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The Noether Formula

§1. The classical Noether formula ([GH], [H])

Let X/k be a smooth projective surface ($k = \mathbb{C}$).
We then have the following numerical invariants:

$$\chi(\mathcal{O}_X) = \sum_{i=0} (-1)^i h^i(X, \mathcal{O}_X) \qquad \text{Euler-Poincaré characteristic}$$

$$c_i(X) = c_i(T_{X/k}) = (-1)^i c_i(\Omega^1_{X/k}) \in H^{2i}(X, \mathbb{C})$$

i^{th} Chern class

so $c_1^2, c_2 \in H^4(X, \mathbb{C}) = \mathbb{Z}$.

These are related by the formula

$$\boxed{12\chi(\mathcal{O}_X) = c_1^2 + c_2} \qquad \text{Noether's formula} \tag{1}$$

This follows easily from the Hirzebruch-Riemann-Roch Theorem (cf. [H], p. 433). Another proof, more explicit, is given in [GH], ch. 4 §6 (pp. 600 ff). Below we shall see yet another proof.

In order to present an analogue of this for arithmetic surfaces, it is useful to first re-write the above formula in the relative case,

$$f : X \to C,$$

in another form which uses only the dualising sheaf $\omega_{X/C}$ (and $\Omega^1_{X/C}$).

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§2. The relative version (semi-stable case) (cf. [Sz])

Let C/k be a smooth, proj. irred. curve of genus q

$f : X \to C$ a surjective (hence flat) morphism, where X/k is as above.

Assume: 1) The generic fibre X_η is a smooth, geometrically connected curve of genus $g \geq 1$ over $\kappa(\eta)$ = function field of C.

2) $f : X \to C$ is a semi-stable curve over C (of genus g).

This means:

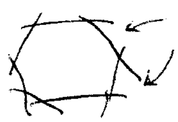
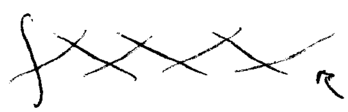
Def. A semi-stable (resp. stable) curve of genus g over a scheme S is a proper, flat morphism $f : X \to S$ such that each fibre X_s ($s \in S$) satisfies:

1) X_s is a geom. connected, reduced curve of arith. genus $p_a(X_s) = g$.

2) Each singularity of X_s is a node (a double point of order 2): X

* 3) Each irreducible component $X_{s,i}$ of $X_s \otimes \overline{\kappa(s)}$ with $X_{s,i} \simeq \mathbb{P}^1$ meets the curve in 2 (resp. 3) other components.

Ex. 1) A smooth, proper morphism $f : X \to S$ with geom. conn. fibres of dim 1 is $\overset{\text{stable}}{\text{semi-stable}}$.

2)  or  $\leftarrow \mathbb{P}^1$ are semi-stable, but  $\leftarrow \mathbb{P}^1$

(nodal cubic) (n-gon polygon) $\leftarrow$ not stable is not.

Basic Facts: 1) $f_* \mathcal{O}_X = \mathcal{O}_C$

2) The relative dualizing sheaf, $\omega_{X/C}$ is invertible and $f_* \omega_{X/C}$ is loc. free of rank g. Moreover,

$$(2) \qquad \begin{aligned} R^1 f_* \mathcal{O}_X &\simeq (f_* \omega_{X/C})^\vee \\ R^1 f_* \omega_{X/C} &\simeq (f_* \mathcal{O}_X)^\vee = \mathcal{O}_C \end{aligned} \qquad \text{(relative duality)}$$

Note: These two facts are true for any semistable curve $f : X \to S$.

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3) We have the exact sequences:

(3a) $$0 \to f^*\Omega^1_{C/k} \to \Omega^1_{X/k} \to \Omega^1_{X/C} \to 0$$

(3b) $$0 \to \Omega^1_{X/C} \to \omega_{X/C} \to \mathcal{T} \to 0 ,$$

where $\mathcal{T}$ is a torsion sheaf.

From these facts one easily relates the invariants of §1 to invariants involving $\omega_{X/C}$ and $\Omega^1_{X/C}$ (and g and q):

<u>Lemma</u> 1: a) $\chi(\mathcal{O}_X) = \deg_C(f_*\omega_{X/C}) + (1-g)(1-q)$,

b) $c_1^2(X) = (\omega_{X/C}\cdot\omega_{X/C}) + 8(1-g)(1-q)$,

c) $c_2(X) = c_2(\Omega^1_{X/C}) + 4(1-g)(1-q)$.

<u>Pf</u>. a) $\chi(\mathcal{O}_X) = \chi(f_*\mathcal{O}_X) - \chi(R^1f_*\mathcal{O}_X)$

$= \chi(\mathcal{O}_C) - \chi((f_*\omega_{X/C})^\vee)$ by 1) and 2)

$= (1-q) - (-\deg(f_*\omega_{X/C}) + g(1-q))$ by 2) and R-R on C

$= \deg f_*\omega_{X/C} + (1-g)(1-q)$.

b) From (3a), (3b) we obtain: $\omega_X := \bigwedge^2\Omega^1_{X/k} = f^*\omega_C \otimes \omega_{X/C}$.
Thus:

$$c_1^2 = (-1)^2(\omega_X\cdot\omega_X) = (\omega_{X/C}\cdot\omega_{X/C}) + 2(f^*\omega_C\cdot\omega_{X/C}) + \underbrace{(f^*\omega_C\cdot f^*\omega_C)}_{0}$$

$$= (\omega_{X/C}\cdot\omega_{X/C}) + 2(2q-2)(2g-2)$$

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c) From (3a) we obtain

$$c_2(X) = c_1(f^*\Omega^1_C)\cdot c_1(\Omega^1_{X/C}) + c_2(\Omega^1_{X/C})$$
$$= (2q-2)(2g-2) + c_2(\Omega^1_{X/C}).$$

If we substitute the above expressions for $\chi(\mathcal{O}_X), c_1^2, c_2$ in the Noether formula, we obtain:

$$(4) \qquad \boxed{12 \deg f_* \omega_{X/C} = (\omega_{X/C}\cdot\omega_{X/C}) + c_2(\Omega^1_{X/C})}$$

"Relative Noether formula"

We next want a better expression for $c_2(\Omega^1_{X/C})$. For this, let

δ_y = # singularities of X_y $(y \in C)$

$\Delta_{X/C} = \sum \delta_y \cdot y \in \mathrm{Div}(C)$, an effective divisor on C

$\delta = \deg(\Delta_{X/C}) = \sum_{y \in C} \delta_y \geq 0$

Then we have by (3b) and a local calculation (cf. [Sz], p. 48):

$$(5) \qquad \boxed{c_2(\Omega^1_{X/C}) = \chi(\Sigma) = \delta = \deg(\Delta)}$$

Note that this shows that $c_2(\Omega^1_{X/C})$ can be computed fibre-by-fibre!

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§3. The Mumford-Noether formula ([M], [Sa], [MB])

By using the Deligne-pairing we can re-write the (relative) Noether formula once more.

Recall that the Deligne-pairing is a bilinear map

$$\langle \, , \rangle : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathrm{Pic}(C)$$

defined by the formula

(6) $$\langle \mathcal{L}, \mathcal{L}' \rangle = \det Rf_*(\mathcal{L} \otimes \mathcal{L}') \otimes (\det Rf_* \mathcal{L})^{-1} \otimes \det(Rf_* \mathcal{L}')^{-1} \otimes \det Rf_* \mathcal{O}_X .$$

By comparing this with the cohomological formula for the intersection product (cf. [H], Ex V.1.1)

(7) $$(\mathcal{L} . \mathcal{L}') = \chi(\mathcal{L} \otimes \mathcal{L}') - \chi(\mathcal{L}) - \chi(\mathcal{L}') + \chi(\mathcal{O}_X)$$

one easily sees that

(8) $$(\mathcal{L} . \mathcal{L}') = \deg_C(\langle \mathcal{L}, \mathcal{L}' \rangle).$$

(Use the fact that

(9) $$\chi(\mathcal{L}) = \deg_C(\det Rf_* \mathcal{L}) + \overbrace{(n+1-g)}^{\chi(\mathcal{L}_\eta)} \chi(\mathcal{O}_C),$$

where $n = \deg(\mathcal{L}|X_\eta)$ denotes the generic degree.)

We can thus rewrite the Noether formula in the following form:

(10) $$\boxed{\deg_C((\det f_* \omega_{X/C})^{\otimes 12}) = \deg_C(\langle \omega_{X/C}, \omega_{X/C} \rangle \otimes \mathcal{O}(\Delta_{X/C}))}$$ Relative Noether formula II

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This equation therefore asserts that the degrees of two invertible sheaves on C are the same. This raises the question: are the two invertible sheaves themselves isomorphic, i.e. is

$$(\det f_* \omega_{X/C})^{\otimes 12} \cong \langle \omega_{X/C}, \omega_{X/C} \rangle \otimes \mathcal{O}(\Delta_{X/C}) \qquad ?$$

This is, in fact, true and was proved by Mumford in 1977 in much greater generality (which we will need here as well):

<u>Theorem 1</u> (Mumford-Noether) Let $f: X \to S$ be a stable curve of genus $g \geq 1$.$^{*)}$ Then there is an (explicit) invertible sheaf $\mathcal{O}(\Delta_{X/S})$ on S and a canonical isomorphism

(11) $$\nu_{X/S} : (\det f_* \omega_{X/S})^{\otimes 12} \xrightarrow{\sim} \langle \omega_{X/S}, \omega_{X/S} \rangle \otimes \mathcal{O}(\Delta_{X/S}),$$

compatible with base change.

[*) or a semi-stable curve over a Dedekind base]

<u>Remark</u>. In [M], Mumford constructs a canonical isomorphism (compatible with base change)

(12) $$\mu_{X/S} : (\det f_* \omega_{X/S})^{\otimes 13} \xrightarrow{\sim} \det f_*(\omega_{X/S}^{\otimes 2}) \otimes \mathcal{O}(\Delta).$$

This isomorphism, however, is virtually the same as $\nu_{X/S}$ because we have

(13) $$\langle \omega_{X/S}, \omega_{X/S} \rangle = \det f_*(\omega_{X/S}^{\otimes 2}) \cdot (\det f_* \omega_{X/S})^{-1}.$$

(The latter identity follows easily from the definition of $\langle\ ,\ \rangle$ because

$$\det Rf_* \mathcal{O}_X = \det(R^1 f_* \mathcal{O}_X)^{-1} = \det f_* \omega_{X/C}, \quad \text{by duality},$$
$$\det Rf_* \omega_{X/S} = \det f_* \omega_{X/S}, \quad \text{since } f_* \omega_{X/S} = \mathcal{O}_S,$$
$$\det Rf_*(\omega_{X/S}^{\otimes 2}) = \det f_*(\omega_{X/S}^{\otimes 2}), \quad \text{since } R^1 f_* \omega_{X/S}^{\otimes 2} = 0. \quad)$$

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Corollary. Suppose $f: X \to S$ is a smooth, proper curve of genus $g \geq 1$. Then there is a canonical isomorphism

(14) $\nu_{X/S}: (\det Rf_*\omega_{X/S})^{\otimes 12} = (\det f_*\omega_{X/S})^{\otimes 12} \xrightarrow{\sim} \langle \omega_{X/S}, \omega_{X/S} \rangle$

which is compatible with base change.

Remark. The above isomorphism $\nu_{X/S}$ (more precisely, $\mu_{X/S}$) plays an important role in string theory in Physics in connection with the (bosonic) Polyakov measure (cf. Beilinson-Manin [BM]).

Following Saito [Sa], we can use the above Corollary to define the invariant δ in another way such that:

1) the definition applies to more general curves (not just semi-stable ones);

2) the definition naturally carries over to Arakelov theory.

For this, we first introduce the following notation.

Notation. Let O_v be a discrete valuation ring with quotient field K, and let M and N be free rank 1 O_v-modules. Then for any K-isomorphism

$$\varphi: M \otimes K \xrightarrow{\sim} N \otimes K$$

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define the relative invariant $\delta(\varphi) = \delta(\varphi; M, N) \in \mathbb{Z}$ of φ by the formula

(15) $$\varphi(M\otimes 1) = \pi^{-\delta(\varphi)}(N\otimes 1) \subset N\otimes K,$$

where π is a prime element of $\mathcal{O}_v$ (i.e. $(\pi) = \mathfrak{m}_v$ is the max'l ideal)

Note: $\delta(\varphi)$ is closely related to the relative invariant $\chi(\varphi(M), N)$ defined in Bourbaki, Comm. Algebra, ch. VII.

We now define the δ-invariant $\delta(X/S)$ in the following situation:

Let $S = \mathrm{Spec}(\mathcal{O}_v)$

$f: X \to S$ be proper, flat with generic fibre a geom. conn. smooth curve of genus $g \geq 1$

Assume: X is regular.

Then: the rel. dual. sheaf $\omega_{X/S}$ exists and is invertible (because the fibres of f are local complete intersections); and properties 1) and 2) of §2 hold.

By the above Corollary, applied to the generic fibre $f_\eta = X_\eta \to \mathrm{Spec}(K)$, we have a canonical isomorphism

(16) $$\nu = \nu_{X_\eta/K}: M\otimes K \xrightarrow{\sim} N\otimes K$$

where $M = (\det Rf_*\omega_{X/S})^{\otimes 12}$ and $N = \langle \omega_{X/S}, \omega_{X/S}\rangle$.

Note that by construction, M and N are free $\mathcal{O}_v$-modules of rank 1.

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We now <u>define</u> the δ-invariant $\delta(f) = \delta(X/S)$ as the relative invariant of ν:

$$(17) \qquad \delta(f) = \delta(\nu, (\det Rf_*\omega_{X/S})^{\otimes 12}, \langle \omega_{X/S}, \omega_{X/S} \rangle).$$

Note that if X/S is semi-stable, then we have by Theorem 1 that

$$(18) \qquad \boxed{\delta(X/S) = \delta_\nu \; (= \#\text{sing. of } X_s \otimes \overline{k(s)})}$$

local Noether formula

It is immediate that this formula implies the (relative) Noether formula (10) because we have quite generally:

<u>Lemma 2</u>: Let $\mathcal{L}, \mathcal{L}'$ be two invertible sheaves on a curve C and let $\varphi : \mathcal{L}_\eta \xrightarrow{\sim} \mathcal{L}'_\eta$ be an isomorphism of the generic stalks. Then we have

$$(19) \qquad \deg_C(\mathcal{L}) = \deg_C(\mathcal{L}_1 \otimes \mathcal{O}(\Delta(\varphi))),$$

where $\Delta(\varphi) = \sum_{x \in C} \delta(\varphi; \mathcal{L}_x, \mathcal{L}'_x) \cdot x \in \mathrm{Div}(C)$.

<u>Proof</u>. Easy exercise.

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<u>Remark</u>. There is a generalization of the local Noether formula (18) to the general case considered above:

$$\boxed{\delta(X/S) = -\mathrm{Art}(X/S) := -\left(\chi_{et}(X_\eta \otimes \bar{K}) - \chi_{et}(X_s \otimes \overline{k(s)}) + Sw_s H^*_{et}(X_\eta \otimes \bar{K}, \mathbb{Q}_\ell)\right)} \tag{20}$$

Artin conductor — etale Euler characteristic — (alternating sum of) Swan conductors

This formula is due to Saito [Sa] and builds on the work of Bloch [Bl] who first defined and studied the Artin conductor $\mathrm{Art}(X/S)$.

Note that the above formula not only includes (18) as a special case (as Saito shows), but also (for $g=1$) reduces to <u>Ogg's formula</u>:

conductor (ell. curve) = expression involving # of comp'ts of Néron model E.

In fact, this seems to be the 1st proof of Ogg's formula which is valid in characteristic 2!

Note also that it follows from (20) that we always have

$$\delta(X/S) \geq 0, \tag{21}$$

because $\chi_{et}(X_s \otimes \overline{k(s)}) \geq \chi_{et}(X_\eta \otimes \bar{K})$ and $Sw_s H^*_{et}(\cdot\cdot) = -Sw_s H^1(\cdot\cdot) \leq 0$. Moreover, the case $\delta(X/S) = 0$ can also be analysed (cf. [Sa]).

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§4. The arithmetic Noether formula ([F], [Sa], [MB])

Suppose now that $f: X \to S = \mathrm{Spec}(\mathcal{O}_K)$ is a regular arithmetic surface over a number field K (with geom. irred. fibre of genus $g \geq 1$).

We can then define the invariant $\hat{\delta}(X/S) \in \mathbb{R}$ in a manner analogous to the geometric case:

<u>Notation</u>. Let $\bar{\mathcal{L}} = (\mathcal{L}, h)$ and $\bar{\mathcal{L}}' = (\mathcal{L}', h') \in \widehat{CH}^1(S)$ be two metrized line bundles, and let $\varphi: \mathcal{L} \otimes K \to \mathcal{L}' \otimes K$ be a K-vector space isomorphism. Then the relative invariant $\Delta(\varphi) = \Delta(\varphi; \bar{\mathcal{L}}, \bar{\mathcal{L}}') \in \widehat{\mathrm{Div}}(S)$ is the Arakelov divisor defined by

$$(21) \qquad \Delta(\varphi) = \sum_{v \in \bar{S}} \delta_v(\varphi; \bar{\mathcal{L}}, \bar{\mathcal{L}}')\, v ,$$

where, for a finite place $v \in S$,

$$(22a) \qquad \delta_v(\varphi, \bar{\mathcal{L}}, \bar{\mathcal{L}}') = \delta_v(\varphi; \mathcal{L} \otimes \mathcal{O}_v, \mathcal{L}' \otimes \mathcal{O}_v)$$

is the relative invariant defined above (cf. equation (15)), and, for an infinite place $v \in S_\infty$, $\delta_v(\varphi) = \delta_v(\varphi, \bar{\mathcal{L}}, \bar{\mathcal{L}}')$ is defined by

$$(22b) \qquad e^{\delta_v(\varphi)} = \|\varphi\|_v = \frac{\|\varphi(s)\|_{\mathcal{L}' \otimes_v \mathbb{C}}}{\|s\|_{\mathcal{L} \otimes_v \mathbb{C}}} , \qquad (s \text{ local section of } \mathcal{L} \otimes_v \mathbb{C})$$

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where $\varphi_v = \varphi \otimes_v \mathbb{C} : \mathcal{L} \otimes_v \mathbb{C} \xrightarrow{\sim} \mathcal{L}' \otimes_v \mathbb{C}$ is the induced isomorphism.

With this definition one easily checks that the arithmetic analogue of Lemma 2 holds:

(23) $$\deg_{\bar{S}} \bar{\mathcal{L}} = \deg_{\bar{S}} \left(\bar{\mathcal{L}}' \otimes \mathcal{O}(\Delta(\varphi)) \right).$$

We now apply this to the invertible $\mathcal{O}_K$-sheaves

$$\mathcal{L} = (\det Rf_* \omega_{X/S})^{\otimes 12} = \left(\det R\Gamma(X, \omega_{X/S})^{\otimes 12}\right)^{\sim}$$

$$\mathcal{L}' = \langle \omega_{X/S}, \omega_{X/S} \rangle = \det R\Gamma(X, \omega_{X/S}^{\otimes 2}) (\det R\Gamma(X, \omega_{X/S}))^{-1},$$

which carry a natural metrization given by the Faltings volume. Now by Mumford's theorem (cf. Corollary to Th. 1) we have the can. iso.

$$\nu = \nu_{X_\eta/K} : \mathcal{L} \otimes K \to \mathcal{L}' \otimes K,$$

so we can define the 2^nd arithmetic Chern class:

(24) $$\hat{\delta}(X/S) = \deg_{\bar{S}}(\underbrace{\Delta(\nu, \bar{\mathcal{L}}, \bar{\mathcal{L}}')}_{\bar{\Delta}(X/S)}).$$

We thus obtain from (23) the following analogue of the Noether formula:

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$$(25)\quad \boxed{12 \deg_{\bar{S}} (\det Rf_* \omega_{X/S}) = (\omega_{X/S} \cdot \omega_{X/S}) + \hat{\delta}(X/S);}$$

"arithmetic Noether formula"

here we have used the fact that the Arakelov intersection pairing on X is related to the Deligne pairing by the formula

$$(26) \qquad (\mathcal{L} . \mathcal{L}') = \deg_{\bar{S}} \langle \mathcal{L}, \mathcal{L}' \rangle .$$

<u>However</u>: equation (25) is really just a tautology; just as in the case of surfaces, what is still missing is the analogue of the "local Noether formula" (equations (18) or (20)) at the archimedean places. To this end we introduce:

<u>Notation</u>. Let X be a Riemann surface and $J = \mathrm{Pic}^{g-1}_X$. Then for each $\mathcal{L} \in \mathrm{Pic}^{g-1}(X)$ we had constructed a can. iso.

$$\varphi_{\mathcal{L}} : \det R\Gamma(X, \mathcal{L}) \longrightarrow \mathcal{O}_J(-\Theta)_{\mathrm{cl}(\mathcal{L})}$$

(cf. H. Kurke's talk). Now $\mathcal{O}_J(-\Theta)$ comes equipped with a can. hermitian metric (induced by that on $\mathcal{O}_J(\Theta)$). On the other hand, $\det R\Gamma(X, \mathcal{L})$ is endowed with the <u>normalized</u> Faltings volume. Because of this normalization, $\varphi_{\mathcal{L}}$ is no longer

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an isometry; however, we have (by construction of the Faltings volume) that

(27) $$e^{\delta(X)/8} = \| \varphi_{\mathcal{L}} \|$$

is independent of $\mathcal{L} \in \mathrm{Pic}^{g-1}(X)$.

We now have the following result due to Faltings [F] (up to a constant) and Moret-Bailly [MB]:

<u>Theorem 2</u>. For each $v \in S_\infty$ we have

(28) $$\boxed{\delta_v(X/S) = \delta(X \otimes_v \mathbb{C}) - 4g \log(2\pi)}$$ archimedean local Noether formula

(Here $\delta_v(X/S)$ denotes the v-component of the Arakelov divisor $\bar{\Delta}(X/S) = \sum \delta_v(X/S) \cdot v = \Delta(\nu; \mathcal{L}, \mathcal{L}')$ defined in (24) above.)

<u>Remark</u>. Faltings [F] and Moret-Bailly [MB] formulate the "arithmetic Noether formula" (25) only for semi-stable curves; the above formulation is (equivalent to) that of Saito [Sa].

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§5. Proof sketch of Theorem 2 ([MB])

Step 0: characterizing the Mumford isomorphism $\nu_{X/S}$ (or $\mu_{X/S}$).

Since $\nu_{X/S}$ is to be compatible with base change, it is enough to specify $\nu_{X/S}$ for the universal stable curve:

Def. The universal stable curve of genus $g \geq 1$ is a stable curve

$$\bar{\pi}: \bar{\mathcal{C}} \to \bar{\mathcal{M}}_g \qquad \text{of genus } g$$

such that for every scheme S and every stable curve $f: X \to S$ of genus g there is a unique morphism

$$g: S \to \bar{\mathcal{M}}_g \qquad \text{such that} \quad X = S \times_{\bar{\mathcal{M}}_g} \bar{\mathcal{C}} \text{ and } f = pr_1.$$

(i.e. we have a cartesian diagram
$$\begin{array}{ccc} X & \xrightarrow{g'} & \bar{\mathcal{C}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ S & \xrightarrow{g} & \bar{\mathcal{M}}_g \end{array}$$
)

Fundamental problem: no such schemes $\bar{\mathcal{M}}_g$, $\bar{\mathcal{C}}$ exist.

However: we can embed the category Sch of schemes in a larger "category" AlgStacks of algebraic stacks:

$$\underline{Sch} \hookrightarrow \underline{AlgStacks}$$

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in such a way that:

1) virtually all the concepts of schemes (e.g. smooth, proper, ... morphisms etc.) can be extended to Alg Stacks, so that for all intents and purposes we can think of objects and morph's of Alg Stacks as schemes and their morphisms;

2) "univ. objects" exist in Alg Stacks.

Basic Facts ([DM], [DR]):

1) The univ. stable curve $\bar{\pi} : \bar{\mathcal{C}} \to \bar{\mathcal{M}}_g$ exists (as alg. stacks)

2) $\bar{\mathcal{M}}_g$ is proper and smooth over Spec($\mathbb{Z}$) with geom. irred. fibres.

stable reduction theorem

deformation theory of stable curves

3) Let $\mathcal{M}_g \subset \bar{\mathcal{M}}_g$ denote the open substack where $\bar{\pi}$ has smooth fibres (so $\pi = \bar{\pi}|_{\mathcal{C}} : \mathcal{C} = \bar{\pi}^{-1}(\mathcal{M}_g) \to \mathcal{M}_g$ is proper, smooth). Then

$$\Delta = \bar{\mathcal{M}}_g \setminus \mathcal{M}_g$$

is a divisor on $\bar{\mathcal{M}}_g$ with normal crossings.

Now let

$$\bar{K} = \omega_{\bar{\mathcal{C}}/\bar{\mathcal{M}}_g}$$ denote the rel. dualizing sheaf.

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Then Theorem 1 is a consequence of the following more precise result:

Theorem 1'. There exists an isomorphism

(29) $$\bar{\nu} = \nu_{\bar{\mathcal{C}}/\bar{\mathcal{M}}_g} : (\det \pi_* \bar{K})^{\otimes 12} \xrightarrow{\sim} \langle \bar{K}, \bar{K} \rangle \otimes \mathcal{O}_{\bar{\mathcal{M}}_g}(\Delta)$$

of invertible sheaves on $\bar{\mathcal{M}}_g$. Moreover, $\bar{\nu}$ is unique up to a sign.

Pf (sketch): The existence is (essentially) proven in [M]. The uniqueness (up to sign) follows because by property 2) we have

(30) $$H^0(\bar{\mathcal{M}}_g, \mathbb{G}_m) = \{-1, +1\}.$$

(Note: for any two isom. inv. sheaves $\mathcal{L} \simeq \mathcal{L}'$ on a scheme (or alg. stack) X one has a (non-can.) bijection:
$$\mathrm{Iso}(\mathcal{L}, \mathcal{L}') \leftrightarrow H^0(X, \mathbb{G}_m). \quad)$$

Corollary'. Let $K = \omega_{\mathcal{C}/\mathcal{M}_g} = \bar{K}|_{\mathcal{C}}$. Then there is an isom.

(31) $$\nu : (\det \pi_* K)^{\otimes 12} \xrightarrow{\sim} \langle K, K \rangle$$

of invertible sheaves on $\mathcal{M}_g$. Moreover, ν is unique up to a sign.

Pf. Existence: restrict $\bar{\nu}$ to $\mathcal{M}_g$. Uniqueness: follows from:

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Lemma: $H^0(\mathcal{M}_g, \mathbb{G}_m) = \{-1, +1\}$

Pf. [MB], Lemma (2.2.3).

Step 1: The isomorphism $\sigma_{g,N}: \omega_g^{\otimes N} \xrightarrow{\sim} \mathrm{Th}_g^{\otimes 2N}$

Let $f: \mathfrak{X}_g \to \tilde{A}_g$ denote the univ. abelian variety of dim. g with a symmetric theta divisor Θ_g (defining a principal polarization); these exists as alg. stacks. Put:

$$\left.\begin{aligned} \omega_g &= e^*(\Omega^g_{\mathfrak{X}_g/\tilde{A}_g}) \\ \mathrm{Th}_g &= e^*(\mathcal{O}_{\mathfrak{X}_g}(\Theta_g)) \end{aligned}\right\} \text{ sheaves on } \tilde{A}_g$$

where $e: \tilde{A}_g \to \mathfrak{X}_g$ denotes the unit section.

One then proves (cf. [MB]):

Theorem 3. There is some $N \geq 1$ such that $\omega_g^{\otimes N} \simeq \mathrm{Th}_g^{\otimes 2N}$ (in fact, one can take $N = 4$). Moreover, any such iso.

$$\sigma_{g,N}: \omega_g^{\otimes N} \simeq \mathrm{Th}_g^{\otimes 2N}$$

is unique up to sign on each (of the two) irreducible components of $\tilde{A}_g$.

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Each complex point $x \in \tilde{A}_g(\mathbb{C})$ (i.e. $x : \mathrm{Spec}(\mathbb{C}) \to \tilde{A}_g$) corresponds to an ab. var. $A = A_x / \mathbb{C}$ (+ theta divisor Θ_x on A_x).

Then the pullbacks $\begin{cases} x^*\omega_g = e^*\Omega^g_{A_x/\mathbb{C}} \\ x^*\mathrm{Th}_g = e^*\mathcal{O}_{A_x}(\Theta_x) \end{cases}$ carry canonical hermitian structures, and one has (cf. [MB]):

Theorem 4. $\| x^* \sigma_{g,N} \| = (2\pi)^{Ng}$.

Step 2. Construction of an isomorphism $\tilde{\tau}_N$.

Let $\tilde{\pi} : \tilde{\mathcal{C}} \to \tilde{\mathcal{M}}_g$ denote the univ. smooth curve of genus g with a theta-characteristic ℓ.

(i.e. $\ell \in \mathrm{Pic}^{g-1}_{\tilde{\mathcal{C}}/\tilde{\mathcal{M}}_g}(\tilde{\mathcal{M}}_g)$ s.th. $2\ell \simeq \Omega^1_{\tilde{\mathcal{C}}/\tilde{\mathcal{M}}_g}$).

Then the forget map

$q : \tilde{\mathcal{M}}_g \to \mathcal{M}_g$ is finite, flat of degree 2^{2g}.

Claim: For some $N \geq 1$ there is an isom.

(32) $$\tilde{\tau}_N : q^*(\det \pi_* K)^{\otimes(12N)} \xrightarrow{\sim} q^* \langle K, K \rangle^{\otimes N}$$

such that for any $x \in \tilde{\mathcal{M}}_g(\mathbb{C})$ we have

(33) $$\| x^* \tilde{\tau}_N \| = (2\pi)^{-4Ng} e^{N\delta},$$

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where $\delta = \delta(x^*\tilde{\mathcal{C}})$ denotes the Faltings invariant of the Riemann surface $x^*\tilde{\mathcal{C}} = \{x\} \times_{\mathcal{M}_g} \tilde{\mathcal{C}}$.

Rmk. This is the main step; it uses the can. iso.

$$\det R\pi_*\mathcal{L} \xrightarrow{\sim} \ell^* \mathcal{O}_J(-\Theta) \qquad (\ell = cl(\tilde{x}))$$

considered above (in defining δ) and the iso.

$$\ell^* \Omega^g_{J/B} \xrightarrow{\sim} \det \tilde{\pi}_* \omega_{\tilde{\mathcal{C}}/B} \qquad (B = \mathcal{M}_g).$$

By applying Th. 3 one finds N to construct τ_N; the calculation of the norms uses Th. 4.

Step 3: Conclusion.

Put $N' = 2^{2g} N$, where N is as in step 2. Taking norms yields an isom.

(34) $$\tau : (\det \pi_* K)^{\otimes 12 N'} \xrightarrow{\sim} \langle K, K \rangle^{\otimes N'}$$

with norm

(35) $$\| x^* \tau \| = (2\pi)^{-4gN'} e^{N'\delta(x)} \quad \text{for any } x \in \mathcal{M}_g(\mathbb{C}).$$

By the above Lemma we must have:

(36) $$\tau = \pm \nu^{\otimes N'},$$

so it follows from (35) and (36) that

(37) $$\| x^* \nu \| = (2\pi)^{-4g} e^{\delta(x)},$$

which proves Theorem 2.

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