vartheta_K(z) - \vartheta_K(2z) + 2\vartheta_K(4z) + 3\vartheta_K(9z) - 3\vartheta_K(18z) + 6\vartheta_K(36z).$$

Similarly, $c'_1 = 1, c'_2 = -1, c'_4 = 2$ and so

$$\vartheta_{\chi^2} = \vartheta_{36}(z) - \vartheta_{36}(2z) + 2\vartheta_{36}(4z).$$

Note that the $L$-functions of $\vartheta_1$ and $\vartheta_{\chi^2}$ are

$$L(s, \vartheta_1) = C(s, 1)L(s, (1)_{pr})$$

$$= (1 - 2^{-s} + 2^{1-2s})(1 + 3^{1-2s})\zeta_K(s),$$

$$L(s, \vartheta_{\chi^2}) = C(s, \chi^2)L(s, (\chi^2)_{pr})$$

$$= (1 - 2^{-s} + 2^{1-2s})L(s, \psi_{-3})L(s, \psi_{12}),$$

where $\zeta_K(s) = L(s, \vartheta_K)$ is the Dedekind zeta-function of the field $K := \mathbb{Q}(\sqrt{D}) = \mathbb{Q}(i).$
5. Main Results II: CM-forms

Definition: Let $f \in M_k(N, \psi)$ be a $\mathbb{T}(N)$-eigenfunction with eigencharacter $\lambda_f : \mathbb{T}(N) \to \mathbb{C}$. We say that $f$ has CM (complex multiplication) by a Dirichlet character $\theta$ if

$$\lambda_f(T_p)\theta(p) = \lambda_f(T_p), \quad \text{for all } p \nmid N\text{cond}(\theta),$$

or, equivalently, if

$$\lambda_f(T_p) = 0, \quad \text{for all } p \nmid N\text{cond}(\theta) \text{ with } \theta(p) \neq 1.$$

Notation: We let $M_k^{CM}(N, \psi; \theta)$ denote the space generated by all $\mathbb{T}(N)$-eigenfunctions $f \in M_k(N, \psi)$ which have CM by $\theta$.

Theorem 4: For every discriminant $D < 0$ we have that

$$(5) \quad \Theta(D) = M_1^{CM}(|D|, \psi_D) := M_1^{CM}(|D|, \psi_D; \psi_D).$$

Corollary:

$$(6) \quad \dim \Theta(D) = \dim M_1^{CM}(|D|, \psi_D) = \sum_{f|f_D} 2^{\omega(f)}h_{D/f^2},$$

where $\omega(f)$ denotes the number of distinct prime divisors of $f$. Moreover, the dimensions of the Eisenstein part and of the cuspidal part of $M_1^{CM}(|D|, \psi_D)$ are given by

$$\dim E_1^{CM}(|D|, \psi_D) = \sum_{f|f_D} 2^{\omega(f)}g_{D/f^2},$$

$$\dim S_1^{CM}(|D|, \psi_D) = \sum_{f|f_D} 2^{\omega(f)}(f_{D/f^2} - g_{D/f^2}).$$

Remark: There is no (known) formula for $\dim M_1(|D|, \psi_D)$. 
6. Example: $D = -144$ (again)

Put $\mathcal{D} = \{-4, -16, -36, -144\}$. Then by definition

$$\Theta(D) := \langle \vartheta_q : \Delta(q) \in \mathcal{D} \rangle.$$ 

By Theorem 4:

$$\Theta(D) = M_1^{CM}(144, \psi_{-144}) = E_1^{CM}(144, \psi_{-144}) \oplus S_1^{CM}(144, \psi_{-144}).$$

Since $h_D = g_D$, for $D \in \mathcal{D}$, $D \neq -144$, the Corollary $\Rightarrow$

$$\dim S_1^{CM}(144, \psi_{-144}) = \frac{1}{2}(2^0(h_{-144} - g_{-144})) = \frac{1}{2}(4 - 2) = 1,$$

and so it follows that

$$S_1^{CM}(144, \psi_{-144}) = \mathbb{C}\vartheta_x(z) = \mathbb{C}\eta(12z)^2.$$ 

Moreover, from the Corollary we also have that

$$\dim E_1^{CM}(144, \psi_{-144}) = 2\omega(6)g_{-4} + 2\omega(3)g_{-16} + 2\omega(2)g_{-36} + g_{-144}$$

$$= 4 \cdot 1 + 2 \cdot 1 + 2 \cdot 2 + 2 = 12.$$ 

Now $E_1^{CM}(144, \psi_{-144})$ contains two $T(144)$-eigenspaces given by the eigenfunctions (primitive forms) $\vartheta_K$ and $\vartheta_{36}$:

$$M_1(144, \psi_{-144})[\lambda_{\vartheta_K}] = \langle \vartheta_K(nz) : n|36 \rangle,$$

$$M_1(144, \psi_{-144})[\lambda_{\vartheta_{36}}] = \langle \vartheta_{36}(nz) : n|4 \rangle,$$

which have dimension $d(36) = 9$ and $d(4) = 3$, respectively.

Thus

$$E_1^{CM}(144, \psi_{-144}) = M_1(144, \psi_{-144})[\lambda_{\vartheta_K}] \oplus M_1(144, \psi_{-144})[\lambda_{\vartheta_{36}}].$$
7. Ingredients

1) Dedekind’s Isomorphism:

\[ \lambda_D : \text{Cl}(D) \xrightarrow{\sim} \text{Pic}(\mathcal{O}_D), \]

where \( \mathcal{O}_D = \mathbb{Z} + \mathbb{Z}\frac{D + \sqrt{D}}{2} \subset \mathcal{O}_K \) is the order of discriminant \( D \) (and/or of conductor \( f_D \) in \( K \)).

2) A classification of the invertible ideals of \( \mathcal{O}_D \):

\[ \Rightarrow \] the multiplicativity of \( a_n(\chi) \),
the value of \( c_n(\chi) \) for \( n \mid D \), etc.

3) A study of the conductor of \( \chi \in \text{Cl}(D)^* \) : via the isomorphism

\[ I_K(f_D\mathcal{O}_K)/P_{K,\mathbb{Z}}(f_D) \xrightarrow{\sim} \text{Pic}(\mathcal{O}_D), \]

one can identify each \( \chi \in \text{Cl}(D)^* \) with a Hecke character \( \tilde{\chi} \) on the group \( I_K(f_D\mathcal{O}_K) \) of fractional ideals prime to the ideal \( f_D\mathcal{O}_K \). A key fact is:

\[ \chi \text{ is primitive on } \text{Cl}(D) \Leftrightarrow \tilde{\chi} \text{ is primitive mod } f_D\mathcal{O}_K. \]

4) Genus theory (Gauss/Kronecker/Weber): identifies quadratic characters \( \chi \in \text{Cl}(D)^* \) with certain Dirichlet characters.

5) Extended Atkin-Lehner (newform) theory: this describes:

1) the characters \( \lambda \in \mathbb{T}(N)^* = \text{Hom}(\mathbb{T}(N), \mathbb{C}) \) of the Hecke algebra \( \mathbb{T}(N) \subset \text{End}(M_k(N, \psi)) \) in terms of primitive eigenfunctions (newforms);

2) the structure of the \( \mathbb{T}(N) \)-eigenspace associated to \( \lambda \):

\[ M_k(N, \psi)[\lambda] = \{ f \in M_k(N, \psi) : f|_kT_n = \lambda(T_n)f, \forall (n, N) = 1 \} \]
For Theorem 4, we also need:

6) (a) The Deligne/Serre theory of Galois representations

\[ \rho_f : G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{C}) \]

attached to \( T(N) \)-eigenfunctions \( f \in M_1(N, \psi) \).

(b) A characterization of characters of ring class fields via (strongly) dihedral Galois representations of \( G_\mathbb{Q} \) (= reinterpretation of a result of Bruckner (1966)).

(c) A characterization of CM forms via their associated Galois representations (→ Theorem 5 below).
8. Galois representations

Deligne/Serre (1974): If \( f \in M_1(N, \psi) \) is a normalized \( \mathbb{T}(N) \)-eigenfunction, then \( \exists! \) Galois representation

\[
\rho_f : G_\mathbb{Q} \rightarrow \text{GL}_2(\mathbb{C})
\]

such that for all primes \( p \nmid N \)

\[
\text{tr}(\rho_f(Fr_p)) = \lambda_f(T_p) = a_p(f), \\
\text{det}(\rho_f(Fr_p)) = \psi(p).
\]

Furthermore, \( \rho_f \) is irreducible \( \iff \) \( f \) is a cusp form.

Definition: An Galois representation \( \rho : G_\mathbb{Q} \rightarrow \text{GL}_2(\mathbb{C}) \) is called strongly dihedral if \( \text{Im}(\rho) \cong D_n \) is a dihedral group (\( n \geq 3 \)). Moreover, \( \rho \) is said to be of dihedral type if \( \text{Im}(\rho)/Z(\text{Im}(\rho)) \cong D_n \) is a dihedral group (\( n \geq 2 \)).

Theorem 5: Let \( f \in S_1(N, \psi) \) be a newform.

(a) \( f \) has CM by some character \( \Theta \iff \rho_f \) is of dihedral type.

(b) \( f \) has CM by \( \psi \iff \rho_f \) is strongly dihedral.

Theorem 6: Let \( \rho : G \rightarrow \text{GL}_2(\mathbb{C}) \) be Galois representation.

(a) (Hecke) If \( \rho \) is of dihedral type and is odd, then \( \rho = \rho_f \) for some \( f \in S_1(N, \psi) \).

(b) (Bruckner, 1966) \( \rho \) is strongly dihedral if and only if the field \( \text{Fix(Ker(\rho))} \) is contained in some ring class field.

Remark: Theorems 3, 5, 6 \( \Rightarrow \) Theorem 4.