Tensor Products of Galois Representations

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1. Introduction

- Let: $K$ be a number field, $G_K = \text{Gal}(\overline{K}/K)$, $A/K$ an abelian variety over $K$, $d = \dim(A)$, $T_\ell(A) = \lim \leftarrow A[\ell^n]$, the $\ell$-adic Tate-module, $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, viewed as a $\mathbb{Q}_\ell[G_K]$-module, $\rho_{A/K,\ell} : G_K \to \text{Aut}(T_\ell(A)) \subseteq \text{Aut}(V_\ell(A)) \cong \text{GL}_{2d}(\mathbb{Q}_\ell)$, $\overline{\rho}_{A/K,\ell} : G_K \to \text{Aut}(A[\ell]) \cong \text{GL}_{2d}(\mathbb{F}_\ell)$, the associated Galois representations.

- Faltings (1983): The homomorphism

$$\tau_{A/K,\ell} : \text{End}_K(A) \otimes \mathbb{Q}_\ell \to \text{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A))$$

is an isomorphism. In particular, the intertwining number

$$(\rho_{A/K,\ell}, \overline{\rho}_{A/K,\ell})_{G_K} := \dim_{\mathbb{Q}_\ell} \text{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A))$$

is independent of the choice of $\ell$ (and equals $\dim_{\mathbb{Q}} \text{End}_K^0(A)$).
1. Introduction

- Zarhin (1977), (1985); Faltings (1984): For almost all $\ell$

$$\bar{\tau}_{A,\ell} : \text{End}_K(A) \otimes \mathbb{F}_\ell \to \text{End}_{\mathbb{F}_\ell[G_K]}(A[\ell])$$

is an isomorphism. Thus

$$(\bar{\rho}_{A/K,\ell}, \bar{\rho}_{A/K,\ell})_{G_K} := \dim_{\mathbb{F}_\ell} \text{End}_{\mathbb{F}_\ell[G_K]}(A[\ell])$$

does not depend on $\ell$, if $\ell \gg 0$.

- Question 1: Let $B/K$ be another abelian variety. Are there analogous results for $\rho_{A,B,K,\ell} := \rho_{A/K,\ell} \otimes \rho_{B/K,\ell}$ and for

$$\bar{\rho}_{A,B,K,\ell} := \bar{\rho}_{A/K,\ell} \otimes \bar{\rho}_{B/K,\ell}$$?
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- **Remark:** This question arises naturally when one studies the Hasse-Weil zeta function of a product of two curves, and is closely connected with the *Tate Conjecture*. For this, recall that if $X/K$ is a smooth, projective variety, then the *Tate Conjecture* $T^r(X)$ states that the cycle map

$$cyc_X^r : \mathcal{A}^r(X) \otimes \mathbb{Q}_\ell \rightarrow (H^{2r}_{et}(\overline{X}, \mathbb{Q}_\ell)(r))^G_K$$

is an isomorphism, where $\mathcal{A}^r(X)$ denotes the group of cycles of codimension $r$ on $X$, modulo homological equivalence. Indeed, $T^2(A \times B \times A^* \times B^*) \Rightarrow$ the analogue of Faltings’ Theorem holds for $\rho_{A,B,K,\ell} = \rho_{A/K,\ell} \otimes \rho_{B/K,\ell}$.

In this, the ring $\text{End}_K(A)$ is replaced by a certain (abstract) ring of correspondences which contains $\text{End}_K(A) \otimes \text{End}_K(B)$. This leads to the following question:
• Question 2: Let $\tau_{A,B,K,\ell} = \tau_{A/K,\ell} \otimes \tau_{B/K,\ell}$, so

$$
\tau_{A,B,K,\ell} : \text{End}_K(A) \otimes \text{End}_K(B) \otimes \mathbb{Q}_\ell \to \text{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A,B)),
$$

where $V_\ell(A,B) = V_\ell(A) \otimes_{\mathbb{Q}_\ell} V_\ell(B)$.

When is $\tau_{A,B,K,\ell}$ an isomorphism? In other words, when is

$$
(1) \left( \rho_{A,B,K,\ell}, \rho_{A,B,K,\ell} \right)_{G_K} := \dim_{\mathbb{Q}_\ell} (\text{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A,B)))
$$

$$
\overset{?}{=} \dim_{\mathbb{Q}} \text{End}_K^0(A) \dim_{\mathbb{Q}} \text{End}_K^0(B)?
$$

Similarly: when is

$$
(2) \left( \bar{\rho}_{A,B,\ell}, \bar{\rho}_{A,B,\ell} \right)_{G_K} = \dim_{\mathbb{Q}} \text{End}_K^0(A) \dim_{\mathbb{Q}} \text{End}_K^0(B)?
$$

A first (naive) guess is that the following holds.
• Hypothesis $H_{A,B,K}$: The following are equivalent:
  (i) Formula (1) holds for all primes $\ell$;
  (i′) Formula (1) holds for one prime $\ell$;
  (ii) $\text{Hom}_K(A, B) = 0$.

• Observation: While $H_{A,B,K}$ holds for some abelian varieties $A/K$ and $B/K$, it is not true in general. There are (at least) two classes of counterexamples:
  (i) $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties which have a common internal twist (in the sense of Ribet);
  (ii) $A/K$ and $B/K$ are CM elliptic curves which are defined over $\mathbb{Q}$ and $K$ is a suitable real quadratic field.
Thus, a better guess is the following:
1. Introduction - 6

- **Hypothesis \( \overline{H}_{A,B} \):** The hypothesis \( H_{A,B,K} \) holds whenever \( K \) is large enough, i.e., whenever

\[
\text{End}_{\overline{K}}(A) = \text{End}_{\overline{K}}(A) \quad \text{and} \quad \text{End}_{\overline{K}}(B) = \text{End}_{\overline{K}}(B).
\]

- **Observation:** If \( \overline{H}_{A,B} \) holds for \( A/K \) and \( B/K \), and if (ii) holds, then for every finite extension \( L/K \) and prime \( \ell \) we have an induced isomorphism

\[
\tilde{\tau}_{A,B,L} : (\text{End}_{\overline{K}}(A) \otimes \text{End}_{\overline{K}}(B))^{G_L} \otimes \mathbb{Q}_\ell \cong \text{End}_{\mathbb{Q}_\ell[G_L]}(V_\ell(A, B)).
\]

Thus, \( H_{A,B,L} \) holds if and only if

\[
(\text{End}_{\overline{K}}(A) \otimes \text{End}_{\overline{K}}(B))^{G_L} = \text{End}_L(A) \otimes \text{End}_L(B).
\]
2. Main Results

- **Theorem 1.** If $A$ and $B$ are isogenous (over $\overline{\mathbb{Q}}$) to products of elliptic curves, then $H_{A,B}$ holds.

- **Definition:** A *modular abelian variety* $A/K$ is a quotient of the Jacobian variety $J_1(N)_K$ of the modular curve $X_1(N)/K$, for a suitable $N$.

- **Theorem 2.** If $A$ and $B$ are modular abelian varieties, then $H_{A,B}$ holds.

- **Remark:** Both Theorem 1 and Theorem 2 are special cases of a more general theorem. For this, I introduce the class of abelian varieties of *generalized $GL_2$-type* (see below). These include:
  - products of elliptic curves
2. Main Results - 2

- K. Murty’s abelian varieties of type (T) (1983)
- K. Ribet’s abelian varieties $A/\mathbb{Q}$ of $\text{GL}_2$-type (1992); these include the Shimura quotients $A_f$, where $f \in S_2(\Gamma_1(N))^{\text{new}}$.
- Theorem 3. If $A$ and $B$ are abelian varieties of generalized $\text{GL}_2$-type, then $H_{A,B}$ holds.
- Corollary 1: If $A/K$ and $B/K$ are abelian varieties of generalized $\text{GL}_2$-type, then $H_{A,B,K}$ holds $\iff$

$$\left(\text{End}_{\overline{K}}(A) \otimes \text{End}_{\overline{K}}(B)\right)^{G_K} = \text{End}_K(A) \otimes \text{End}_K(B).$$

- Corollary 2: If $A/K$ and $B/K$ are abelian varieties of generalized $\text{GL}_2$-type whose $\mathbb{Q}$-endomorphisms are defined over $K$, then $H_{A,B,K}$ holds.
2. Main Results - 3

- **Theorem 4**: If $A/K$ and $B/K$ are elliptic curves without CM, then the following conditions are equivalent:
  
  (i) Formula (1) holds for all primes $\ell$;
  (i') Formula (1) holds for one prime $\ell$;
  (ii) $\text{Hom}_K(A, B) = 0$;
  (iii) Formula (2) holds for almost all primes $\ell$.

- **Remarks**: 1) The equivalence of the first 3 conditions is a special case of Corollary 2 above.
  2) The proof of the equivalence of (i) and (iii) uses a result of Frey/Jarden (2002), together with a modification of the representation theoretic results of §5.
3. Analysis of Condition (1)

- **Notation:** If $V$ is a $\mathbb{Q}_\ell[G_K]$-module, let $\overline{V} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ denote the associated $\overline{\mathbb{Q}_\ell}[G_K]$-module. Here $\overline{\mathbb{Q}_\ell}$ denotes an algebraic closure of $\mathbb{Q}_\ell$.

- **Lemma 1:** $\tau_{A,B,K,\ell}$ is an isomorphism (i.e., condition (1) holds for $A, B, K, \ell$) if and only if the following two conditions hold:
  
  (i) (Irreducibility) If $V \subset \overline{V}_\ell(A)$ and $W \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}_\ell}[G_K]$-submodules, then $V \otimes W$ is also irreducible.
  
  (ii) (Multiplicity 1) If $V_i \subset \overline{V}_\ell(A)$ and $W_i \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}_\ell}[G_K]$-submodules (for $i = 1, 2$), then

  $$V_1 \otimes W_1 \simeq V_2 \otimes W_2 \iff V_1 \simeq V_2 \text{ and } W_1 \simeq W_2.$$
3. Analysis of Condition (1) - 2

- **Counterexamples to $H_{A,B,K}$:**
  1) Let $E_i/\mathbb{Q}$ be two elliptic curves with CM by $F_i$, where $F_1 \neq F_2$. If $K = (F_1 F_2)^+$, then $H_{E_1,E_2,K}$ does not hold. Here $\text{Hom}_{\mathbb{Q}}(E_1, E_2) = 0$, but (1) does not hold (for any $\ell$) because $\dim_{\mathbb{Q}} \text{End}_K(E_i) = 1$ and $(\rho_{E_1,E_2,K,\ell}, \rho_{E_1,E_2,K,\ell})_{G_K} = 2 \neq 1$. (Here $V_\ell(E_1, E_2)$ is reducible, so Property (i) fails.)
3. Analysis of Condition (1) - 3

- Counterexamples to $H_{A,B,K}$:
  2) Let $E_i / \mathbb{Q}$ be two modular (non-CM) elliptic curves with associated newforms $f_i \in S_2(\Gamma_0(N_i))$, and assume that $E_1$ and $E_2$ are not $\mathbb{Q}$-isogenous. Moreover, let $\chi$ be a Dirichlet character of order $m > 2$, and let $g_i$ be the newform associated to the twist $(f_i)_\chi$ of $f_i$ by $\chi$. If $A_i = A_{g_i} / \mathbb{Q}$ is the Shimura quotient associated to $g_i$, then $H_{A_1,A_2,\mathbb{Q}}$ does not hold.

  Indeed, $A_i \otimes \overline{\mathbb{Q}} \sim E_i^{\phi(m)} \otimes \overline{\mathbb{Q}}$, so $\text{Hom}_{\overline{\mathbb{Q}}}(A_1, A_2) = 0$, but (1) does not hold. (Here Property (i) holds, but (ii) fails because $A_1$ and $A_2$ have “simultaneous inner twists”.)
4. Abelian Varieties of Generalized GL$_2$-type

- **Definition:** A $\mathbb{Q}_\ell[G_K]$-module $V$ has *restricted GL$_2$-type* if $V = \oplus V_i$ is a direct sum of two-dimensional $\mathbb{Q}_\ell[G_K]$-modules $V_i$ such that each $V_i$ is of one of the following two types:
  (I) $V_i$ is irreducible and $\det V_i = \chi_\ell$,
  where $\chi_\ell$ is the cyclotomic $\ell$-adic character on $G_K$.
  (II) $V_i \cong \overline{V}_\ell(E_i)$, for some CM elliptic curve $E_i/K$.

- **Definition:** An abelian variety $A/K$ has *generalized GL$_2$-type* if there is a finite extension $L/K$ such that
  (i) $\text{End}_L^0(A) = \text{End}_\mathbb{Q}^0(A)$;
  (ii) $\overline{V}_\ell(A)$ has restricted GL$_2$-type as a $G_L$-module, $\forall \ell$. 
Remark: The class \( (\text{genGL}_2)_K \) of abelian varieties \( A/K \) of generalized \( GL_2 \)-type is closed under products. Moreover, if \( A \in (\text{genGL}_2)_K \) and if \( B \subset A \), then \( B, A/B \in (\text{genGL}_2)_K \).

Lemma 2: If \( A \in (\text{genGL}_2)_K \), then there is a decomposition \( A \sim A^{nCM} \times A^{CM} \) such that for any \( L/K \) with (i) we have that

(a) \( A^{CM} \otimes L \sim \) product of CM elliptic curves \( E_i/L \), and \( \overline{V}_\ell(A^{CM}) \) is a direct sum of 1-dimensional \( GL \)-modules;

(b) Each \( GL \)-irreducible component \( V \) of \( \overline{V}_\ell(A^{nCM}) \) has dimension 2 and is strongly irreducible, i.e. \( V|_U \) is irreducible, \( \forall \) open \( U \leq GL \). Moreover, \( \overline{V}_\ell(A^{nCM}) \) has no internal twists, i.e., if \( V_i \) are two irreducible submodules of \( \overline{V}_\ell(A^{nCM}) \), then

\[ V_1 \simeq V_2 \otimes \chi, \text{ for some } \chi \in \text{Hom}(GL, \overline{Q}_\ell^\times) \Rightarrow \chi = 1. \]
5. Representation Theory: non-CM Case

- Let: $k = \overline{\mathbb{Q}}_\ell$ and $G = G_K$. Here we study $k[G]$-modules $V$ satisfying the following property:

\[(3) \quad V \text{ is strongly irreducible of dimension 2.}\]

(Recall: this means that $V|_U$ is irreducible, $\forall$ open $U \leq G$.)

- Theorem 5 (Irreducibility Criterion): If $V, W$ satisfy (3), then $V \otimes W$ is irreducible $\iff$

\[(4) \quad V \not\cong W \otimes \chi, \quad \text{for all } \chi \in \text{Hom}(G, k^\times).\]

- Remark: By using Schur’s Lemma, this follows easily from a result of D. Ramakrishnan (2000) on adjoint representations.
Theorem 6 (Cancellation Criterion): If $V_i, W_i$ satisfy (3) for $i = 1, 2$, and if

$$V_i \otimes W_j \text{ is irreducible, for all } i, j \in 1, 2,$$

then $V_1 \otimes W_1 \simeq V_2 \otimes W_2 \iff \exists \chi \in \text{Hom}(G, k^\times)$ such that

$$(6) \quad V_1 \simeq V_2 \otimes \chi \quad \text{and} \quad W_1 \simeq W_2 \otimes \chi^{-1}.$$

Remarks: 1) In view of Lemmas 1 and 2, Theorems 5 and 6 imply Theorem 3 in the non-CM case (i.e. when $A \sim A^{nCM}$.)

2) The proof of Theorem 6 uses the following identity (which was also used in Ramakrishnan’s proof):

$$\wedge^2(V \otimes W) \simeq (S^2 V \otimes \wedge^2 W) \oplus (\wedge^2 V \otimes S^2 W).$$

(As usual, $S^2 V$ denotes the symmetric square of $V$.)
6. Representation Theory: CM Case

- Recall: If $E/K$ is a CM elliptic curve with $F := \text{End}_K^0(E) \neq \mathbb{Q}$, then $F \subset K$ and $F$ is an imaginary quadratic field. Moreover,

$$\overline{V}_\ell(E) \simeq \psi_1 \oplus \psi_2, \quad \text{with } \psi_i \in \text{Hom}(G_K, \overline{\mathbb{Q}}^\times_\ell).$$

In addition, $\psi_1 \psi_2 = \chi_\ell$.

- Lemma 3: Let $E_i/K$ be an elliptic curve with CM by $F_i \subset K$, and let $\overline{V}_\ell(E_i) = \psi_{i1} \oplus \psi_{i2}$, where $i = 1, 2$. Assume that $F_1 \neq F_2$. If $p$ is a prime which splits completely in $K$, then

$$\mathbb{Q}(\psi_{1i} \psi_{2j}(\sigma_{\mathfrak{p}})) \simeq F_1 F_2, \quad \forall i, j = 1, 2,$$

where $\sigma_{\mathfrak{p}} \in G_K$ is a Frobenius element at $\mathfrak{p} \mid p$. 
Remarks: 1) Using Lemma 3, it follows easily that Property (ii) holds if $A = A^{CM}$ and $B = B^{CM}$. Since Property (i) is trivial, we thus see that Theorem 3 holds in this case. Combining this with the results of §5, this proves Theorem 3 because it is easy to verify Properties (i) and (ii) for the “mixed terms” $V_i \otimes \psi_j$.

2) By using a more general version of the Irreducibility Criterion (Theorem 5) and the results of Ribet (1980), one can also show:

**Theorem 7:** If $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties with $\text{Hom}_{\mathbb{Q}}(A, B) = 0$, then Property (i) holds, i.e.,

$$V \otimes W$$

is $G_{\mathbb{Q}}$-irred., if $V \subset \overline{V}_\ell(A)$, $W \subset \overline{V}_\ell(B)$ are $G_{\mathbb{Q}}$-irred.