§1. Naive Heights

Let $K$ be a number field, $\overline{K}$ its alg. closure
$V/K$ a smooth projective variety over $K$
$\mathcal{H}(V) = \{ h : V(\overline{K}) \to \mathbb{R} \}$ set of real-valued functions on $V(\overline{K})$
$\mathcal{H}(V) = \mathcal{H}(V)/\text{bounded functions}$
$B(h, L, c) := \{ \exists f \in V(L) : h(f) \leq c \}$, for $h \in \mathcal{H}(V)$, $KcLc\overline{V}$, $c \in \mathbb{R}$.

Recall (cf. Janssen's talk): there is a homeo. (of ab. groups)

$h_V : \text{Pic}(V) \to \overline{\mathcal{H}}(V)$

$\mathcal{L} \mapsto h_{\mathcal{L}} = h_V\mathcal{L}$, $\mathcal{L}$ set of functions

such that:
1) for any morphism $f : V \to W$ and $\mathcal{L} \in \text{Pic}(W)$ we have
$h_{f^*\mathcal{L}} = h_{\mathcal{L}} \circ f$;
2) for $V = \mathbb{P}^n_K$, $\mathcal{L} = \mathcal{O}(1)$ we have
$h_L = \text{class containing the naive height on } \mathbb{P}^n_K$

From these properties follows:
3) (Finiteness property) $\mathcal{L}$ ample $\Rightarrow$ for each $h \in h_\mathcal{L}$ and
$L/K$ finite we have that $\#B(h, L, c) < \infty$, $c \in \mathbb{R}$.
§2. Abelian varieties

Def. An abelian variety \( A/K \) is a proj. (smooth) algebraic group variety.

Thus, 3 morphisms
\[ m : A \times A \to A \quad (\text{multiplication}) \]
\[ c : A \to A \quad (\text{inversion}) \]
\[ 0 \in A(K) \quad (\text{identity}) \]

satisfying the group axioms.

Ex. 1) \( A = E \) elliptic curve (mult: chord/tangent method)

2) \( A = J_C \), Jacobian of a curve \( C \)
\( (K = \mathbb{C}) \)
3) a torus \( \mathbb{C}^g / \Lambda \to \mathbb{P}^N_a \)

Basic facts: 1) Each ab. var. is commutative
\( A(L) \) is a comm. gp.

2) \( f : A_1 \to A_2 \) morph of ab. var's \( \Rightarrow f = h + f(0) \)
where \( h \) is a homomorphism; moreover \( h(A_1) \) is an ab. subvariety of \( A_2 \).

3) Each ab. var. is a quotient of Jacobians of curves.

4) \( A(K) \) is divisible
\( A[n](K) \) a finite...
§ 3. Statement of the Theorem

Theorem 1. (Take) Let $A/K$ be an ab. var. Then there exists a unique lift

$$
\hat{h}_A : \text{Pic}(A) \rightarrow \mathcal{H}(A) \quad (\xi \mapsto \hat{h}_A\xi)
$$

(c.i.e. $\hat{h}_A\xi$)

of $\hat{h}_A\xi$ such that for each $\xi \in \text{Pic}(A)$ we have

$$
\hat{h}_A\xi(P) = \langle P, P \rangle_\xi + \langle P \rangle_\xi
$$

where $\langle , \rangle : A(\bar{K}) \times A(\bar{K}) \rightarrow \mathbb{R}$ is a bi-additive map, $\langle , \rangle : A(\bar{K}) \rightarrow \mathbb{R}$ is a herm.

\[\text{Cor. 1)} \quad \hat{h}_A \text{ is a homomorphism: } \hat{h}_A(\xi_1 \circ \xi_2) = \hat{h}_A\xi_1 + \hat{h}_A\xi_2 \]

\[\text{2)} \quad \text{If } f : A \rightarrow B \text{ is a homeomorphism, then } \hat{h}_f = \hat{h}_\xi \circ f \]

\[\text{3)} \quad \text{If } \xi \circ \xi = \xi \text{ is symmetric then } \langle \xi \rangle = 0, \text{ i.e. } \hat{h}_\xi \text{ is a quadratic form.} \]

\[\text{4)} \quad \text{If } \xi \text{ is ample and symmetric then } \hat{h}_\xi(P) \geq 0 \text{ and } \hat{h}_\xi(P) = 0 \iff P \in A(K)_{tor}, \text{ and } \{P \in A(K) : \hat{h}_\xi(P) < C \} \text{ is finite.} \]
§4. Quadratic functions on ab. groups

Let $A, B$ be (abstract) ab. groups

$$f : A \rightarrow B$$ a map.

Define

$$
\Delta f : A \times A \rightarrow B \quad ("1st \, difference")
$$

$$
\Delta f(x, y) = f(x+y) - f(x) - f(y)
$$

and $\Delta^2 f : A \times A \times A \rightarrow B$

$$
\Delta^2 f(x, y, z) = f(x+y+z) - f(x+y) - f(x+z) - f(y+z) + f(x) + f(y) + f(z).
$$

Def. $f$ is a quadratic (resp. linear) function if $\Delta^2 f \equiv 0$

(resp. if $\Delta f \equiv 0$)

Ex. 1) $f$ linear $\iff f$

2) $f = q(x, y) + l(x)$ is quadratic if $q(x, y) : A \times A \rightarrow B$

is bi-additive and $l$ is linear.

Lemma 1: If $B = \mathbb{R}$, then every quadratic function $f : A \rightarrow \mathbb{R}$

has a unique decomposition

$$f = qf + l$$

where $l$ is linear and $qf$ is a quadratic form, i.e.
\[ q_f(x) = b_f(x, x) \] where \( b_f \) is bi-additive.

**Pf.** Easy.

**Def.** A function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is **quasi-quadratic** if \( \Delta^2 f \) is bounded (on \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \)).

**Lemma 2 (Tak).** If \( f \) is quasi-quadratic then \( \hat{f}: \mathbb{R} \rightarrow \mathbb{R} \) and that

1) \( \hat{f} \) is quadratic
2) \( f = \hat{f} + 0(1) \).

**Pf.** Uniqueness is clear, for a bounded quadratic function is constant.

Existence: check that

\[ b(x, y) = \frac{1}{2} \lim_{n \to \infty} \Delta f(2^n x, 2^n y) / 4^n \]

converges and is bi-additive. Similarly,

\[ l(x) = \lim_{n \to \infty} \left[ \frac{f(2^n x) - q(2^n x)}{2^n} \right] \]

converges and is linear, where \( q(x) = \frac{1}{2} b(x, x) \). Finally, one checks that

\[ f = q + l + 0(1) \]

and so the assertion follows.
§5. The Theorem of the Cube

This is:

Theorem 2. Let $A$ be an abelian variety and $V$ be any variety. Then for any $L \in \text{Pic}(A)$ the function

$$f \mapsto f^*L$$

is quadratic.

Rmk. By definition this means: for $f, g, h: V \to A$

$$\Delta^*(L)(f \circ g \circ h) = \Theta_x$$

$$\begin{align*}
(f \circ g \circ h)^*L &\circ (f \circ g)^*L^{-1} \circ (f \circ h)^*L^{-1} \circ (g \circ h)^*L^{-1} \circ f^*L \circ g^*L \circ h^*L
\end{align*}$$

\[ \text{pf. } [\text{Hu}], \text{ p. 58.} \]

\[ \text{Benz technique: base change & cohomology} \]
§6. Proof of Theorem 1:

By Lemma 2 (and Lemma 1) it is enough to show that $h_x$ is quasi-quadratic.

For this, let $p_i : A^3 \to A$ denote the $i$th projection.

$$h_{p_i x} = h_x \circ p_i + O(1)$$

( as functions on $A(k)^3$)

$$h_{(p_i+p_j) x} = h_x (p_i + p_j) + O(1)$$

etc.

we see that

$$\Delta_2 h_x = h \Delta_2 (x)(p_i, p_j, p_k) + O(1)$$

By the theorem of the cube, $\delta : \Delta_2 (x)(p_i, p_j, p_k) = O_A^3$, so

$$h_0 = O(1),$$

which proves that $h_x$ is quasi-quadratic. //

P. of (a): 1), 2) immediate

3) Write $h_x = q + l$. By hypothesis: $h_x(-x) = h_{-x}(x)$

$= h_x(x)$, so $q(x) + l(x) = q(-x) + l(-x) = q(x) - l(x) \Rightarrow l(x) = 0$.

4) When $A$ very ample, so $l = q + O(1)$ for $q : A \to P^n$

Then $h_x$ is bounded below, hence so is the quasi-linear form $h_x \leq \langle x, x \rangle$, $h_x \geq 0$. Moreover, if $h_x(p) = 0$ then $h_x(nP) = 0 \forall n \in \mathbb{Z}$

$\Rightarrow h_x$ finite set (finiteness property) $\Rightarrow P \in A(k)$.

This is:

Theorem 3. Let $A/K$ be an ab. var. over a number field $K$.
Then $A(K)$ is a finitely generated group.

Lemma 3: ("Infinite descent"): Suppose $A$ is an ab. group such that:

1) $m > 1$ s. th. $A/mA$ is finite
2) There exists a non-trivial quadratic form $h: A \rightarrow \mathbb{R}$ s. th. the set
   
   $H_c = \{ a \in A : h(a) \leq c \}$ is finite for $c > 0$.

Then $A$ is finitely generated.

Proof. Let $a_1, \ldots, a_r$ be coset representatives of $A/mA$ and put $c = \max_{i \leq r} h(a_i)$.

Claim. $h(a_1, \ldots, a_r) \in H_c$ generates $A$.

First note that $h > 0$, so

1) $h(x + a_i) = 2h(x) + 2h(a_i) - h(x + a_i) \leq 2h(x) + 2c$.

Moreover,

2) $h(mx) = m^2 h(x)$. 
Now let \( x \in \mathfrak{L} \), and write:
\[
x = mx_1 + a_1 \quad x_1 \in \mathfrak{L}
\]
\[
x_1 = mx_2 + a_2
\]
\[
\vdots
\]
\[
x_n = mx_{n+1} + a_{n+1}
\]

Then \( h(x_j) = \frac{1}{m^2} h(mx_j) = \frac{1}{m^2} h(x_{j-1} - a_j) \leq \frac{1}{m^2} [2h(x_{j-1}) + 2c] \)

so
\[
h(x_n) \leq \left( \frac{2}{m^2} \right)^n h(x) + \left( \frac{1}{m^2} + \frac{2}{m^4} + \frac{2}{m^6} + \ldots \right) 2c
\]
\[
\leq 2^n h(x) + c
\]

For \( n \rightarrow 0 \), \( x_n \in \mathfrak{L}_{c+1} \), and so the assertion follows.

To apply this, lemma we need:

**Theorem 4** (weak Kordell-Wiel): If finite ext'n \( L \supset K \) and \( m^L \cdot K \). \( A(L)/mA(L) \) is fr. g.

**Proof (sketch):** Choose \( L \) s.t. \( A[M](K) \subset A(L) \) (\( \Rightarrow \) \( m^L < C_L \)).

Now use

**Fundamental Lemma:** \( m^L = L(\mathfrak{M}_L^{-1} A(L)) \) is an ab. ext'n of \( L \) of exponent \( m \) which is unramified outside a finite set of places of \( L \), hence is finite over \( L \).
PF. [Hu] , Appendix B

Put $G = \text{Gal}(L^u/L)$. Then there exists

\[ \psi : A(L)/mA(L) \rightarrow \text{Hom}(G, A[L]) \rightarrow \text{finite gp} \]

given by $\psi(x)(g) = g(x) - y$ where $y \in [L]^{-1}(x)$

Note: It is unknown how to characterize the range of $\psi$, so weak HW is not effective (hence also HW itself is not effective).

PF. of HW: choose $L$ according to Th. $\psi'$, so $A = A(L)$

satisfies (i) of Lemma 3.

Choose an ample divisor $L \in \text{Pic}(A)$ (since $A$ is proj.)

Then $L^u = \omega_L$ is symplectic and ample, so it satisfies (ii) of Lemma 2.

(ii). Thus by the lemma, $A(L)$ is f.g., hence so is $A(K)(K)$.

References

[CS7] Cornell, Silverman, Arithmetic Geometry - Springer Verlag