The Ring of Modular Correspondences

1. Introduction.

Let: \( \Gamma(N) = \text{Ker}(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})) \), the principal congruence subgroup of level \( N \),
\( X(N) = \Gamma(N) \backslash \mathfrak{H}^* \), the associated modular curve,
\( J(N) = \text{Jac}(X(N)) \), its Jacobian variety,
\( \mathbb{E}_N = \text{End}^0(J(N)) = \text{End}(J(N)) \otimes \mathbb{Q} \), the endomorphism algebra of \( J(N) \) (Ring of correspondences),
\( \mathbb{M}_N = \sum \mathbb{Q} \mathcal{T}_A \subset \mathbb{E}_N \), the subring of modular correspondences.

Here, the above sum runs over all matrices \( A \in \text{GL}_2(\mathbb{Q})^+ \), and
\( \mathcal{T}_A = \mathcal{T}^{(N)}_A : J_N \to J_N \),
is the endomorphism defined by the curve (correspondence) \( C_A \) on \( X_N \times X_N \) which is the image of the curve
\( \tilde{C}_A = \{(z, A(z)) : z \in \mathcal{H}^* \} \subset \mathcal{H}^* \times \mathcal{H}^* \).

Note: Modular correspondences were introduced by Klein (1879) and were studied by him and by Hurwitz (1883–87). The book of Klein/Fricke (1893) gives a systematic exposition of the theory. Their discussion suggests the following:

Questions: 1) When is every correspondence on \( X_N \) modular, i.e., when is \( \mathbb{M}_N = \mathbb{E}_N \)?
2) How large can \( \delta_N := \dim \mathbb{E}_N - \dim \mathbb{M}_N \) be?
3) What is the growth rate of \( \dim \mathbb{E}_N \) (and of \( \dim \mathbb{M}_N \)) as \( N \to \infty \)?
Remarks: 1) We have the trivial upper bound
\[ \dim \mathbb{E}_N \leq 2g_N^2, \]
where \( g_N \) denotes the genus of \( X(N) \), i.e.
\[ g_N = 1 + \frac{1}{24} \phi(N)\psi(N)(N - 6) \approx N^3, \]
where \( \psi(N) = N \prod_{p|N}(1 + \frac{1}{p}) \) is the Dedekind \( \psi \)-function. Thus, we shall measure the growth rate in terms of \( g_N \).

2) As we shall see, the answer to Question 3) sheds some light on the more general question of determining the growth rate of the function
\[ d_g := \max \{ \dim \text{End}(J_X) : X/\mathbb{C} \text{ is a curve of genus } g \} \leq 2g^2. \]
This question is partially related to the questions asked by Ellenberg (2001) concerning the growth rate of certain subalgebras of \( \text{End}(J_X) \).

3) Since \( X(N) \) has a canonical model \( X(N)/\mathbb{Q} \) over \( \mathbb{Q} \), we could also ask the corresponding questions for the subalgebra
\[ \mathbb{E}_N^\mathbb{Q} := \text{End}(J(N)/\mathbb{Q}) = \text{End}(J(N))^{\text{Gal}(\mathbb{Q}/\mathbb{Q})} \]
of endomorphisms which are defined over \( \mathbb{Q} \). It turns out that this situation is much easier to analyze.
2. Main Results.

Let: \( \mathcal{K} = \{ \mathbb{Q}(\sqrt{-n}) \}_{n \geq 1} \) be the set of imag.-quad. fields,
\( h(D) \) the class number of (forms of) discriminant \( D \), so
\( h_K = h(d_K) \) is the class number of \( K \), where \( d_K = \text{disc}(K) \).

**Theorem 1:** If \( N \geq 1 \), then
\[
\mathbb{M}_N = \mathbb{E}_N \iff h(N^2/d_K) = 1, \ \forall K \in \mathcal{K} \text{ with } d_K|N.
\]
\[
\iff \text{either: } 4 \nmid N \text{ and } p \equiv 1 \pmod{4}, \ \forall p|N, p \neq 2,
\]
\[
\text{or: } N \in \{3, 4, 6, 7, 8, 9, 11, 14, 19, 43, 67, 163\}.
\]

**Remark:** The second equivalence uses the resolution of the class number 1 problem (Heegner, Stark).

**Examples:**
(a) \( \mathbb{M}_N = \mathbb{E}_N \), if \( N \leq 11 \) or if \( N = 13, 14, 17, 25, \ldots \)

(b) \( \mathbb{M}_N \neq \mathbb{E}_N \), if \( N = 12, 15, 16, 18, 20, 21, 22, 23, 24, \ldots \)

**Theorem 2:** If \( N \geq 30 \), then
\[
3^{4/3} \sqrt[3]{4} \ g_N^{4/3} \leq \dim \mathbb{M}_N \leq \psi(N)g_N + O(g_N^{1+\varepsilon})
\]
\[
\leq \log \log(g_N)g_N^{4/3} + O(g_N^{1+\varepsilon}).
\]

**Remark:** This uses the recent result of Solé/Planat (2011):
\[
\psi(N) \leq e^{\gamma} \log \log(N)N, \text{ if } N \geq 30.
\]

**Theorem 3:** For any \( \varepsilon > 0 \)
\[
\delta_N := \dim \mathbb{E}_N - \dim \mathbb{M}_N = O(g_N^{4/3+\varepsilon}) \text{ but } \delta_N \neq O(g_N^{4/3-\varepsilon}).
\]
Thus
\[
\dim \mathbb{E}_N = O(g_N^{4/3+\varepsilon}).
\]
If we restrict $N$ to primes and/or to prime powers, then more can be said.

**Theorem 4:** If $N = p$ is prime, then

$$\dim(\mathbb{M}_p) = 2\sqrt[3]{3} \ g_p^{4/3} + O(g_p).$$

Moreover, for any $\varepsilon > 0$

$$\dim(\mathbb{E}_p) = 2\sqrt[3]{3} \ g_p^{4/3} + O(\log(g_p)^2 g_p).$$

**Remark:** This follows from a preprint of K.-Mohit, which gives an explicit formula for $\dim \mathbb{M}_p$.

**Theorem 5:** If $N = p^r$ is a prime power with $p \equiv 3 \pmod{4}$, then

$$\delta_N = 24 \sqrt[3]{3} \ \frac{h(-p)^2}{p^2} \ g_N^{4/3} + O(\log(p)g_N).$$

**Remarks:**
1) Thus, the “error term” $\delta_N$ is almost as large as (the lower bound for) $\dim \mathbb{E}_N$.

**Theorem 6:** (a) Every $\mathbb{Q}$-rational endomorphism $f \in \mathbb{E}^\mathbb{Q}_N$ is modular, i.e.,

$$\mathbb{E}^\mathbb{Q}_N = \mathbb{M}^\mathbb{Q}_N := (\mathbb{M}_N)^{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}.$$ 

(b) If $N > 12$, then

$$g_N < \dim \mathbb{E}^\mathbb{Q}_N \leq \sigma_0(N^2)g_N;$$

In particular, $\dim \mathbb{E}^\mathbb{Q}_N = O(g_N^{1+\varepsilon})$, for all $\varepsilon > 0$.

**Remark:** Theorem 6 follows easily from the results of K. (2008).
3. Basic Ingredients.

**Main Steps:** 1) Use Atkin-Lehner theory etc., to study the action of $\overline{M}_{N,C} := M_N \otimes \mathbb{C}$ on the space

$$V = S_2(\Gamma(N)),$$

and use this to determine the algebra structure of

$$\overline{M}_{N,C} = \text{the image of } M_{N,C} \text{ in } \text{End}_\mathbb{C}(V).$$

2) Determine $\text{Ker}(M_{N,C} \to \overline{M}_{N,C})$. For this, it is useful to split $V$ as

$$V = V^{nCM} \oplus V^{CM}$$

where $V^{CM}$ is the subspace of forms with complex multiplication (CM) and $V^{nCM}$ the subspace of forms without CM, and to study the action on these spaces separately. This leads to the decomposition

$$\overline{M}_{N,C} = \overline{M}_{N,C}^{nCM} \oplus \overline{M}_{N,C}^{CM}.$$

3) Using the results of Ribet (1980), study the structure of $E_N$. For this, note that we have an isogeny decomposition

$$J(N) \sim J(N)^{nCM} \times J(N)^{CM}$$

which induces algebra decompositions

$$E_N = E_N^{nCM} \oplus E_N^{CM} \text{ and } M_N = M_N^{nCM} \oplus M_N^{CM},$$

where $E_N^{nCM} = \text{End}(J(N)^{nCM}) \otimes \mathbb{Q}$, $E_N^{CM} = \text{End}(J(N)^{CM}) \otimes \mathbb{Q}$, and $M_N^{nCM} = E_N^{nCM} \cap M_N$ and $M_N^{CM} = E_N^{CM} \cap M_N$. 
4. The Structure of $\overline{M} = \overline{M}_{N,\mathbb{C}}$.

**Notation:** Let $\mathcal{N}(V)$ denote the set of normalized newforms (of all levels) in $V = S_2(\Gamma(N))$.

**Observation:** The group $A_N^*$ of characters on $A_N := (\mathbb{Z}/N\mathbb{Z})^\times$ acts on $\mathcal{N}(V)$. (This action is induced by twisting.) Let

$$\overline{\mathcal{N}}(V) = \mathcal{N}(V)/A_N^* \text{ (orbit space)}.$$

**Theorem 7:** $V$ has multiplicity one as an $\overline{M}$-module. More precisely, if $V(f)$ denotes the $\overline{M}$-module generated by $f \in \mathcal{N}(V)$, then $V(f)$ is irreducible and one has the decomposition

$$V = \bigoplus_{f \in \overline{\mathcal{N}}(V)} V(f)$$

into pairwise non-isomorphic $\overline{M}$-modules. Thus

$$\dim_{\mathbb{C}} Z(\overline{M}) = |\overline{\mathcal{N}}(V)| = |\mathcal{N}^{nCM}|/\phi(N) + 2|\mathcal{N}^{CM}|/\phi(N),$$

where $\mathcal{N}^{nCM} = \mathcal{N}(V) \cap V^{nCM}$ and $\mathcal{N}^{CM} = \mathcal{N}(V) \cap V^{CM}$.

**Theorem 8:** If $N \geq 5$, then

$$|\mathcal{N}(V)| = \frac{1}{24} \phi(N)^2(\psi(N) - 6),$$

and hence $\dim_{\mathbb{C}} Z(\overline{M}) \leq \frac{1}{12} \phi(N)(\psi(N) - 6)$.

**Theorem 9 (K.-Mohit):** We have that

$$\dim_{\mathbb{C}} V(f) \leq \psi(N), \text{ for all } f \in \mathcal{N}(V).$$

**Remark:** In our preprint we give in a precise formula for $\dim V(f)$. (This uses the results of Atkin/Li (1978).)
Corollary: If $N \geq 5$, then
\[
\sqrt[3]{\frac{3}{4}} g_N^{4/3} \leq \dim_{\mathbb{C}} \bar{M} \leq \psi(N) g_N.
\]

Remark: This follows from Theorems 7-9 because of the following simple fact:

Lemma: Let $A \subset \text{End}_{\mathbb{C}}(V)$ be a $\mathbb{C}$-algebra such that $V$ is semisimple and has multiplicity one as an $A$-module. Then
\[
\frac{g^2}{z} \leq \dim_{\mathbb{C}} A \leq Mg,
\]
where $g = \dim_{\mathbb{C}} V$, $z = \dim_{\mathbb{C}}(Z(A))$, and
\[
M = \max\{\dim(V_i) : V_i \subset V \text{ is irreducible}\}.
\]

Remark: More precisely, if
\[
V = \bigoplus_i V_i
\]
is the decomposition of $V$ into irreducible $A$-modules, then
\[
\dim A = \sum (\dim V_i)^2.
\]
5. The Kernel of $\mathcal{M}_{N,\mathbb{C}} \to \overline{\mathcal{M}}_{N,\mathbb{C}}$.

**Theorem 10:** We have that

$$\dim_{\mathbb{C}}(\text{Ker}(\mathcal{M}_{N,\mathbb{C}} \to \overline{\mathcal{M}}_{N,\mathbb{C}})) = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{N,\mathbb{C}}^{CM}$$

and so

$$\dim \mathcal{M}_N = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{N,\mathbb{C}}^{ncM} + 2 \dim_{\mathbb{C}} \overline{\mathcal{M}}_{N,\mathbb{C}}^{CM}.$$ 

**Remark:** This result is proved by studying the $\mathcal{M}_N$-structure of the module $H^1(X(N), \mathbb{C}) \simeq V \oplus V^*$. Here $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the contragredient $M_{N,\mathbb{C}}$-module. A key ingredient is:

**Theorem 11:** If $f \in \mathcal{N}(V)$, then

$$V(f)^* \simeq_{\mathcal{M}} V(f) \iff f \in \mathcal{N}^{nCM}.$$ 

**Theorem 12:** The number of CM-newforms in $V$ is

$$|\mathcal{N}(V)^{CM}| = \frac{\phi(N)}{2} \sum_{d_K|N} h(N^2/d_K),$$

where the sum is over all (fundamental) discriminants of imaginary quadratic fields $K \in \mathcal{K}$ with $d_K|N$. Thus

$$\dim_{\mathbb{C}} Z(\overline{\mathcal{M}}_{N,\mathbb{C}}^{CM}) = \sum_{d_K|N} h(N^2/d_K),$$

and

$$\dim \mathcal{M}_{N,\mathbb{C}}^{CM} = O(N^{3+\varepsilon}) = O(g_N^{1+\varepsilon}), \quad \forall \varepsilon > 0.$$ 

In particular,

$$\dim \mathcal{M}_N = \dim \overline{\mathcal{M}}_N + O(g_N^{1+\varepsilon}).$$
6. The Structure of $\mathbb{E}_N$.

**Notation:** If $K \in \mathcal{K}$, let $V_K^{CM}$ denote the subspace of $V$ of forms with CM by $\psi_K = \psi_{d_K} = \left(\frac{d_K}{\cdot}\right)$, and put $s_K = \dim \mathbb{C} V_K^{CM}$.

**Theorem 13:** If $N \geq 5$, then

$$J(N)^{CM} \sim \prod_{d_K | N} E_K^{s_K},$$

where $E_K/\mathbb{C}$ is any elliptic curve with $\text{End}^0(E_K) = K$. Thus

$$\mathbb{E}_N^{CM} \simeq \bigoplus_{d_K | N} M_{s_K}(K)$$

and so $\dim \mathbb{E}_N^{CM} = 2 \sum_{d_K | N} (s_K)^2$.

**Remark:** This follows from the results of Shimura (1976), Ribet (1980) and Theorem 11. Moreover, from the results of Ribet and Theorems 7 and 10 we obtain:

**Theorem 14:** $\mathbb{E}_N^{nCM} = \mathbb{M}_N^{nCM}$.

**Corollary:** If $N \geq 5$, then

$$\dim Z(\mathbb{M}_N) - \dim Z(\mathbb{E}_N) = 2 \sum_{d_K | N} (h(N^2/d_K) - 1),$$

and

$$\dim \mathbb{E}_N - \dim \mathbb{M}_N \leq 2 \sum_{d_K | N} s_K h(N^2/d_K)(h(N^2/d_K) - 1).$$

**Remark:** This corollary implies Theorem 1!