Tensor Products of Galois Representations

Ernst Kani
Queen’s University
Kingston, Ontario

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1. Introduction

Motivation 1: The Hasse-Weil $\zeta$-Function of Products

Recall: The zeta function $\zeta_{X/K}(s)$ of a smooth, projective variety $X$ over a number field $K$ is given (up to finitely many Euler factors) by the zeta function

$$\zeta_X(s) := \prod_{x \in |X|} (1 - N(x)^{-s})^{-1},$$

of any projective model $X/\mathcal{O}_K$ of $X/K$. (This product converges for $\Re(s) > \dim X + 1$.) Write: $\zeta_{X/K}(s) \sim \zeta_X(s)$.

Example: If $X = \text{Spec}(K)$, then $\zeta_{X/K}(s) \sim \zeta_{\text{Spec}(\mathcal{O}_K)}(s) = \zeta_K(s)$, the Dedekind $\zeta$-function of $K$.

Main Principle: $\zeta_{X/K}(s)$ “encodes the arithmetic” of $X$: this is expressed in terms of specific conjectures (see below).

Fact (Grothendieck/Artin/Serre): We have the factorization

$$\zeta_{X/K}(s) \sim \prod_{m=0}^{2d} L_m(X, s)^{(-1)^m} = \frac{L_0(X, s) \cdots L_{2d}(X, s)}{L_1(X, s) \cdots L_{2d-1}(X, s)}$$

where $d = \dim X$ and each $L_m(X, s) = L(\rho_m, s)$ is the $L$-function associated to a suitable (rational) compatible system $\rho_m = \{\rho_{m,\ell}\}_{\ell}$ of $\ell$-adic Galois representations

$$\rho_{m,\ell} = \rho_{X,m,\ell} : G_K = \text{Gal}(\overline{K}/K) \to \text{Aut}_{\mathbb{Q}_\ell}(V_{X,m,\ell}).$$

(Explicitly: $V_{X,m,\ell} = H^m_{et}(X, \mathbb{Q}_\ell)$.)
**Example 1:** If $X/K$ is a curve, then

$$\zeta_{X/K}(s) \sim \zeta_K(s)\zeta_K(s-1)L_1(X,s)^{-1}.$$  

Here $L_1(X,s) = L(\rho_{J_X/K},s)$ is the $L$-function associated to the system $\rho_{J_X/K} = \{\rho_{J_X/K,\ell}\}_\ell$ of Galois representations $\rho_{J_X/K,\ell} : G_K \to \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(J_X))$ afforded by the Tate space of the Jacobian $J_X/K$ of $X$:  

$$V_\ell(J_X) = T_\ell(J_X) \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell, \text{ where } T_\ell(J_X) = \lim_{\leftarrow} J_X[\ell^n].$$

**Example 2:** If $X = X_1 \times X_2$ is a product surface, then (by the Künneth formula) we have

$$\zeta_{X/K}(s) \sim \frac{\zeta_{X_1}(s)\zeta_{X_2}(s)\zeta_{X_1}(s-1)\zeta_{X_2}(s-1)}{\zeta_K(s)\zeta_K(s-1)^2\zeta_K(s-2)}L(\rho_1 \otimes \rho_2, s).$$

Here $\rho_1 \otimes \rho_2 = \{\rho_{J_{X_1}/K,\ell} \otimes \rho_{J_{X_2}/K,\ell}\}_\ell$ is the system of Galois representations afforded by the $\ell$-adic tensor product modules

$$V_\ell(J_{X_1}, J_{X_2}) := V_\ell(J_{X_1}) \otimes_{\mathbb{Q}_\ell} V_\ell(J_{X_1}).$$

Thus, the “new contribution” of the $\zeta$-function $\zeta_X(s)$ is the part coming from the $L$-function $L(\rho_1 \otimes \rho_2, s)$ associated to the tensor product representation $\rho_1 \otimes \rho_2$.

**Remark:** In the case that $X_1 = X_2 = E$ is an elliptic curve, then $V_\ell(E,E) \simeq S^2(V_\ell(E)) \oplus \text{det}(V_\ell(E))$, so $\rho_1 \otimes \rho_1$ is essentially the symmetric square representation.
Motivation 2: The Tate Conjectures

Conjecture 1 (Hasse/Weil): $\zeta_{X/K}(s)$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$.

Refinement: Each $L_m(X, s)$ has a meromorphic continuation to $\mathbb{C}$. (Note: $L_m(X, s)$ converges for $\Re(s) > 1 + \frac{m}{2}$.)

Conjecture 2 (Tate): The order of the pole of $L_{2m}(X, s)$ at $s = m+1$ equals the rank of the group $\mathcal{A}^m(X/K) = Z^m(X)/\equiv$ of codimension $m$-cycles on $X/K$ modulo numerical equivalence:

$$-\text{ord}_{s=m+1}L_{2m}(X, s) = \text{rk}(\mathcal{A}^m(X/K)).$$

Furthermore, for all primes $\ell$,

$$(T^m(X)) \text{ rk}(\mathcal{A}^m(X/K)) = \dim_{\mathbb{Q}_\ell}(H^m_{\text{et}}(X, \mathbb{Q}_\ell)(m))^{G_K}.$$

Remarks: 1) Tate (1963) also has a conjectural interpretation of $\text{ord}_{s=1}L_1(X, s)$ which generalizes the Birch/Swinnerton-Dyer Conjecture.

2) In the case that $X = A \times \hat{A}$, where $A/K$ is an abelian variety and $\hat{A}$ is its dual, Conjecture $T^1(A \times \hat{A})$ is equivalent to the following statement which was proved by Faltings (1983).

Theorem 0 (Faltings): If $A/K$ is an abelian variety, then

1) $\text{rk}(\text{End}_K(A)) = \dim_{\mathbb{Q}_\ell}\text{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A)), \ \forall$ primes $\ell$.

More precisely, we have a natural $\mathbb{Q}_\ell$-linear ring isomorphism

2) $\tau_{A/K, \ell} : \text{End}_K(A) \otimes \mathbb{Q}_\ell \sim \text{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A))$. 
2. Some Questions

**Notation:** If \( \rho_i : G_K \to \text{Aut}_{\mathbb{Q}_\ell}(V_i) \) are two \( \ell \)-adic Galois representations, then put

\[
(\rho_1, \rho_2)_{G_K} := \dim_{\mathbb{Q}_\ell} \text{Hom}_{\mathbb{Q}_\ell[G_K]}(V_1, V_2).
\]

This is often called the intertwining number of \( \rho_1 \) and \( \rho_2 \).

**Example:** If \( A/K \) is an abelian variety, then by Faltings

\[
(\rho_{A/K, \ell}, \rho_{A/K, \ell})_{G_K} = \text{rk}(\text{End}_K(A)).
\]

**Notation:** If \( A/K \) and \( B/K \) are two abelian varieties, let

\[
V_\ell(A, B) = V_\ell(A) \otimes_{\mathbb{Q}_\ell} V_\ell(B)
\]

be the tensor product of the Tate spaces and let

\[
\rho_{A,B,K,\ell} := \rho_{A/K, \ell} \otimes \rho_{B/K, \ell} : G_K \to \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(A, B))
\]

be the associated tensor product representation.

**Question 1:** Is there a “Faltings Theorem” for \( \rho_{A,B,K,\ell} \), i.e., is there an arithmetic interpretation of \( (\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} \)?

**Observation:** The Tate Conjecture \( (T^2(A \times B \times \hat{A} \times \hat{B})) \) implies:

\[
\exists \text{ a subgroup } \mathfrak{A}_{A,B,K} \leq \mathfrak{A}^2((A \times B \times \hat{A} \times \hat{B})/K) \text{ such that }
\]

\[
(\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} = \text{rk}(\mathfrak{A}_{A,B,K}), \quad \forall \ell.
\]

In particular, the left hand side is independent of \( \ell \).

**Subproblem:** Can \( \mathfrak{A}_{A,B,K} \) be interpreted in terms of endomorphisms of abelian varieties?
**Remark:** It is not difficult to see that we have a natural embedding

\[ \text{End}_K(A) \otimes \text{End}_K(B) \hookrightarrow \mathcal{A}_{A,B,K}. \]

This naturally raises the following question.

**Question 2:** When is \( \text{rk}(\text{End}_K(A) \otimes \text{End}_K(B)) = \text{rk}(\mathcal{A}_{A,B,K})? \)

Combining Questions 1 and 2 leads to:

**Question 3:** Let \( \tau_{A,B,K,\ell} = \tau_{A/K,\ell} \otimes_{\mathbb{Q}_\ell} \tau_{B/K,\ell}, \) so \( \tau_{A,B,K,\ell} \) can be viewed as a ring homomorphism

\[ \tau_{A,B,K,\ell} : \text{End}_K(A) \otimes \text{End}_K(B) \otimes \mathbb{Q}_\ell \to \text{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A, B)) \]

via the identification

\[ \text{End}_{\mathbb{Q}_\ell}(V_\ell(A, B)) \simeq \text{End}_{\mathbb{Q}_\ell}(V_\ell(A)) \otimes_{\mathbb{Q}_\ell} \text{End}_{\mathbb{Q}_\ell}(V_\ell(B)). \]

When is \( \tau_{A,B,K,\ell} \) an isomorphism? In other words, when is

\[ (3) \quad (\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} = \text{rk}(\text{End}_K(A)) \text{rk}(\text{End}_K(B))? \]

A first (naive) guess is that the following holds.

**Hypothesis** \( H_{A,B,K} : \) The following are equivalent:

(i) Formula (3) holds for all primes \( \ell; \)

(i') Formula (3) holds for one prime \( \ell; \)

(ii) \( \text{Hom}_{\overline{K}}(A, B) = 0. \)

**Observation:** While \( H_{A,B,K} \) holds for some abelian varieties \( A/K \) and \( B/K, \) it is not true in general. There are (at least) two classes of counterexamples.
Counterexamples to Hypothesis $H_{A,B,K}$:

1) $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties which have a common internal twist (in the sense of Ribet);
2) $A/K$ and $B/K$ are CM elliptic curves which are defined over $\mathbb{Q}$ and $K$ is a suitable real quadratic field.

Remark: Note that $\text{End}_K(A) \neq \text{End}_\mathbb{Q}(A)$ in both cases. Thus, a better guess is the following:

Hypothesis $\overline{H}_{A,B}$: The hypothesis $H_{A,B,K}$ holds whenever $K$ is large enough, i.e., whenever

$$\text{End}_K(A) = \text{End}_\mathbb{Q}(A) \text{ and } \text{End}_K(B) = \text{End}_\mathbb{Q}(B).$$

Observation: If $\overline{H}_{A,B}$ holds for $A/K$ and $B/K$, and if (ii) holds (i.e., if $\text{Hom}_K(A, B) = 0$), then for every finite extension $L/K$ and prime $\ell$ we have an induced isomorphism

$$\tilde{\tau}_{A,B,L} : (\text{End}_K(A) \otimes \text{End}_K(B))^{G_L} \otimes \mathbb{Q}_\ell \cong \text{End}_{\mathbb{Q}_[G_L]}(V_\ell(A, B)).$$

From this it follows that

$$\mathfrak{A}_{A,B,K} \otimes \mathbb{Q} = (\text{End}_K(A) \otimes \text{End}_K(B))^{G_L} \otimes \mathbb{Q},$$

and so we obtain a solution of our subproblem in this case. In particular, $H_{A,B,L}$ holds if and only if

$$(\text{End}_K(A) \otimes \text{End}_K(B))^{G_L} \otimes \mathbb{Q} = \text{End}_L(A) \otimes \text{End}_L(B) \otimes \mathbb{Q}.$$
3. Main Results.

**Theorem 1.** If \( A \) and \( B \) are isogenous (over \( \overline{\mathbb{Q}} \)) to products of elliptic curves, then \( H_{A,B} \) holds.

**Definition:** A *modular abelian variety* \( A/K \) is an abelian variety which is isogenous to a quotient of the Jacobian variety \( J_1(N)_K \) of the modular curve \( X_1(N)_K \), for a suitable \( N \).

**Theorem 2.** If \( A \) and \( B \) are modular abelian varieties, then \( H_{A,B} \) holds.

**Remark:** Both Theorem 1 and Theorem 2 are special cases of a more general theorem. For this, I introduce the class of abelian varieties of *generalized GL\(_2\)-type* (see below). These include:

- products of elliptic curves
- K. Murty’s abelian varieties of type (T) (1983)
- K. Ribet’s abelian varieties \( A/\mathbb{Q} \) of GL\(_2\)-type (1992); these include the Shimura quotients \( A_f \), where \( f \in S_2(\Gamma_1(N))^{new} \).

**Theorem 3.** If \( A \) and \( B \) are abelian varieties of generalized GL\(_2\)-type, then \( H_{A,B} \) holds.

**Corollary 1:** If \( A/K \) and \( B/K \) are abelian varieties of generalized GL\(_2\)-type, then \( H_{A,B,K} \) holds \( \iff \)

\[(\text{End}_K(A) \otimes \text{End}_K(B))^{G_K} = \text{End}_K(A) \otimes \text{End}_K(B).\]

**Corollary 2:** If \( A/K \) and \( B/K \) are abelian varieties of generalized GL\(_2\)-type whose \( \overline{\mathbb{Q}} \)-endomorphisms are defined over \( K \), then \( H_{A,B,K} \) holds.
**Example:** If $A/K$ and $B/K$ are isogenous to products of elliptic curves without CM, then $H_{A,B,K}$ holds by Corollary 2. In other words, the following conditions are equivalent:

(i) Formula (3) holds for all primes $\ell$;

(i') Formula (3) holds for one prime $\ell$;

(ii) $\text{Hom}_K(A, B) = 0$;

**Remark:** Recall that formula (3) was the following identity:

$$(\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} = \text{rk}(\text{End}_K(A)) \text{rk}(\text{End}_K(B)).$$
4. Analysis of Condition (3).

**Notation:** If $V$ is a $\mathbb{Q}_\ell[G_K]$-module, let $\overline{V} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ denote the associated $\overline{\mathbb{Q}}_\ell[G_K]$-module. Here $\overline{\mathbb{Q}}_\ell$ denotes an algebraic closure of $\mathbb{Q}_\ell$.

**Lemma 1:** $\tau_{A,B,K,\ell}$ is an isomorphism (i.e., condition (3) holds for $A, B, K, \ell$) if and only if the following two conditions hold:

1. (Irreducibility) If $V \subset \overline{V}_\ell(A)$ and $W \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}}_\ell[G_K]$-submodules, then $V \otimes W$ is also irreducible.

2. (Multiplicity 1) If $V_1 \subset \overline{V}_\ell(A)$ and $W_1 \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}}_\ell[G_K]$-submodules (for $i = 1, 2$), then

$$V_1 \otimes W_1 \simeq V_2 \otimes W_2 \iff V_1 \simeq V_2 \text{ and } W_1 \simeq W_2.$$

**Counterexamples to $H_{A,B,K}:$**

1) Let $E_i/\mathbb{Q}$ be two elliptic curves with CM by $F_i$, where $F_1 \neq F_2$. If $K = (F_1F_2)^+$, then $H_{E_1,E_2,K}$ does not hold. Here $\text{Hom}_{\overline{\mathbb{Q}}}(E_1, E_2) = 0$, but (3) does not hold (for any $\ell$) because $\dim_{\mathbb{Q}} \text{End}_K(E_i) = 1$ and $(\rho_{E_1,E_2,K,\ell}, \rho_{E_1,E_2,K,\ell})G_K = 2 \neq 1$. (Here Property (1) fails.)

2) Let $E_i/\mathbb{Q}$ be two modular (non-CM) elliptic curves with associated newforms $f_i \in S_2(\Gamma_0(N_i))$, and assume that $E_1$ and $E_2$ are not $\overline{\mathbb{Q}}$-isogenous. Moreover, let $\chi$ be a Dirichlet character of order $m > 2$, and let $g_i$ be the newform associated to the twist $(f_i)_\chi$ of $f_i$ by $\chi$. If $A_i = A_{g_i}/\mathbb{Q}$ is the Shimura quotient associated to $g_i$, then $H_{A_1,A_2,\mathbb{Q}}$ does not hold.

Indeed, $A_i \otimes \overline{\mathbb{Q}} \simeq E_i^{\phi(m)} \otimes \overline{\mathbb{Q}}$, so $\text{Hom}_{\overline{\mathbb{Q}}}(A_1, A_2) = 0$, but (3) does not hold. (Here Property (1) holds, but (2) fails because $A_1$ and $A_2$ have “simultaneous inner twists”.)
5. Abelian Varieties of Generalized \(GL_2\)-type

**Definition:** A \(\bar{\Q}_\ell[G_K]\)-module \(V\) has *restricted \(GL_2\)-type* if \(V = \bigoplus V_i\) is a direct sum of two-dimensional \(\bar{\Q}_\ell[G_K]\)-modules \(V_i\) such that each \(V_i\) is of one of the following two types:

(I) \(V_i\) is irreducible and \(\det V_i = \chi_\ell\), where \(\chi_\ell\) is the cyclotomic \(\ell\)-adic character on \(G_K\).

(II) \(V_i \cong \overline{V}_\ell(E_i)\), for some CM elliptic curve \(E_i/K\).

**Definition:** An abelian variety \(A/K\) has *generalized \(GL_2\)-type* if there is a finite extension \(L/K\) such that

(i) \(\End^0_L(A) = \End^0_{\Q}(A)\);

(ii) \(\overline{V}_\ell(A)\) has restricted \(GL_2\)-type as a \(G_L\)-module, \(\forall \ell\).

**Remark:** The class \((\text{genGL}_2)_K\) of abelian varieties \(A/K\) of generalized \(GL_2\)-type is closed under products. Moreover, if \(A \in (\text{genGL}_2)_K\) and if \(B \leq A\), then \(B, A/B \in (\text{genGL}_2)_K\).

**Lemma 2:** If \(A \in (\text{genGL}_2)_K\), then there is a decomposition \(A \sim A^{nCM} \times A^{CM}\) such that for any \(L/K\) with (i) we have that

(a) \(A^{CM} \otimes L \sim \text{product of CM elliptic curves } E_i/L\), and \(\overline{V}_\ell(A^{CM})\) is a direct sum of 1-dimensional \(G_L\)-modules;

(b) Each \(G_L\)-irreducible component \(V\) of \(\overline{V}_\ell(A^{nCM})\) has dimension 2 and is *strongly irreducible*, i.e. \(V|_U\) is irreducible, \(\forall\) open \(U \leq G_L\). Moreover, \(\overline{V}_\ell(A^{nCM})\) has no internal twists, i.e., if \(V_i\) are two irreducible submodules of \(\overline{V}_\ell(A^{nCM})\), then

\[
V_1 \simeq V_2 \otimes \chi, \text{ for some } \chi \in \text{Hom}(G_L, \bar{\Q}_\ell^\times) \quad \Rightarrow \quad \chi = 1.
\]

Let: \( k = \mathbb{Q}_\ell \) and \( G = G_K \). Here we study \( k[G] \)-modules \( V \) with:

(4) \( V \) is strongly irreducible of dimension 2.

(Recall: this means that \( V|_U \) is irreducible, \( \forall \) open \( U \leq G \).)

**Theorem 5 (Irreducibility Criterion):** If \( V, W \) satisfy (4), then \( V \otimes W \) is irreducible \( \iff \)

(5) \( V \not\cong W \otimes \chi, \) for all \( \chi \in \text{Hom}(G, k^\times) \).

**Remark:** By using Schur’s Lemma, this follows easily from a result of D. Ramakrishnan (2000) on adjoint representations.

**Theorem 6 (Cancellation Criterion):** If \( V_i, W_i \) satisfy (4) for \( i = 1, 2 \), and if

(6) \( V_i \otimes W_j \) is irreducible, for all \( i, j \in 1, 2 \),

then \( V_1 \otimes W_1 \cong V_2 \otimes W_2 \iff \exists \chi \in \text{Hom}(G, k^\times) \) such that

(7) \( V_1 \cong V_2 \otimes \chi \) and \( W_1 \cong W_2 \otimes \chi^{-1} \).

**Remarks:**

1) In view of Lemmas 1 and 2, Theorems 5 and 6 imply Theorem 3 in the non-CM case (i.e, when \( A \sim A^{nCM} \)).

2) The proof of Theorem 6 uses the following identity (which was also used in Ramakrishnan’s proof):

\[ \wedge^2(V \otimes W) \cong (S^2V \otimes \wedge^2W) \oplus (\wedge^2V \otimes S^2W). \]

(As usual, \( S^2V \) denotes the symmetric square of \( V \).)
7. Representation Theory: CM Case.

Recall: If $E/K$ is a CM elliptic curve with $F := \text{End}_K^0(E) \neq \mathbb{Q}$, then $F \subset K$ and $F$ is an imaginary quadratic field. Moreover,

$$\overline{V}_\ell(E) \simeq \psi_1 \oplus \psi_2, \text{ with } \psi_i \in \text{Hom}(G_K, \mathbb{Q}_\ell^\times).$$

In addition, $\psi_1 \psi_2 = \chi_\ell$.

Lemma 3: Let $E_i/K$ be an elliptic curve with CM by $F_i \subset K$, and let $\overline{V}_\ell(E_i) = \psi_{i1} \oplus \psi_{i2}$, where $i = 1, 2$. Assume that $F_1 \neq F_2$. If $p$ is a prime which splits completely in $K$, then

$$\mathbb{Q}(\psi_{i1}\psi_{2j}(\sigma_{\mathfrak{p}})) \simeq F_1F_2, \text{ } \forall i, j = 1, 2,$$

where $\sigma_{\mathfrak{p}} \in G_K$ is a Frobenius element at $\mathfrak{p} | p$.

Remarks: 1) Using Lemma 3, it follows easily that Property (2) holds if $A = A^{CM}$ and $B = B^{CM}$. Since Property (1) is trivial, we thus see that Theorem 3 holds in this case. Combining this with the results of §6, this proves Theorem 3 because it is easy to verify Properties (1) and (2) for the “mixed terms” $V_i \otimes \psi_j$. 2) By using a more general version of the Irreducibility Criterion (Theorem 5) and the results of Ribet (1980), one can also show:

Theorem 7: If $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties with $\text{Hom}_{\mathbb{Q}}(A, B) = 0$, then Property (1) holds, i.e.,

$V \otimes W$ is $G_\mathbb{Q}$-irred., if $V \subset \overline{V}_\ell(A), W \subset \overline{V}_\ell(B)$ are $G_\mathbb{Q}$-irred.