ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Abelian Varieties/ $\mathbb{C}$ and Theta-Divisors

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These are preliminary lecture notes, intended only for distribution to participants
Thus the map
\[ X \rightarrow W^*/\Lambda, \]
\[ x \mapsto (\omega \rightarrow \frac{1}{2}\omega \mod \Lambda) \]
is well-defined, and one check that this is an isomorphism. Thus one has the canonical identification
\[ X \simeq H^1(X, \Omega')^*/\Phi_1(X, \mathbb{Z}) \]

Note that these two descriptions are inverse to each other via the canonical identification
\[ T_0(X) \cong \mathbb{C} \cong H^1(X, \Omega'), \]
which is obtained by dualizing the map
\[ T_0(X) \rightarrow H^1(X, \Omega')^*, \quad \omega \mapsto \omega \]
where \( \omega \) denotes the translation-invariant holomorphic 1-form defined by \( (\omega)_x = T^*_x(\omega) \). Here \( T_x : X \rightarrow X \) denotes the translation map \( T_x(y) = x + y \).

**Fact 2.** \( H^0(X, \mathbb{Z}) \cong \text{Alt}^0(\Lambda, \mathbb{Z}), \quad \forall \Gamma \neq 0 \)

Let \( \pi : V \rightarrow X \) denote the projection map. Then
\( (V, \gamma) \) is clearly the universal covering space of \( X \).
and so we have
\[ H(X) = \Lambda \approx \mathbb{Z}^{2g}. \]
Thus
\[ H'(X, \mathbb{Z}) = \text{Hom}(\pi_1(X), \mathbb{Z}) = \text{Hom}(\Lambda, \mathbb{Z}) = \text{Alt}'(\Lambda, \mathbb{Z}). \]
Furthermore, cupproduct induces a map
\[ \Lambda \text{Hom}(\Lambda, \mathbb{Z}) \rightarrow H'(X, \mathbb{Z}) \]
which one checks to be an isomorphism by applying the Hirzebruch formula to \( C/\Lambda \approx (S')^\# \) (homeomorphism). Thus we obtain the identification
\[ H'(X, \mathbb{Z}) \cong \Lambda \text{Hom}(\Lambda, \mathbb{Z}) = \text{Alt}'(\Lambda, \mathbb{Z}) \]
which, by description (6), satisfies \( \lambda(\Lambda) \subset \Lambda' \).
Conversely, each \( \lambda \in \text{Hom}(V, V') \) with \( \lambda(\Lambda) \subset \Lambda' \)
defines a bundle \( \tilde{\lambda} : X = \mathbb{V}/\Lambda \rightarrow \mathbb{V}/\Lambda' \). Since \( \mathbb{V} \) and \( \mathbb{V}' \) are also the universal covering spaces of \( X \) and \( X' \), it follows that \( \tilde{x} \rightarrow \tilde{\lambda} \) is injective, and so we obtain the indicated equality.

In particular:
1) Every \( h \in \text{Hom}(X, X') \) is a group homomorphism.
2) Every bundle map \( f : X \rightarrow X' \) of the form
\[ f(x) = k(x) + y, \text{ where } k \in \text{Hom}(X, X') \text{ is a homomorphism and } y = f(0). \]
Furthermore:
3) The induced map
\[ \text{Hom} \rightarrow \text{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \]
\[ h \mapsto dh/\Lambda \]
is injective (since \( \Lambda \) contains a \( C \)-base of \( V) \),
and so \( \text{Hom}(X, X') \) is free \( \mathbb{Z} \)-module of finite rank, in fact, we have
\[ \text{rank}_\mathbb{Z} \text{Hom}(X, X') = \text{rank}_\mathbb{Z}(\Lambda, \Lambda') = 4g \times 8. \]
Note that the above map (6) may be (canonical!) identified with the homology map
\[ H_i \xrightarrow{\text{Hom}} \text{Hom}_2 \left( H_i(X, \mathbb{Z}), H_i(X', \mathbb{Z}) \right) \]
via the identification \( \Lambda = \pi_1(X) = H_i(X, \mathbb{Z}) \).

\[ \Lambda : \pi_i(X, x) \to H_i(X, \mathbb{Z}). \]

Fact 4. \( H^n(X, \mathcal{O}_x^\circ) \cong \Lambda^n V^\circ \otimes \Lambda^n \overline{V}^\circ \), where \( V = T_c(X) \).

The identification (3) generalizes to yield short isomorphisms
\[ (8) \quad C_\alpha \otimes \Lambda^n V^\circ \cong \mathbb{R}^n, \]
from which we obtain
\[ (9) \quad H^9(X, \mathcal{O}_x^\circ) \cong H^9(X, \mathcal{O}_x) \otimes \Lambda^n V^\circ. \]

Much more difficult, however, is to show that
\[ (10) \quad H^9(X, \mathcal{O}_x) \cong \Lambda^9 \overline{V}^\circ, \]
where \( \overline{V}^\circ = \text{Hom} \left( \mathcal{O}_x, \mathbb{C} \right) \), from which fact 4 follows in view of (9). (For the proof of (10), cf. [H], pp. 4-8.)

In particular:
\[ H^i(X, \mathcal{O}_x^\circ) = V^\circ \otimes \overline{V}^\circ = \text{Hom}(V, \mathbb{C}), \]
where \( \text{Hom}(V, \mathbb{C}) = \{ H^i \cdot V \otimes V \to \mathbb{C} : H^i \cdot V \text{ cohomologies} \} \)
denotes the space of Hamiltonian forms on \( V \).

Fact 5. The above isomorphisms render the following diagram commutative:

\[ \begin{array}{ccc}
H^i(X, \mathcal{O}_x) & \cong & \Lambda^r(\Lambda, \mathbb{Z}) = \Lambda^r \text{Hom}(\Lambda, \mathbb{Z}) \\
\downarrow & & \downarrow \Lambda^r \\
H^i(X, \mathcal{O}_x) & \cong & \Lambda^r(V^\circ \otimes \overline{V}^\circ) = \bigoplus \Lambda^r V^\circ \otimes \Lambda^r \overline{V}^\circ \\
\downarrow p & & \downarrow \Lambda^r \\
H^i(X, \mathcal{O}_x) & \cong & \Lambda^r(\overline{V}^\circ)
\end{array} \]

Here, \( \alpha \circ \beta \) are the maps induced by the inclusion of sheaves \( \mathcal{O}_x \subset \mathcal{O}_x \), and \( i : \text{Hom}(\Lambda, \mathbb{Z}) \to V^\circ \otimes \overline{V}^\circ \) is the coadj. inclusion. Finally, \( p \) denotes the projection onto the \( p=0, q=0 \) factor.
§2. Line bundles on $X$

To construct line bundles on $X = V / \Lambda$, let us start with the trivial line bundle $L = V \times \mathbb{C}$ on $V$. If $L$ admits a $\Lambda$-action of the form

$$\lambda (v, z) = (v + \lambda, e_\lambda (v) \cdot z),$$

where $\lambda \in \Lambda, v \in V, z \in \mathbb{C}$ and $e_\lambda (v) \in \mathbb{C}^\times$, then we can consider the quotient

$$L(\{e_1\}) = V \times \mathbb{C} / \Lambda,$$

where $e_1$ is a fixed element of $\Lambda$. One easily checks:

1. $\pi_1(L(\{e_1\})) \to V / \Lambda$ is a holomorphic line bundle on $X$.

We now want to arrive at a convenient representation of this cohomology group $H^1(V, \Lambda)$. To this end, let

$$H^1(V, \Lambda) = \{ H \in \text{Hom}(V, \mathbb{C}^\times) : (\text{Im} H)|_{\Lambda, N} \subset \mathbb{Z} \}$$

and for a homomorphism $H \in H^1(V, \Lambda)$ let

$$\mathcal{C}_h^R(H) = \{ \chi : \Lambda \to \mathbb{C}^\times \text{ s.t. } (4)_h \text{ below holds} \}.$$

Here, $\mathcal{C}_h = \{ z \in \mathbb{C} : |z| = 1 \}$ and the condition condition holds is

$$H^1(V, \Lambda) = \mathcal{C}_h^R \times \mathbb{C}^\times,$$

by pulling line bundles on $X$ back to $V$ that carry holomorphic line bundle $L \subset \text{Pic}(X)$ on $X$ arises in this way. Moreover, one checks easily that we have an isomorphism:

$$H^1(V, H^0(V, \Theta^*)) \cong H^1(V, \Theta^*) = \text{Pic}(X).$$

Here, the group on the left is the usual 1st cohomology group $H^1 = Z^1 / B^1$ in group cohomology.

In fact, since every line bundle on $V$ is trivial (because $H^1(V, \Theta^*) = 0$), one sees easily

$x)$ Since $V = 0$ and $H^1(V, \Theta^*) = 0$ ($\delta$-Poincaré lemma) and $H^1(V, \mathcal{O}) = 0$ ($V \times \mathbb{C}^\times$ contractible), it follows from the exact sequence that $H^1(V, \Theta^*) = 0$.
where, as usual, \( \Phi(t) = \exp(2\pi i t) \) and \( \im(H) \).

Note that since \( \Phi(\frac{1}{2} E_1, \lambda^i) = \pm 1 \), each \( \lambda^i \)
is a character (when \( \lambda \in \mathrm{Ch}^k(H) \)), so the \( \lambda^i \)
are "square roots of characters", which justifies the notation \( \mathrm{Ch}^k(H) \).

Consider now a pair \((H, \lambda)\), where \( H \in \mathrm{Hom}(V, \Lambda) \) and \( \lambda \in \mathrm{Ch}^k(H) \). Then, as
a corollary checked,

\[
\ell_n, \lambda, \chi(t) = \chi(\lambda) \exp(-\frac{i}{2} H(V, \lambda) - \frac{i}{2} H(\lambda, \lambda))
\]
is a couple \( \{ \ell_n, \lambda, \chi(t) \} \in Z^1(\Lambda, H^0(V, \Omega^1_v)) \)
and hence gives rise to a holomorphic line bundle

\[
L(H, \lambda) := L(\{ \ell_n, \lambda, \chi(t) \}).
\]

Let

\[
P = P(V, \Lambda) = \{ H, \lambda \} : H, \lambda \text{ as above}\}
\]
de note the set of such pairs. We can make \( P \) into a group via the addition law

\[
(H_1, \lambda_1) + (H_2, \lambda_2) = (H_1 + H_2, \lambda_1 + \lambda_2).
\]

We then have:

**Theorem 2.1 (Appell - Humbert).** The map \( (H, \lambda) \mapsto L(H, \lambda) \)
induces a group homomorphism

\[
L : P(V, \Lambda) \rightarrow \mathrm{Pic}(X) = H^0(X, \Omega^1).
\]

More precisely, we have the following commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\mathrm{Hom}(V, \Lambda) & P(V, \Lambda) & \mathrm{Hom}(V, \Lambda) \\
\downarrow & \downarrow & \downarrow \\
\mathrm{Pic}^0(X) & \mathrm{Pic}(X) & \mathrm{Pic}(X) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

where

\[
\begin{align*}
\ell_n & \downarrow \quad L \downarrow \quad S \downarrow \\
0 & \rightarrow \mathrm{Hom}(V, \Lambda) & P(V, \Lambda) & \rightarrow \mathrm{Hom}(V, \Lambda) & \rightarrow 0 \\
0 & \rightarrow \mathrm{Pic}^0(X) & \rightarrow \mathrm{Pic}(X) & \rightarrow \mathrm{Hom}(H^0(Z, \Omega) \rightarrow H^1(X, \Omega)) & \rightarrow 0 \\
0 & \rightarrow 0 & 0 & \rightarrow 0 & \rightarrow 0
\end{align*}
\]
in which
\[ \text{Hom}(V, \Lambda)^{\text{mod}} = \{ H \in \text{Hom}(V) : (\text{Im}H)(\Lambda \times \Lambda) \subset \mathbb{Z} \} \]
\[ \text{Pic}^0(X) = \ker (c_1 : H^1(X, \mathcal{O}^*) \to \text{Pic}(X) \to H^2(X, \mathbb{Z})) \]
\[ \alpha(x) = (0, x) \in \text{Pic}(V, \Lambda) \]
\[ \beta(H, x) = \beta(H) \in \text{Hom}(V, \Lambda) \]
\[ \lambda(x) = L(0, x) \]
\[ \rho(H) = \text{Im}(H)|_{\Lambda \times \Lambda} \in \text{Alg}(\Lambda, \mathbb{Z}) = H^2(X, \mathbb{Z}) \]

In particular, we have the following formula for the first Chern class of \( L(H, x) \):
\[ (\gamma) \quad c_1(L(H, x)) = E \mid_{\Lambda \times \Lambda} \in \text{Alg}(\Lambda, \mathbb{Z}) = H^2(X, \mathbb{Z}), \]
where, as before, \( E = \text{Im}(H) \).

Remark 2: Recall that if \( L \in \text{Pic}(X) \) is a line bundle on a complex space \( X \), then its Chern class is defined as
\[ c_1(L) = \delta(L), \]
where
\[ \delta : H^1(X, \mathcal{O}^*) = \text{Pic}(X) \to H^2(X, \mathbb{Z}) \]
is the boundary map of the long exact sequence induced by the exponential sequence
\[ (\delta) \quad 0 \to \mathbb{Z} \to \mathbb{C} \overset{\delta}{\to} \mathcal{O}^* \to 0. \]

Pt. 2. Sketch of the proof. Clearly, the diagram \( (\delta) \) commutes and has exact rows.

Using facts 5 of §1 one sees easily that \( \rho \) is an isomorphism.

To see that \( \lambda \) is injective, use the fact that if \( f \in H^0(V, \mathcal{O}^*) \) is bounded, then \( f \) is constant.

The surjectivity of \( \lambda \) follows by a suitable diagram chase and observing that \( \gamma : H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}) \)
is surjective (cf. fact 5).

Since \( \rho \) and \( \lambda \) are isomorphisms, and the rows are exact, it follows that \( L \) is also an isomorphism.
1. The dual torus $\hat{X} = \text{Pic}^0(X)$

In the course of proving the A-H Theorem we had established the isomorphism
$$\text{Hom}(\Lambda, \mathbb{C}^*) \cong \text{Pic}^0(X).$$

Note that $\hat{X} = \text{Hom}(\Lambda, \mathbb{C}^*)$ itself is a complex torus (also of dimension $g$), so the group $\text{Pic}^0(X)$ carries a natural structure.

On the other hand, from the long exact sequence associated to the exponential sequence we obtain
$$\hat{X} = \text{Pic}^0(X) = \text{ker}(\partial) = \text{ker}(\text{hom}(H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}_X)))$$
$$= \text{ker}(\text{hom}(H^1(X, \mathcal{O}_X) / H^1(X, \mathcal{O}_X)))$$

which shows that $\hat{X}$ is again a complex torus.

2. The theorem of the square

The pullback $T_x^* L$ of a line bundle $L \in \text{Pic}(X)$ w.r.t. the translation map $T_x : X \to X$, $T_x(y) = xy$, is given explicitly as follows:

(10) $T_x^* L \otimes L \cong L(H, \mathcal{E}(\epsilon_0, 1) X)$, for $x \in X$. From this we see that for any $L \in \text{Pic}(X)$ and $x \in X$ we have

(11) $\phi_L(x) := T_x^* L \otimes L^{-1} \in \text{Pic}^*(X)$, so that $\phi_L$ defines a map

$\phi_L : X \to \text{Pic}^*(X) = \hat{X}$.

The theorem of the square asserts that this is a homomorphism, i.e., that

(12) $T_x^* y(L) \otimes L = T_x^* (L) \otimes T_y^*(L)$.

Again, this follows readily from (9) (and $-H$):

Write $L = L(H, \mathcal{E})$, then for $x \in \mathbb{C}^*$, we have $T_x^* L \otimes L \cong L(H, \mathcal{E}(\epsilon_0, 1) X) \otimes L(H, \mathcal{E})$
$$\cong L(2H, \mathcal{E}(\epsilon_0, 1) X) \otimes L(1, \mathcal{E})$$

$$= L(H, \mathcal{E}(\epsilon_0, 1) X) \otimes L(1, \mathcal{E})$$
\[ = L(H, e^{-E(\cdot, \cdot)}) X) \cdot L(H, e^{-E(\cdot, \cdot)}) X \]
\[ = T_x^* L \otimes T_y^* L, \quad \text{which proves (1)}. \]

We can also easily determine the kernel of \( \phi \):
\[ K(L) := \ker(\phi_L) = \{ x \in X : T_x^* L \cong L \}. \]

Indeed, since \( e(E(\cdot, \cdot)) = 1 \Leftrightarrow E(\cdot, \cdot) \in \mathbb{Z}, \forall x \in X \), it follows that
\[ K(L) = \mathcal{V}(H)/\Lambda, \]
where \( \mathcal{V}(H) = \{ v \in V : E(\cdot, \lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda \} \).

In particular, we see
\[ K(L) \text{ is finite } \iff \mathcal{V}(H) \text{ is a lattice} \]
\[ \iff (E(\cdot, \cdot), E(\cdot, \cdot)) \text{ is non-degenerate}. \]

3. The Theorem of the Cube

The line bundles \( L(H, X) \) satisfy the following functoriality property: If \( L = L(H, X) \) is a line bundle on \( X' = Y/\Lambda' \) and \( \lambda : X \to X' \) is induced by \( \lambda \in \text{Hom}_G(V, V') \), then
\[ (\lambda^* L(H, X)) = L(\lambda^* H, \lambda^* X). \]

We can use this to prove the theorem of the cube.

**Theorem 2.3.** Given a complex space \( Y \) and holomorphic maps \( f, g, h : Y \to X \), where \( X \) is a complex torus. Then, for any \( L \in \text{Pic}(X) \) we have
\[ (f + g + h)^* (L) \otimes (f^* (L) \otimes g^* (L) \otimes h^* (L)) \]
\[ = (f + g)^* (L) \otimes [(f + h)^* (L) \otimes g^* (L)]. \]

To prove this, consider the line bundle
\[ \Omega^i (L) = \bigotimes_{i \neq j, \ldots} (m_i^* L)^{\otimes (\alpha_{ij})}, \]
on \( X^* \), where \( m_i^* : X^* \to X \).
map \( m_i(x_1, \ldots, x_n) = \sum x_i \). Then (16) is clearly equivalent to the assertion

\[(13) \quad (f, g, h) \in \mathcal{D}_c(L) \Rightarrow \mathbb{C} \, .\]

where \((f, g, h) : Y \rightarrow x_1 \times x_2 \times \cdots \times x_n \). Now in fact we have

\[(15) \quad \mathcal{D}_c(L) \cong \bigodot_{x \in x_n} \, \forall \, n \geq 3 \, ,
\]

because for \( L = L(H, X) \) we have

\( \mathcal{D}_c(L) \cong L(\mathcal{D}_c(H), \mathcal{D}_c(X)) \), and \( \mathcal{D}_c(H) \)

and \( \mathcal{D}_c(X) \) are clearly compact to be trivial.

Hence, for any map \( h : x' \rightarrow x \)

\( \mathcal{D}_c(h) = \mathcal{D}_c \left( m_i(h) \right) \),

and this is easily seen to be trivial.)

\[\text{Remark 24. For the line bundle } \mathcal{D}_c(L) \text{ etc., cf.} \]
\[\text{[M-82], p. 12 ff.} \]

§3. Theta-functions

We now turn to examining the holomorphic sections of the line bundles \( L \rightarrow L(H, X) \). By general principles of quasiregular spaces and algebras, we have a natural correspondence

\[(1) \quad \mathcal{H}^0(X, \mathcal{L}(H, X)) \cong \mathcal{H}^0(V, V \times C)^{\mathfrak{g}, \mathfrak{f}}
\]

of the space of holomorphic sections of \( L(H, X) \) with the space of \( \mathfrak{g} \)-equivariant sections of \( V \times C \) (via the \( \mathfrak{h} = \mathfrak{n} \times \mathfrak{z} \)-action).

Now we can identify

\( \mathcal{H}^0(V, V \times C) \cong \mathcal{H}^0(V, \mathcal{O}) = \{ \text{holomorphic } f : V \rightarrow \mathcal{O} \} \),

\( s : V \rightarrow V \times C \mapsto f_s \), \( f_s(v) = \pi_2(g(v)) \),

but this identification is incompatible with the group action. However, it is immediate that

\( s \in \mathcal{H}^0(V, V \times C)^{\mathfrak{g}, \mathfrak{f}} \iff f = f_s \) satisfies.

\[(2) \quad f(\sigma + \lambda) = e^{(H, X)} f(\sigma), \quad \forall \sigma \in V, \quad \forall \lambda \in \mathfrak{h} \, .
\]

Thus we have a natural identification:
(2) \( H^0(X, L(H, X)) = \text{Th}(H, X) \),

where

\[
\text{Th}(H, X) = \{ \text{holo. } f : V \to \mathbb{C} \text{ satisfying (2)} \}.
\]

**Definition.** The functions \( f \in \text{Th}(H, X) \) are called (normalized) theta functions (with respect to \((H, X)\)).

**Remark 3.0** If we consider more general cocycles \( 1_{e_3} \in Z'(\Lambda, H^0(Y, \Theta)) \) then an analogous assertion holds, i.e.

\[
H^0(X, L(1_{e_3})) = \text{Th}(1_{e_3}),
\]

where the space on the right denotes the space of holomorphic functions \( f : V \to \mathbb{C} \) satisfying

\[
(f(v + \lambda) = e_2(v) f(v), \forall v \in V, \lambda \in \Lambda).
\]

Such functions \( f \) are called unnormalized theta functions.

We first make some preliminary observations about \( \text{Th}(H, X) \) (cf. [MI], pp. 25-6):

1) If \( R = \text{Rad}(H) = \{ u \in \mathbb{V} : H(u, w) = 0, \forall w \in V \} \)
\[
= \{ u \in \mathbb{V} : E(v, w) = 0, \forall w \in V \}
\]
denotes the radical of \( H(v, w) \) and \( \overline{H} : \overline{V} \times \overline{V} \to \mathbb{C} \) the induced (non-degenerate) Hermitian form on \( \overline{V} = V/R \), then \( \overline{X} = V/R \) is a lattice in \( \overline{V} \) and \( X = \Lambda \to \mathbb{C} \) admits a map \( \overline{X} : \overline{X} \to \mathbb{C} \) such that \( (\overline{H}, \overline{X}) \in \mathcal{P}(\overline{V}, \overline{X}) \). Then, if \( p : V \to \overline{V} \) denotes the projection map, one checks that

\[
\text{Th}(\overline{H}, \overline{X}) \cong \text{Th}(H, X) \quad f \mapsto pf
\]

is a bijection.

2) If \( H \) is not positive, then

\[
\text{Th}(H, X) = 0
\]
3) By 1) and 2) we see:

If $L \cong L(H, X)$ is ample, then

$H$ is positive-definite (i.e., non-degenerate).

**Theorem 3.1.** A line bundle $L(H, X)$ is ample

if and only if $H \in \text{Herm}(V) = H^1(X)$ is positive-

definite.

In particular, $X$ is projective if $H$ is pos.

definite Hermitian form $H$ on $V$ with $V(\lambda x, x) < 2$.

- **Sketch (via Kodaira embedding theorem):**
  
  We had already seen that $L(H, X)$ ample $\Rightarrow$ $H$ positive.

  Conversely, suppose $H$ is ample. Via our
  identifications (foc 4.5) it follows that $H$
  $= c_1(L) \in H^2_{\text{top}}(X) = H^2(X, \mathbb{C})$ defines a positive
  $(1, 1)$-form. Thus $L(H, X)$ is positive line bundle
  in the sense of Kodaira (df. [G-HI], p. 148)
  and hence, by the Kodaira embedding theorem.

  ([G-HI], p 181), $L(H, X)$ is ample.

**Remark 3.2.** In place of using Kodaira's embedding

theorem, one can also deduce Th. 3.1 from the

following and more precise statement:

**Theorem 3.3.** Let $L = L(H, X)$ be a

line bundle such that $H$ is positive-definite.

Then $H^0(X, L^\otimes k)$ has no zero points for

$k > 2$ and yields a projective embedding for

$k \geq 3$.

(Will not prove)

This proof depends in part on having a

suitable base at one's disposal. Here the first

step is given by

**Theorem 3.4.** (Riemann--Roch): If $L = L(H, X)$

is positive (i.e., $H$ is positive-definite), then

(6) $h^0(X, L) = \sqrt{\text{det} (E_{\lambda \Lambda})} = \# \mathbb{K}(L),$

where $E = \text{Im}(\gamma)$. Thus, for any $n > 1$ we have:
\( \dim H^0(X, L^m) = \eta^g \dim H^0(X, L). \)

**Remark 35.** It is possible to deduce this theorem from the Hirzebruch–Riemann–Roch theorem:

\[ \chi(\sigma(L)) = \deg(\text{ch}(L) \cdot \text{td}(X)) \]

once one knows in addition:

1) \( \omega_x \equiv \Omega_x \quad (\Rightarrow \text{td}(X) = 1) \)

2) \( H^q(X, \Theta(L)) = 0 \), \( \forall q > 0 \).

This follows from Kodaira's vanishing theorem since \( \omega_x \equiv \Omega_x \) and \( L \equiv \text{positive} \).

3) \( c_1(L)^g = \eta^g \sqrt{\det(\text{Hess}(\varphi_{\omega}))} \).

Thus, Th 3.3 is truly a "Riemann–Roch theorem". However, the proof sketched below is much more elementary and explicit in that a canonical basis of \( H^0(X, L) \) will be constructed. Since this basis lies at the heart of the theory of theta-

functions, we sketch the constructive. First:

**Broué outline of proof of R–R:**

1) For a suitable \( \chi_0 \in \text{Ch}^1(H) \), construct a "base" theta–function \( \varphi_0 \in \Theta(H, X) \).

2) There exists \( \varphi \in \text{V} \) such that \( \chi = \chi_0 \ast (\varphi, \varphi) \).

Then the (modified) translate \( \varphi - t^*\varphi_0 \) lies in \( \Theta(H, X) \).

3) There is a finite subgroup \( \mathbb{K}_2 \subset \mathbb{K}(L) \)

such that \( \{ t^* \varphi \}_{t \in \mathbb{K}_2} \) is a basis of \( \Theta(H, X) \).

**Remark 36.** As we shall see, the "base \( \Theta \) function" \( \varphi_0 \) above is a suitable modification of the classical Riemann's \( \Theta \)–function which is defined as follows.

Let \( T \in \mathbb{F}_g := \{ T \in \mathbb{H}_g(\mathbb{C}) : T^t = T \, \text{(i.e. symmetric)} \}

and \( \text{Im} \, T > 0 \, \text{(i.e. positive)} \).
be an element of the Siegel upper 1/2-space \( \mathfrak{H}_g \). Then the Riemann \( \Theta \)-function is defined by

\[
\Theta(z, T) = \sum_{\tau \in \mathbb{H}} e^{\frac{1}{2}(\tau^T T \tau + \tau^T \bar{z})}, \quad z \in \mathbb{C}.
\]

(Note that since \( \text{Im} T > 0 \), the terms of the sum are bounded by \( e^{-c \text{Im} \tau} \), with \( c > 0 \), so this series converges absolutely.)

Thus, in case \( g = 1 \), we have \( f_g = f = \) usual upper 1/2-plane, and

\[
\Theta(z, T) = \sum_{\tau \in \mathbb{H}} e^{\pi i (\tau^T z + 2\tau \bar{z})}
\]

is precisely Jacobi's \( \Theta \)-function. It is this latter function, which Jacobi denoted by \( \Theta \) "by accident", that gives the theory its name: "Theta".

As a function of \( z \in \mathbb{C} \), the transformation laws of \( \Theta(z, T) \) are as follows:

\[
\begin{align*}
(8a) \quad & \Theta(z + \tau, T) = \Theta(z, T) \\
(8b) \quad & \Theta(z + Tm, T) = e^{-\frac{1}{2} \text{Im}(Tm + m^T \bar{z})} \Theta(z, T), \\
& \text{for } \tau, m \in \mathbb{Z}^g, \ i \in \mathbb{Z}.
\end{align*}
\]

To see that this is a theta function, let:

\[
\begin{align*}
\Lambda &= \mathbb{Z}^g + T \mathbb{Z}^g \subset \mathbb{C}^g \quad \text{(lattice)} \\
T &= X + iY, \quad X, Y \in \mathbb{H}_g(\mathbb{R}) \\
H(z_1, z_2) &= z_1^* Y^{-1} z_2, \quad \text{pos. def. Hermitian} \\
\text{Herm}(\mathbb{C}^g, \Lambda) \\
\chi_0(T + Tm) &= e^{\frac{1}{2} \text{Re} m}, \quad \chi_0 \in C_c^\infty(H).
\end{align*}
\]

Then \( \Theta \) is "almost" in \( \text{Th}(H, \chi_0) \); put

\[
\begin{align*}
(9a) \quad & \Theta_0(z) = e^{-\frac{i}{4} z^* Y^{-1} z} \Theta(z, T), \\
& \text{then we have} \\
(11) \quad & \Theta_0 \in \text{Th}(H, \chi_0).
\end{align*}
\]

\[\text{cf. [LL], p. 140.}\]
A sketch of Th. 3.4 (Lefschetz–Roch)

Lemma 3.7 (Frobenius) Let $H \in \text{Hom}(V, V)$ be positive definite. Then there exists a basis $\lambda_1, \cdots, \lambda_g$ of $V$ such that the (Gram) matrix of $E = \text{Im} H$ w.r.t. basis is

$$J_{\delta} := \begin{pmatrix} 0 & \Delta_{\delta} \\ -\Delta_{\delta} & 0 \end{pmatrix},$$

where $\Delta_{\delta} = \text{diag}(s_1, \ldots, s_g)$ with $s_i \in \mathbb{N}$ and $s_1 | s_2 | \cdots | s_g$. Furthermore, the vector $\delta = (s_1, \ldots, s_g)$ is uniquely determined by $H$.

For a vector $\delta = (s_1, \ldots, s_g)$ as in the lemma, put

$$K_1(\delta) = \bigoplus_{i=1}^g \mathbb{Z}/s_i\mathbb{Z},$$

$$K_1(\delta)^* = \text{Hom} \left( K_1(\delta), \mathbb{C} \right),$$

$$K(\delta) = K_1(\delta)^* \oplus K_1(\delta).$$

Note that $K(\delta)$ carries a unique symplectic form $\langle \cdot, \cdot \rangle$ defined by

$$\langle (h_1, x_1), (h_2, x_2) \rangle = h_1(x_2) - h_2(x_1).$$

Lemma 3.8. If $\{\lambda_1, \ldots, \lambda_g\} \rightarrow \text{Hom}_c V$ as in Lemma 3.7, then

$$V(H) = \bigoplus_{i=1}^g \mathbb{C} \lambda_i \oplus \bigoplus_{i=1}^g \mathbb{C} \lambda_i^c$$

and hence there is a symplectic isomorphism

$$\lambda : K(\delta) \rightarrow K(L) = V(H)/\Lambda.$$

Furthermore, if we put $e_i = \lambda_i(\delta^c)$, $1 \leq i \leq g$, then $\{e_1, \ldots, e_g\}$ is a $C$-basis of $V$ and if we write

$$\lambda = \sum_{i=1}^g \omega_{ij} e_j$$

then the $g \times g$ - matrix $\Omega = (\omega_{ij})$ has the form

$$\Omega = \begin{pmatrix} \Delta_{\delta} & T \\ -\Delta_{\delta} & -T \end{pmatrix},$$

where $T \in \mathfrak{g}_g$.

\[a\) Here, $K(L)$ has a symplectic structure via $\langle x, y \rangle \in C(E(\delta^c))$.\]
Remark 3.9 Conversely, if we fix a C-basis $\alpha_1, \ldots, \alpha_g$ of $V$ and let $\beta = (\beta_1, \ldots, \beta_g)$ and $T$ be given as in Lemma 3.8. Put

\begin{equation}
\Omega = \begin{pmatrix} \alpha_1 & \cdots & \alpha_g \\ \cdots & \cdots & \cdots \\ \alpha_g & \cdots & \alpha_1 \end{pmatrix}
\end{equation}

and $\Lambda = \mathbb{Z}^g \Omega (\beta_i) \subset V$.

Then, if we put

\begin{equation}
\Pi = \begin{pmatrix} \alpha_1 & \cdots & \alpha_g \\ \cdots & \cdots & \cdots \\ \alpha_g & \cdots & \alpha_1 \end{pmatrix}
\end{equation}

and $\Pi = \mathbb{Z}^g \Omega (\beta_i) \subset V$.

Choose a symplectic basis $\lambda_1, \ldots, \lambda_g$ of $\Lambda$ as in Lemma 3.7 and put

\begin{equation}
V_1 = \mathbb{C} \lambda_1 ; \quad V_2 = \mathbb{C} \lambda_2 + i (R - V, \lambda_1)
\end{equation}

Thus $V = V_1 \oplus V_2$. With respect to this decomposition define $\chi_0 : \Lambda \to \mathbb{C}^1$, by

\begin{equation}
\chi_0(\lambda) = \frac{1}{2} \Re (\beta_i, \lambda_i),
\end{equation}

where $\beta = \lambda_1, \ldots, \lambda_g$ is the above mentioned decomposition. Furthermore, let

\begin{equation}
B = (\Pi | V_1 \times V_2) \otimes \Pi,
\end{equation}

and put

\begin{equation}
\delta_0 (\nu, \tau) = e^{(-\frac{i}{2} B(\nu, \tau))} \sum e^{(\frac{i}{2} (H - B)(n - \frac{i}{2}, \lambda)}
\end{equation}

Then $\delta_0 \in \text{Th}(H, \chi_0)$; cf. [B-L] and [H1], p.2.

step 2. to $\delta_0 \in L(H, \chi)$ for suitable $\omega \in V$.

Lemma 3.10 If $f \in \text{Th}(H, \chi)$ then $\omega \in \text{Th}(H, \chi \circ (E \omega, \cdot ))$.

where

\begin{equation}
(\tilde{f}(\omega))(\nu) = e^{(\frac{i}{2} H(\nu, \omega))} f(\nu + \omega)
\end{equation}
Thus, since $\kappa_0 : \Lambda \to E_1^*$ is a character and $E_{1,\Lambda}$ non-degenerate, we can find $\omega = \omega_0 \in \Lambda$.

$\lambda = \kappa_0 \circ (E(\omega, \cdot))$

and so

$$\nu = t^* \delta_0 \in \text{Th}(H, \pi).$$

Remark 3.12. By using the theory of theta groups, one can give a representation-theoretic interpretation of this basis and show that it is unique (once such an identification $\Lambda$ has been chosen). From this one can then, following Mumford, build up an algebraic theory of theta functions (cf. [12] and [13], [14]).

Theorem 3.11. Let $\nu \in \text{Th}(H, \pi)$ be as above,

and let $\lambda : K(\mathfrak{g}) \to K(\text{L}(\mathfrak{g}, \pi))$ be an $\mathfrak{g}$-linear form.

as in Lemma 3.8. Then

$$\{ t_{\mathfrak{g}}(\lambda) \delta_{g} \}_{g \in K(\mathfrak{g})}$$

is a basis of $\text{Th}(H, \pi)$. Note that $t_{\mathfrak{g}}(\lambda) = \sqrt{\kappa(\mathfrak{g})}$.

Note. Once this basis has been set up properly, the proof is not difficult; cf. [14], pp. 27-31, [13], pp. 318-20.
Polarizations and moduli spaces

The proof of the K - R theorem of the previous section is closely related to the construction of moduli spaces: there are complex varieties which parametrize abelian varieties together with some additional structure such as a polarization, we are now define.

Definition. A polarization of an abelian variety $X$ is a homomorphism

$$\phi : X \rightarrow \hat{X} = \text{Pic}^0(X)$$

which is of the form

$$\phi = \phi_L$$

for some ample line bundle $L \in \text{Pic}(X)$.

Remark. Since $\text{Ker}(L) = \text{Ker}(\phi)$ is finite and

$$\dim X = \dim \hat{X},$$

it follows that $\phi_L$ is surjective.

If $L = L(H, X)$ and $L' = L(H', X)$

are two line bundles on $X$, then

$$\phi_L = \phi_{L'} \iff L \otimes L' \in \text{Pic}^0(X) \iff H = H'. $$

Thus, the set of polarizations $\text{Pol}(X) = \{ \phi_L \}$ can be canonically identified with the set

$$\text{Hom}(V, \Lambda) = \{ H \in \text{Hom}(V, \Lambda) : H \text{ pos, } \phi_L \}$$

via

$$H \mapsto \phi_L(H(x)).$$

By Lemma 3.7, each $H$ (or $\phi_L$) has a canonical sequence $S = (s_1, \ldots, s_g)$ attached to it; we have

$$s_1, \ldots, s_g = \sqrt{\text{det} \, \text{End} \Lambda} = \sqrt{\text{Ker}(L)}.$$

This sequence is called the type of the polarization. Note that Lemma 3.8 gives an intrinsic characterization of this seq. in terms of $[K(L) : \text{Ker}(\phi_L)].$

Definition. A polarized abelian variety of type $S = (s_1, \ldots, s_g)$ is a pair $(X, \phi)$ (or, equiv.,

---

This definition differs from that of [K], but is as in [H1] - [H2].
a pair \((X, H)\) where \(X\) is an abelian variety and \(\varphi \in \text{Pol}(X)\) (resp. \(\varphi \in \text{Hom}^+(V, W)\)).

Two such pairs \((X, \varphi)\) and \((X', \varphi')\) are isomorphic if \(\exists h : X \cong X'\) s.t. \(h^* \varphi' = \varphi\),

where \(h^* \varphi' = \varphi h^{-1}\) if \(\varphi' = \varphi L\), \(L \in \text{Aut}(X)\) (or \(h^* H' = H\)).

The moduli space of polarized abelian varieties of dimension \(g\) and type \(\delta = (\delta_1, \ldots, \delta_g)\) is:

\(A_g^{(\delta)} = \text{set of isomorphism classes of polarized abelian varieties \((X, \varphi)\) of type \(\delta\) and dim } X = g\).

By the lemmas of the previous section we see that we have a natural surjective map

\(\Pi_g : f_g \rightarrow A_g^{(\delta)}\)

via

\[\Pi_g \left( (A, B) \in \text{Sp}_g(\mathbb{Z})/\mathbb{T} \right) = (AT + B)(CT + D)^{-1} \in A_g^{(\delta)}\]

More generally, consider the group

\[\mathcal{G} = \{ A \in \text{SL}(g, \mathbb{Z})/\mathbb{T}: A^T \varphi_A A = \varphi \}\]

which also acts on \(f_g\) (in the same way).

\(\Pi_g = \{ A \in \text{SL}(g, \mathbb{Z})/\mathbb{T}: A^T \varphi_A A = \varphi \}\)
Prop. 4.1 The surjection $\pi_2 \to \pi_1$ is $\pi_1$-equivariant and induces a bijection

$$\pi_2 \big/ \phi_2 \to \pi_1 \big/ \phi_1 \, .$$

Thus, $\pi_1 \big/ \phi_2$ may be endowed with the structure of a complex space. In particular, if $\xi = (1, \ldots, 1)$ then $\pi_2 = \text{Spec}(\mathbb{Z}) / \pi_1$ and so we have

$$\pi_1 \big/ \phi_2 \cong \text{Spec}(\mathbb{Z}) / \pi_1 \, .$$

Hint: 1) $\pi_1$ is called the moduli space of principally polarized abelian varieties.

2) It is more difficult to show that $\pi_2$ and $\pi_1 \big/ \phi_2$ are quasi-projective. This is done by "evaluating the canonical bound" (found in the previous section) at $0$, cf. [H2],[H13] and [H2] for details.

35. Symmetric theta divisors

It seems natural to call an effective divisor $D \geq 0$ on $X$ a theta divisor if the pullback

$$\pi^* D = (S)$$

is the divisor of zeros of some (normalized) theta function on $V$. However, such a definition is superfluous since we have, by our identification of theta divisors as sections of line bundles that:

Prop. 5.1 Every effective divisor $D \geq 0$ on $X$ is of the form (1) for a suitable $S$.

For the purposes of these notes, let us therefore make the following (not universally accepted) definition.

Definition. A theta divisor is an effective divisor $D \geq 0$ on $X$ such that its associated
line bundle $L = L(D)$ induces a principal polarization, i.e. an isomorphism $\phi_L : X \to X^*$. For a principal polarization $\phi : X \to X^*$ (or $H$), let $\Theta_H = \phi_H$ the theta divisor.

Remark 5.2: By Riemann-Roch, a divisor $D$ on a theta divisor $\Theta_H$ has $\ell^0(X, L(D)) = 1$. Thus:

2) $\Theta_H + \phi : X \to X^*$.

3) If $H, H'$ are two principal polarizations, then:

- $\Theta_H \cong \Theta_{H'}$, for some $\phi \in \Theta_H$.
- $\Theta_H \cong \Theta_{H'}$ for some $\phi \in \Theta_H$.

4) If $\Theta_H$, then $\Theta_H = \{ T_x \phi : x \in X \}$.

Of particular interest are symmetric theta divisors; these are the divisors $\Theta$ satisfying

$$\Theta'' = i^* \Theta - \Theta$$

(6) If $\Theta_H^{\text{sym}} = \{ T_x \phi : x \in X \}$

Moreover, if $\Omega \in \Theta_H^{\text{sym}}$, then

$$\Theta_H^{\text{sym}} = \{ T_x \phi : x \in X \}$$

(7) $\Theta_H^{\text{sym}} = \{ T_x \phi : x \in X \}$.
where, as usual, \( X[2] = 3 x \in X : 2x = 0 \).

**Prop. 4.** Suppose first that \( \Theta \in \Theta^\infty_0 \). Then
\[
\tau^* \Theta_0 = \tau_0^* \tau^* \Theta_0 = \tau_0^* \Theta_0,
\]
so \( \Theta = \tau^* \Theta_0 \in \Theta_0 \) is symmetric \(\iff\) \( \tau_0^* \Theta_0 = \tau_0^* \Theta_0 \)
\(\iff\) \( \tau_0^* \Theta_0 = \Theta_0 \iff 2x = 0 \iff x \in X[2] \). This proves (1). Thus, to prove (6), it is enough to show that \( \Theta^\infty_0 + \Theta \).

First proof: Given \( H \) (and \( X \)), we can a suitable \( T \) by the period matrix \( \Omega = (T^3 + T^3) \). By example 5.3, \( \Theta \in \Theta_0 \), then \( \Theta \) is a symmetric \( \Theta \)-divisor.

Second proof: Let \( L = L(H, X) \) be a line bundle (for some \( \chi \)). Then \( \tau^* L \cong L(\tau^* H, \tau^* \chi) = L(H, \tau^* \chi) \).

Since \( \tau^* \chi \) is a character and \( E \) is rank one, we can find \( V \) such that
\[
(\tau^* \chi)_{|V} = \chi(E) \epsilon(V, E).
\]

Put \( X = \chi_{|V} \epsilon(E) \epsilon(V, E) \); then \( \tau^* X = X \)
and so \( \tau^* L(H, X) = L(H, X) \). Then, if \( \Theta \) is the divisor of \( \chi \) of \( L(H, X) \) we have \( \tau^* L(\Theta) = \Theta \), so \( \tau^* \Theta = \Theta \).

**Remark 5.5.** 1) The divisors in \( \Theta^\infty_0 \) are often called **theta characteristics** (of the polarization \( H \)).

Note if \( \Theta \) denote the moduli space which classifies \( \Theta \)-divisors \( (H, \Theta) \), where \( \Theta \) is a symmetric \( \Theta \)-divisor (theta char.), then Prop. 5.4 states that the map
\[
\pi : \Theta^\infty_0 \rightarrow \Theta,
\]
\( (X, \Theta) \rightarrow (X, \pi_*(\Theta)) \)
is a surjective cover of degree \( 2^g \).

2) Note that \( \Theta \)-characteristics are not totally homogeneous, for one can distinguish between odd and even characteristics (cf. [M2]).

**Example 5.6.** Recall that the Jacobian \( \mathfrak{J}_C \) of a curve \( C \) comes equipped with a theta divisor \( \Theta \) defined by the theta function \( \Theta_\theta(\tau) \) attached to the period lattice \( \Omega = \mathbb{Z}^g + \mathbb{Z}^g \) of \( C \). (Note that \( \Theta_\theta \) is symmetric!)
On the other hand, by fixing a base point \( p_0 \in C \), the Abel-Jacobi map
\[
\varphi_{p_0} : C^{(g-1)} \to J_C
\]
\[
\Delta \mapsto D - (\gamma_1 - \gamma_2) p_0
\]
defines a divisor \( W_{p_0} = W_{p_0}^{(g-1)} \) (which depends on the choice of \( p_0 \)). Since both have the same Chern class, we see that
\[
\tau_x(W_{p_0}^{(g-1)}) = 0
\]
for some \( x \in J_C \). While \( x \) will depend on the choice of \( T \) (i.e., the theta characteristic), we do have:
\[
2x = \varphi((\gamma_1 - \gamma_2) p_0 - \omega_C)
\]
\[
= -\varphi_{p_0}(\omega_C)
\]
(c.f. [64], p. 340).

86. Hamiltonian structures on line bundles and Arakelov theory

In [7], Faltings defined a canonical, possibly non-Hamiltonian metric on \( L - L(\Theta) \), where
\( \Theta \) is a symmetric \( \Theta \)-divisor (attached to a principal polarization) of an abelian \( X \), as follows:

1) If \( \Theta = (\Theta) \) is the divisor of a theta function \( \Theta = \Theta(z,T) \), then put
\[
||L_{(\Theta)}(z)|| = \sqrt{\text{det}(z)} \cdot \varTheta \left( \frac{1}{2} H(1,2) \right) \delta(1,1)
\]

2) More generally, if \( \Theta = \varphi_\ast \Theta_0 \otimes \mathcal{O}_X \) is another symmetric \( \Theta \)-divisor, then define:
\[
||L_{(\Theta)}(z)|| = ||L_{(\Theta_0)}(z-x)||
\]

As Faltings remarks, these metrics can be characterized by the property they are invariant under translation. In fact, these metrics...
Theorem 1. (Hortz-Baily): There is a unique way of attaching to each pair $(X, L)$, where $X$ is an abelian variety and $L$ a line bundle on $X$, a set $\pi(X, L)$ of positive hermitian $\mathbb{C}^2$ metrics on $L$ such that:

1. If $u : L_1 \cong L_2$ is an isomorphism, then $u(\pi(X, L)) = \pi(X, L)$.
2. $\pi(X, X \times L) = \text{set of constant metrics}$.
3. $\pi(X, L_1) \otimes \pi(X, L_2) \subset \pi(X, L_1 \otimes L_2)$.
4. If $f : X \to X_2$ is a morphism, $L_2 \in \text{Pic}(X_2)$ then $f^* \pi(X, L_2) \subset \pi(X, f^* L_2)$.

Moreover, each $\pi(X, L) \neq \emptyset$, and if $p \in \pi(X, L)$ then $\pi(X, L) = \{ \lambda p : \lambda \in \mathbb{C}^* \}$. Furthermore, $\pi(X, L) = \{ \rho : \text{the curve } \gamma \rho \text{ is translation invariant} \}$.

Ref. [HBI], pp 50-52; cf. also [HBI], p. 48ff.

Remark: This characterization is analogous to Neveu's class of K-functors.

References


