Hurwitz Spaces of Covers
of an Elliptic Curve

1. Introduction

Riemann’s Existence Theorem (RET) (1857):
Every compact Riemann surface $X$ has a non-constant meromorphic function, i.e. $X$ admits a non-constant holo. map to the Riemann sphere $\mathbb{C}_{\infty} = \mathbb{C} \cup \{\infty\}$.
**Consequence:** Every compact Riemann surface is a complex algebraic curve $X_{\mathbb{C}}$ defined by an equation

$$F(x, y) = 0, \quad \text{where } F \in \mathbb{C}[x, y],$$

and the (holomorphic) map $f : X \rightarrow \mathbb{C}_{\infty}$ corresponds to a morphism $f : X_{\mathbb{C}} \rightarrow \mathbb{P}^1_{\mathbb{C}}$ of complex curves of the same degree (and conversely):

$$
\begin{array}{cc}
X & X_{\mathbb{C}} \\
\downarrow f & \leftrightarrow \\
\mathbb{C}_{\infty} & \mathbb{P}^1_{\mathbb{C}}
\end{array}
$$

**Properties of $f$:**

1. $\deg(f) := \max_{y \in \mathbb{C}_{\infty}} (\#(f^{-1}(y))) < \infty.$
2. The set

$$R_f := \{ y \in \mathbb{C}_{\infty} : \#(f^{-1}(y)) < \deg(f) \}$$

of ramification points of $f$ is finite:

$$w := \#R_f < \infty.$$
**Problem (Hurwitz, 1891)** Fix integers $N$ and $w$, and put $Y = \mathbb{C}_\infty$.

1) Investigate the totality $H(Y, N, w)$ of all covers $f : X \to Y$ with $\text{deg}(f) = N$ and $\#R_f = w$.

2) Calculate the number $\#H(Y, N, R)$ of such covers with fixed ramification locus $R_f = R$.

**Remarks:** 1) A **cover** is a non-constant holomorphic map $f : X \to Y$. Throughout, we always consider equivalence classes of covers: 

$$(X_1 \xrightarrow{f_1} Y) \sim (X_2 \xrightarrow{f_2} Y) \iff \exists \phi : X_1 \overset{\sim}{\to} X_2 \text{ with } f_2 \circ \phi = f_1.$$ 

2) As Hurwitz observed, it is useful to refine the above problems by fixing the **ramification type** of the cover. For example, we might want to classify (or count) all **simple covers**:

A cover $f : X \to Y$ is called **simple** if

$$\#(f^{-1}(y)) \geq \text{deg}(f) - 1, \quad \text{for all } y \in Y.$$
Theorem A (Hurwitz, 1891): If \( Y = \mathbb{C}_\infty \), then

(a) \( H(Y, N, w) \) is a “Riemannian space”.

(b) \( H^{simple}(Y, N, w) \) is a connected manifold of dimension \( w \) (provided that \( w \geq 2N - 2 \) and \( 2 | w \)).

(c) The discriminant map

\[
\delta : H^{simple}(Y, N, w) \rightarrow Y^{(w)} \setminus \Delta_w
\]

is finite and unramified. Thus, \#\( H^{simple}(Y, N, R) \) depends only on \( w = \#R \).

Observation (Hurwitz): RET \( \Rightarrow \) the calculation of \#\( H(Y, N, R) \) is a purely group-theoretic problem, albeit one that is “highly complicated” (Hurwitz):

\[
H(Y, N, R) \overset{\sim}{\rightarrow} \text{Hom}'(\pi_1(Y \setminus R), S_N)/S_N.
\]

Hurwitz (1891/1901) found a “satisfactory solution” for calculating \( n_{N, w} := \#H^{simple}(\mathbb{C}_\infty, N, R) \):

\[
n_{2,w} = 1,
\]

\[
n_{3,w} = \frac{1}{3!}(3^{w-1} - 3),
\]

\[
n_{4,w} = \frac{1}{4!}(2^{w-2} - 4)(3^{w-1} - 3), \text{ etc.}
\]
**Question 1:** Is there an intrinsic description of the topology and/or complex structure of the Hurwitz spaces $H(Y, N, w)$?

**Recall:** 1) The points of $H(Y, N, w)$ correspond to covers $f : X \to Y$ of degree $N$ with $w$ ramification points.

2) The topology of $H^{simple}(Y, N, w)$ is induced by the discriminant map

$$\delta : H^{simple}(Y, N, w) \to Y^{(w)} \setminus \Delta_w.$$  

Thus: a neighbourhood of a cover $f \in H(Y, N, w)$ consists (roughly) of those covers whose ramification loci are close to that of $f$.

**Question 2:** Generalizations of Hurwitz spaces?

a) Construct $H(Y, N, w)$ for other Riemann surfaces/complex curves $Y$;

b) Study rationality conditions: over which ground fields $K \subset \mathbb{C}$ are the covers defined?
2. Intrinsic Description of Hurwitz Spaces

**Key Observation** (Grothendieck, 1960): A topological (complex) space $H$ is uniquely characterized by the set of maps $\text{Hom}(T, H)$, as $T$ runs over all topological (complex) spaces.

**In other words:** As a topological space, $H$ is determined by the functor

$$F_H : \text{Top} \to \text{Sets}$$

which is given by $F_H(T) = \text{Hom}_{\text{top}}(T, H)$. (Similarly for complex spaces.)

**Problem:** For each complex space $T$, describe the holomorphic maps

$$T \to H = H^{\text{simple}}(Y, N, w).$$

**Fulton (1969):** Consider families of covers, i.e. covers of curve families $/T$:

$$f : \mathcal{X} \to Y_T = \mathbb{P}^1_T = \mathbb{P}^1 \times T.$$  

Thus: For each $t \in T$, the fibre $f_t : \mathcal{X}_t \to (\mathbb{P}^1_T)_t = \mathbb{P}^1$ of $f$ at $t$ is a cover (of curves) in the previous sense, i.e. $f_t \in H(\ldots)$. 
**Expect:** 1) For each family of covers $f : \mathcal{X} \rightarrow Y_T = \mathbb{P}_T^1$, the assignment $t \mapsto f_t$ defines (naturally) a holomorphic map $[f] : T \rightarrow H$.

2) Each holomorphic map $g : T \rightarrow H$ arises uniquely in this way, i.e. $g = [f]$, for a unique family of covers $f : \mathcal{X} \rightarrow Y_T$ (up to isomorphism).

**Reformulation:** Let

$$H^{simple}(Y_T/T, N, w) = \text{(set of families of simple covers over } T \text{ with } f_t \in H^{simple}(Y, N, w), \forall t)/\sim.$$ 

It is easy to see that the assignment $T \mapsto H^{simple}(Y_T/T, N, w)$ defines a functor

$$\mathcal{H}_{N,w} : \underline{\text{C} - \text{spaces}} \rightarrow \text{Sets},$$

and that

$$\text{Expectation } \Leftrightarrow \mathcal{H}_{N,w} \simeq F_H \quad \text{def} \quad H \text{ represents the functor } \mathcal{H}_{N,w}.$$
**Theorem B (Fulton, 1969):** If $N \geq 3$, then the Hurwitz space $H_{\text{simple}}(Y, N, w)$ (as defined by Hurwitz) represents the above functor $\mathcal{H}_{N,w}$.

This theorem generalizes to the algebraic setting by replacing complex spaces by schemes:

**Theorem C (Fulton, 1969):** If $N \geq 3$, then the functor

$$\mathcal{H}_{N,w} : \textbf{Sch} \rightarrow \textbf{Sets}$$

is representable by a scheme $H_{N,w}/\mathbb{Z}$ of finite type. In particular, for any field $K$ we have

$$H_{N,w}(K) = H_{\text{simple}}(\mathbb{P}^1/K, N, w).$$

In addition, the restriction of the discriminant map to $H_{N,w} \otimes \mathbb{Z}[1/N!] \subset H_{N,w}$,

$$\delta : H_{N,w} \otimes \mathbb{Z}[1/N!] \rightarrow (\mathbb{P}_\mathbb{Z}[1/N!]^1)^{(w)} \setminus \Delta_w,$$

is finite and etale.

**Remark:** Little seems to be known about the geometric structure of $H_{N,w}$.

**Aim:** Study analogues of these results in the case that $Y = E$ is an elliptic curve (and $w = 2$).
**Remark:** In recent years, there have been an abundance of results and applications of Hurwitz spaces:

1) Inverse Galois theory: Fried, Völklein, . . .

- Fried, Völklein, Harbater, Debes, Wevers, . . .: studied moduli spaces of other types of covers $/\mathbb{P}^1$.

2) Moduli problems of curves: Fulton, Mumford and Harris, . . .

- used $H_{N,w}$ to study the geometry of $M_g$, the moduli space of curves of genus $g$.

3) String theory: Gromov/Witten, Dijkgraaf, . . .

also: Cordes/Moore/Ramgoolan, Kontsevich, . . .
3. The Case $Y = E$ and $w = 2$

**Reference:** IEM Preprint No. 9 (2001), IEM Essen.
(See also www.mast.queensu.ca/~kani)
– to appear in: Collectanea Mathematica

**Let** $E/K$ **be an elliptic curve** over a field $K$ ($\text{char} \neq 2$). Fix $N \geq 2$ prime to $\text{char}(K)$.

**Note:** If $(X \xrightarrow{f} E) \in H^{\text{simple}}(E/K, N, 2)$, then by the Riemann-Hurwitz relation

$$2g_X - 2 = N(2g_E - 2) + w = w = 2 \Rightarrow g_X = 2.$$

**More generally:** Study the set $H^{(2)}(E/K, N)$ of all genus 2 covers of degree $N$ of $E/K$:

$$f : X \to E, \quad \deg(f) = N \text{ and } g_X = 2.$$ 

Similarly, study the set $H^{(2)}(E_T/T, N)$ of families of such covers:

$$f : \mathcal{X} \to E_T = E \times T, \quad f_t \in H^{(2)}(E_t/K(t), N).$$ 

As before, the assignment $T \mapsto H^{(2)}(E_T/T, N)$ defines a functor

$$\mathcal{H}^{(2)}_{E/K,N} : \text{Sch} \to \text{Sets}.$$
**Theorem 1.** If $N$ is odd, then $\mathcal{H}_{E/K,N}^{(2)}$ is representable by a smooth, quasi-projective surface $H_{E/K,N}^{(2)}$ over $K$ which has (over $\overline{K}$) 

$$\sum_{d|N} \sigma(d) - \sigma(N)$$

irreducible components. Thus $H_{E/K,N}^{(2)}$ is irreducible if and only if $N$ is prime.

**Remarks:** 1) The above result does not extend to the case that $N$ is even. However, a slightly weaker result is true in that case: the functor $\mathcal{H}$ is coarsely representable by such a variety.

2) The reason that $H$ breaks up into components is the following:

Each $X \xrightarrow{f} E$ factors as $X \xrightarrow{f'} E' \xrightarrow{u_f} E$, where $u_f : E' \to E$ is the max. unramified subcover of $f$.

Thus: $H_{E/K,N}^{(2)}$ is a union of components which are indexed by subgroups $G \leq E$ with $\#G|N$ (and $\#G \neq N$); explicitly, $G = \text{Ker}(\hat{u}_f)$. 
**Definition:** A cover $f : X \to E$ is called minimal if $\text{deg}(u_f) = 1$.

**Theorem 2.** For every $N \geq 3$ (prime to $\text{char}(K)$), the functor $\mathcal{H}_{E/K,N}^{(\text{min})}$ which classifies minimal genus 2 covers is representable by a smooth, irreducible quasi-projective surface $H_{E/K,N}^{(\text{min})}$ over $K$.

More precisely, we have

$$H_{E/K,N}^{(\text{min})} \otimes_K \overline{K} \cong E \times H_{E/K,N}^{(\text{min})}$$

where $H_{E/K,N} \subset X(N)$ is an open subvariety (curve) of the modular curve $X(N)$ of (full) level $N$.

**Remarks:**
1) If $K = \mathbb{C}$, then $X(N) = \Gamma(N) \backslash \mathcal{H}^*$, which is a Galois cover of $X(1) \cong \mathbb{P}^1$ of degree

$$\overline{sl}(N) := |\text{SL}_2(\mathbb{Z}/N\mathbb{Z})|/\{\pm 1\}|.$$  

2) The reason that $E$ appears as a factor of $\mathcal{H}_{E/K,N}^{(\text{min})}$ is due to the fact that the group $E(K)$ acts on $E$ and hence on $H_{E/K,N}^{(2)}$ etc. via translation: $f \mapsto T_x \circ f$.

**Thus:** introduce and study normalized covers.
**Definition:** A cover \( f : X \to E \) with \( g_X = 2 \) is called normalized if it is minimal and if

\[
f(W) \subset E[2] \text{ and } #(f^{-1}(0_E) \cap W) = \begin{cases} 3 & N \text{ odd} \\ 0 & \text{else} \end{cases}
\]

where \( W = \text{Fix}(\sigma_X) \) denotes the set of 6 Weierstrass points of \( X \). (Here: \( \sigma_X \) is the hyperelliptic involution of \( X \)).

**Notes:**
1) If \( f : X \to E \) is minimal, then \( \exists! y \in E(K) \) such that \( T_y \circ f : X \to E \) is normalized.
2) If \( f \) is normalized, then \( f \circ \sigma_X = [-1]_E \circ f \).
Thus \( \text{Disc}(f) \) is symmetric with respect to \( [-1]_E \), i.e. \([[-1]^*_E \text{Disc}(f) = \text{Disc}(f)\).

**Example:** Let

\[
E : \quad y^2 = (x - a)(x - b)(x - c), \quad abc \neq 0
\]

\[
X : \quad s^2 = (t^2 - a)(t^2 - b)(t^2 - c).
\]

Then the cover \( f : X \to E \), given by \( f^*x = t^2, f^*y = s \), is normalized and of degree 2.

**Theorem 3.** For every \( N \geq 3 \) (as above), the functor \( \mathcal{H}_{E/K,N} \) which classifies normalized genus 2 covers is representable by a smooth, irreducible affine curve \( H_{E/K,N}/K \) such that \( H_{E/K,N} \otimes \overline{K} \subset X(N) \).
**Theorem 4:** Let

\[ D_{E/K,N} = X(N)/\overline{K} \setminus (H_{E/K,N} \otimes \overline{K}) \]

denote the degeneracy locus. Then

\[ \#D_{E/K,N} \leq \frac{1}{12N}(5N + 6)s\overline{l}(N), \]

and equality holds if and only if \( \text{char}(K) \nmid N! \).


**Theorem 5:** The assignment \((X \xrightarrow{f} E) \mapsto \text{Disc}(f)\) is represented by a quasi-finite morphism

\[ \delta = \delta_{E/K,N} : H_{E/K,N} \to \mathbb{P}^1_K \simeq (E^{(2)})_{\text{sym}}. \]

Furthermore, if \( \text{char}(K) \nmid N! \), then \( \delta \) is finite and unramified outside of \( \pi_E(E[2]) \subset \mathbb{P}^1 \).

**Theorem 6:** If \( \text{char}(K) \nmid N! \), then

\[ \deg(\delta_{E/K,N}) = \frac{1}{6}(N - 1)s\overline{l}(N). \]

**Remarks:**
1) This degree can be viewed as a measure of non-rigidity of coverings (\( \to \) Völklein).
2) H. Völklein proved Theorem 6 for \( N = 3, 5, 7 \) by using group theory (and a computer).
4. Some applications

(a) **Rationality Questions** ($K$ a number field)

Since $g_X(N) \geq 2$ for $N \geq 7$, we have by Faltings’ theorem (= Mordell’s Conjecture):

**Corollary 1:** $\# \mathcal{H}_{E/K,N}(K) < \infty$, if $N \geq 7$.

**Question:** Is $\mathcal{H}_{E/K,N}(K) = \emptyset$, for $N \gg 0$?

This is false (even for $N$ prime), for there exist curves $X/K$ with $\infty$’ly many $f_N : X \to E$ for which $N = \deg(f_N)$ is prime.

**Conjecture (**)** For each $E/K$ there exist only finitely many genus 2 curves $X/K$ which have a (minimal) morphism $f : X \to E$ of degree $N \geq 7$.

**Remark:** ABC conj. $\Rightarrow$ Asym. Fermat $\Rightarrow$ Conj. (*). Moreover, the converse: Conj. (*) $\Rightarrow$ Asym. Fermat is “almost true”: it implies a slightly weaker version of Frey’s Conjecture 5 (which by Frey and Wiles is equivalent to the Asymptotic Fermat Conjecture (for $K = \mathbb{Q}$).)**
(b) Moduli

**Question:** For which curves $Y/K$ does there exist a (minimal) morphism $f : Y \to E$ of degree $N$?

**Corollary 2:** For every $N$ there exists a morphism

$$\mu_N : H_{E(N)/X'(N), N} \to M_2$$

to the moduli space of curves of genus 2. Moreover:

a) $\text{Im}(\mu_N) = \text{Humbert surface with Inv. } \Delta = N^2$;

b) $\text{deg}(\mu_N) = 2s\ell(N)$; more precisely,

$$\text{Im}(\mu_N) \sim Z^{sym}_{N,-1} := (X(N) \times X(N))/\langle \Delta_{N,-1}, \tau \rangle,$$

where $\tau(x, y) = (y, x)$ and

$$\Delta_{N,-1} = \{(g, \alpha_{-1}(g)) : g \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}\},$$

where $\alpha_{-1}(g) = Q_{-1}gQ_{-1}^{-1}$ with $Q_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

**In particular,** the normalization (and compactification) of the Humbert surface $\text{Im}(\mu_N)$ is the symmetric Diagonal Quotient Surface $Z^{sym}_{N,-1}$. 
(c) **Counting Covers:** \((K = \overline{K})\)

**Corollary 3:** If \(N \geq 2\) and \(\text{char}(K) \nmid N\), then for every \(R \subset E\) with \(#R = 2\) we have

\[
c_N := \sum_{f \in H^s(E/K,N,R)} \frac{1}{|\text{Aut}(f)|} = \frac{1}{3} (N \sigma_3(N) - N^2 \sigma_1(N)),
\]

where \(\sigma_k(N) = \sum_{d|N} d^k\). Thus, if \(\text{char}(K) = 0\), then \(F_2(q) := \sum c_N q^N\) is a quasi-modular form of weight 6; explicitly we have

\[
F_2(q) = \frac{1}{51840} (10E_2^3 - 6E_2E_4 - 4E_6),
\]

where \(E_k = 1 + b_k \sum_{n \geq 1} \sigma_{k-1}(n)q^n\) with \(b_2 = -24, b_4 = 240\) and \(b_6 = -504\).

**Remarks:**
1) The identity (1) was first proven by R. Dijkgraaf (1995) by using the methods of mirror symmetry (→ B. Mazur).

2) Theorem 6 \(\Rightarrow\) Corollary 3 by using the identities

\[
\sum_{n|N} \sigma_1(n)sl(N/n) = \sigma_3(N),
\]

\[
\sum_{n|N} n\sigma_1(n)sl(N/n) = N^2 \sigma_1(N).
\]
(d) Curves with minimal degeneration:

Let \( H = H_{E/K,N} \subset X = X(N) \) be the moduli space, 
\( f : Y_N \to E_H = E \times H \) the universal cover, 
\( p : \overline{Y}_N \to X \) the minimal model of \( Y_N \) over \( X \), 
\( h_{\overline{Y}_N/X} = \deg_X(p_\ast \omega_{Y_N/X}^0) \) its modular height.

**Corollary 4:** The curve \( \overline{Y}_N/X(N) \) is semi-stable and has bad reduction at \( X \setminus H \). Furthermore, its Jacobian \( J = J_N \) has bad reduction at \( X(N)_\infty := X(N) \setminus X'(N) \), and its modular height is
\[
h_{\overline{Y}_N/X} = h_{J/X} = \frac{1}{2}(2g_{X(N)} - 2 + \#X(N)_\infty).
\]

In particular, for \( N = 3 \) one thus obtains a semi-stable family \( p : \overline{Y}_3 \to \mathbb{P}^1 \) whose Jacobian has precisely 4 places of bad reduction.

**Remarks:**
1) By a theorem of Faltings it follows (in \( \text{char} = 0 \)) that for any such curve we have the inequality
\[
h_{\overline{Y}_N/X} = h_{J/X} \leq \frac{1}{2}(2g_{X(N)} - 2 + \#X(N)_\infty).
\]

2) In a recent preprint E. Viehweg and K. Zuo study the structure of families of abelian varieties with such “minimal degeneration”.
5. The Basic Construction

**Reference:** Frey/K., Curves of genus 2 covering elliptic curves . . . (Texel Conference, 1989)

**Given:**

\[
\begin{array}{c}
X \\
\downarrow f \sim \\
E \\
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow \\
E \\
\end{array}
\quad
\begin{array}{c}
\sim \\
\downarrow \psi : E[N] \sim E^\perp[N]. \\
E^\perp \\
\end{array}
\]

(via the duality theory of \( J_X \).)

**Conversely:** given anti-isometry \( \psi : E[N] \to E'[N] \), one can recover a (normalized) genus 2 cover

\[
f_\psi : X_\psi \to E.
\]

However: the curve \( X_\psi \) may be reducible!

\[ \Rightarrow H_{E/K,N} \subset X_{E/K,N,-1}. \]

**Note:**

1) The moduli space \( X_{E/K,N,-1} \) classifies pairs \( (E', \psi) \), where \( \psi : E[N] \to E'[N] \) is an anti-isometry.

2) This construction also works for families! (Cf. IEM Preprint, op. cit.): \( \Rightarrow \) Theorem 3 \( \Rightarrow \) Theorem 2 \( \Rightarrow \) Theorem 1.
6. Proof of Theorem 6 (Overview)

Remark: The proof of Theorem 6 uses the methods of Arithmetic Algebraic Geometry. More precisely, it uses:

- a study of degenerations of the universal cover

\[ f_{\text{univ}} : X_H \rightarrow E \times H; \]

In other words:

1) study the degeneration of the minimal model \( M(X_H) \) of \( X_H \); this uses the modular height of relative curves.

2) study whether or not \( f_{\text{univ}} \) extends to a cover

\[ f : M(X_H) \rightarrow E \times X(N). \]

- intersection theory on \( M(X_H) \).
7. Study of Degenerations

Let $H = H_{E/\overline{K},N}$ denote the moduli space,
$f_H : Y_H \to E_H = E \times_{\overline{K}} H$ the universal cover,
$X = X(N) \supset H$ the natural compactification,
$\overline{Y}/X$ the minimal model of the generic fibre of $Y_H$.

Facts. 1) The fibres of $\overline{Y}/X$ are semi-stable.

2) $f_H$ extends to a morphism $f = f_X : \overline{Y} \to E_X$ which is finite if and only if $\text{char}(K) \nmid N!$.

Theorem 7: Suppose $\text{char}(K) \nmid N!$. Then:

(a) The fibres $\overline{Y}_x$ of $\overline{Y}/X$ are stable curves with at most one singularity.

(b) $\overline{Y}_x$ is singular if and only if $x \in D_{E/\overline{K},N} = X_\infty \cup X_1$, where $X_\infty$ is the set of cusps of $X$.
(Note that $\#X_\infty = \overline{sl}(N)/N$.)

(c) If $x \in X_\infty$, then $\overline{Y}_x$ is an irreducible curve whose normalization is a curve of genus 1.

(d) If $x \in X_1$, then $\overline{Y}_x = E_{x,1} \cup E_{x,2}$ is the union of two curves of genus 1 which meet transversely in a unique point $P_x$. 
8. Calculation of Intersection Numbers

Let \( F = \kappa(X) \) denote the function field of \( X = X(N) \),
\[ f_F : Y_F \rightarrow E_F \] the generic cover over \( F \),
\[ D_F = \text{Diff}(f_F) \] the different divisor of \( f_F \),
\[ W_{C_F} \in \text{Div}(Y_F) \] the hyperelliptic divisor of \( Y_F \),
\( D \) and \( W \) their respective closures in \( \overline{Y} \),
\[ \omega^0_{Y/X} \] the relative dualizing sheaf of \( p_{\overline{Y}} : \overline{Y} \rightarrow X \).

**Theorem 8:** The modular height of \( \overline{Y}/X \) is

\[ h_{\overline{Y}/X} := \deg((p_{\overline{Y}})_*(\omega^0_{Y/X})) = \frac{1}{12} s\ell(N), \]

and the self-intersection number of \( \omega^0_{Y/X} \) is

\[ (\omega^0_{Y/X})^2 = \frac{7}{5} \#X_1 + \frac{1}{5} \#X_\infty = \frac{1}{12N}(7N - 6) s\ell(N). \]

**Remark:** The proof uses Theorem 4, the Noether formula and Mumford’s formula (which holds if \( g = 2 \)):

\[ h = \omega^2 + \delta_0 + \delta_1 \quad \text{and} \quad 5\omega^2 = \delta_0 + 7\delta_1, \]

where \( h = h_{\overline{Y}/X}, \omega = \omega^0_{Y/X} \), and \( \delta_0 \) (respect. \( \delta_1 \))
is the number of singular points of all fibres which do not (respect. do) disconnect the fibre.
**Theorem 9:** (a) $D$ is an irreducible curve on $\overline{Y}$ which represents the dualizing sheaf: $\omega^0_{\overline{Y}/X} \sim D$.

(b) If $q_1 = pr_1 \circ f|_D : D \rightarrow E$ and $q_2 = pr_2 \circ f|_D : D \rightarrow X$, then $\pi_E \circ q_1 = \overline{\delta}_{E,N} \circ q_2$, where $\overline{\delta} : X \rightarrow \mathbb{P}^1$ is the unique extension of $\delta : H \rightarrow \mathbb{P}^1$. Thus

$$\deg(\overline{\delta}) = \deg(q_1) = (\omega^0_{\overline{Y}/X} \cdot f^*(P \times X)).$$

(c) We have $6D \sim 2W + f^*(E \times A)$, for some $A \in \text{Div}(X)$, and hence

$$\deg(q_1) = \frac{N}{6} \deg(A) = \frac{N}{36}(9(\omega^0_{\overline{Y}/X})^2 - W^2).$$

(d) The self-intersection number of $W$ is

$$W^2 = \frac{6}{7} \#X_1 - \frac{9}{7} \omega^2 = -\frac{3}{4N}(N - 2)\overline{sl}(N).$$

**Remark:** To compute $W^2$, consider the pullback $W^*$ of $W$ to (the desingularization of) $\overline{Y} \times_X X(2N)$, and observe that $W^* = W_1 + \ldots + W_6 + B$, where the $W_i$’s are 6 disjoint sections and $B$ is a fibral divisor supported on the fibres over $X(2N)_\infty$. 