Hurwitz Spaces of Genus 2

Covers of an Elliptic Curve

Ernst Kani

Department of Mathematics and Statistics
Queen's University
Kingston, Ontario
Http://www.mast.queensu.ca/~kani

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1. Introduction

Recall: If X is a compact Riemann surface, then

$$H(X,N,w) := \{\text{(equiv. cl. of) } N\text{-sheeted coverings of } X \text{ with } w \text{ branch points}\}/\simeq$$

has the structure of a (not nec. connected) complex manifold (Hurwitz, 1891).

Theorem A (Hurwitz, 1891): R.E.T $\Rightarrow$

(a) $H^{simple}(\mathbb{P}^1, N, w)$ is a connected manifold.

(b) The discriminant map $\delta : H^{simple}(\mathbb{P}^1, N, w) \to (\mathbb{P}^1)^{(w)} \setminus \Delta_w$ is finite and etale.

Theorem B (Fulton, 1969): $\exists$ fine moduli space $H = H^{simple}(\mathbb{P}^1, N, w)_{/\mathbb{Z}}$ whose fibres H_k/k satisfy (a) and (b) if (and only if) $\mathrm{char}(k) \nmid N!$.

Fried, Völklein, Harbater, Wevers,. . . : studied moduli spaces of other types of covers $/\mathbb{P}^1$.

Aim: Study analogues of these results in the case that $X = E$ is an elliptic curve (and $w = 2$).

2. Normalized Genus 2 Covers

Reference: E.K., Hurwitz spaces of genus 2 covers, IEM Preprint No. 9 (2001), IEM Essen. (See also www.mast.queensu.ca/~kani.)

Let E/K be an elliptic curve over a field K of char $\neq 2$,
$E[2]^{\#} = P_1 + P_2 + P_3$, if $E[2] = \{0_E, E_1, E_2, E_3\}$,
$N \geq 3$ an integer with $(\mathrm{char}(K), N) = 1$,
$f : C \to E$ a genus 2 cover of E/K of degree N,
$\sigma_C \in \mathrm{Aut}(C)$, the hyperelliptic involution of C,
$\pi_C : C \to \langle\sigma_C\rangle\backslash C \simeq \mathbb{P}^1$, the hyperelliptic cover,
$W_C = \mathrm{Diff}(\pi_C)$, the hyperelliptic divisor of C.

Definition. 1) $f : C \to E$ is said to be minimal if f does not factor over an isogeny of E.

2) f is normalized if it is minimal and if the norm (or direct image) of W_C has the form

$$f_*(W_C) = 3[0_E] + E[2]^{\#}, \text{ resp. } f_*(W_C) = 2E[2]^{\#},$$

if $N = \deg(f)$ is odd or even, respectively.

Notes: 1) If $f : C \to E$ is minimal, then $\exists! x \in E(K)$ such that $T_x \circ f : C \to E$ is normalized.

2) If f is normalized, then $f \circ \sigma_C = [-1]_E \circ f$.

3) $(f : C \to E) \rightsquigarrow (\overline{f} : \mathbb{P}^1 \to \mathbb{P}^1)$ →Frey/K./Völklein

3. Results

For any extension field L/K put

$$\mathcal{H}_{E/K,N}(L) = \{f : C \to E_L \text{ norm. genus 2 cover}\}/\simeq.$$

The assignment $L \mapsto \mathcal{H}_{E/K,N}(L)$ naturally extends to all K-schemes to define a Hurwitz functor

$$\mathcal{H}_{E/K,N} : \underline{\mathrm{Sch}}_{/K} \to \underline{\mathrm{Sets}}.$$

Theorem 1: The functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,N}/K$. More precisely, $H_{E/K,N}$ is an open subscheme of a certain twist $X_{E/K,N,-1}$ of the modular curve $X(N)$ of level N. In particular,

$$H_{E/K,N} \otimes \overline{K} \overset{open}{\subset} X(N)_{/\overline{K}},$$

i.e. $D_{E/K,N} := X(N)_{/\overline{K}} \setminus H_{E/K,N} \otimes \overline{K}$ is a finite set (called the degeneracy locus).

Remarks: 1) A similar statement holds for families of elliptic curves E/S, S any scheme (with $\frac{1}{2N} \in S$.)

2) If K is a number field, then Th. 1 and Faltings' theorem show that $\#H_{E/K,N}(K) < \infty$ for $N \geq 7$ (because then $g_{X(N)} \geq 2$).

Theorem 2: We have

$$D_{E/K,N} \leq \tfrac{1}{24}(5N-6)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}),$$

and equality holds if and only if $\mathrm{char}(K) \nmid N!$.

– reinterpretation of results of Crelle J. 485 (1997) + J. No. Th. 64 (1997).

Theorem 3: The assignment $(C \xrightarrow{f} E) \mapsto \mathrm{Disc}(f)$ is represented by a quasi-finite morphism

$$\delta = \delta_{E/K,N} : H_{E/K,N} \to \mathbb{P}^1_K.$$

Furthermore, if $\mathrm{char}(K) \nmid N!$, then δ is finite and etale outside of $\pi_E(E[2]) \subset \mathbb{P}^1$.

Theorem 4: If $\mathrm{char}(K) \nmid N!$, then

$$\deg(\delta_{E/K,N}) = \tfrac{1}{12}(N-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Remark: This degree can be viewed as a measure of non-rigidity of coverings (→ Völklein). Indeed, if

$$\mathrm{Cov}_{E,N,\overline{P}} := \{f \in \mathcal{H}_{E/K,N}(\overline{K}) : \mathrm{Disc}(f) = \pi_E^*\overline{P}\},$$

where $\overline{P} \in \mathbb{P}^1(\overline{K})$, then we have:

Corollary: If $\overline{P} \in \mathbb{P}^1(\overline{K}) \setminus \pi_E(E[2])$, then

$$\mathrm{Cov}_{E,N,\overline{P}} = \tfrac{1}{12}(N-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Note: 1) This number and/or degree is closely connected with the total (weighted) number of genus 2 covers of an elliptic curve $E/\overline{K}$ with fixed (generic) discriminant D (where $\deg(D) = 2$). For $K = \mathbb{C}$, this (total) number can be computed by using mirror symmetry (cf. Dijkgraaf, Mazur).

2) Since δ is ramified at the points $\overline{P} \in \pi_E(E[2])$, it is natural to count the points $P_x \in \mathrm{Cov}_{E,N,\overline{P}}$ according to their ramification degrees, i.e.

$$\deg(\mathrm{Cov}_{E,N,\overline{P}}) := \sum_{P_x} e_\delta(P_x).$$

We then obtain:

Theorem 5: If $\mathrm{char}(K) \nmid N!$ and N is odd, then for $\overline{P} \in \pi_E(E[2])$ we have

$$\deg(\mathrm{Cov}_{E,N,\overline{P}}) = \begin{cases} \frac{3}{16N}(N-1)sl(N) & \text{if } \overline{P} = \bar{0}_E \\ \frac{1}{16N}(N-1)sl(N) & \text{otherwise} \end{cases}$$

where $sl(N) = \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\bar{0}_E = \pi_E(0_E)$.

4. The Basic Construction

Reference: Frey/K., Curves of genus 2 covering elliptic curves … (Texel Conference, 1989)

Given:

$$\begin{tikzcd} C \arrow[d, "f"'] & & & C \arrow[dl] \arrow[dr] & & \\ E \arrow[r, phantom, "\rightsquigarrow"] & {} & E & & E^{\perp} \arrow[r, phantom, "\rightsquigarrow"] & \psi : E[N] \xrightarrow{\sim} E^{\perp}[N]. \end{tikzcd}$$

(via the duality theory of J_C.)

Conversely: given anti-isometry $\psi : E[N] \to E'[N]$, one can recover a (normalized) genus 2 cover

$$f_\psi : C_\psi \to E.$$

However: the curve C_ψ may be reducible!
$\Rightarrow H_{E/K,N} \subset X_{E/K,N,-1}$.

Note: 1) The moduli space $X_{E/K,N,-1}$ classifies pairs (E', ψ), where $\psi : E[N] \to E'[N]$ is an anti-isometry.

2) This construction also works for families! (Cf. E.K., Hurwitz spaces …).

5. Study of Degenerations

Let $H = H_{E/\overline{K},N}$ denote the moduli space,
$f_H : \mathcal{C}_H \to E_H = E \times_{\overline{K}} H$ the universal cover,
$X = X(N) \supset H$ the natural compactification,
$\mathcal{C}/X$ the minimal model of the generic fibre of $\mathcal{C}_H$.

Facts. 1) The fibres of $\mathcal{C}/X$ are semi-stable.

2) f_H extends to a morphism $f = f_X : \mathcal{C} \to E_X$ which is finite if and only if $\mathrm{char}(K) \nmid N!$.

Theorem 6: Suppose $\mathrm{char}(K) \nmid N!$. Then:

(a) The fibres $\mathcal{C}_x$ of C/X are stable curves with at most one singularity.

(b) $\mathcal{C}_x$ is singular if and only if $x \in D_{E/\overline{K},N} = X_\infty \,\dot{\cup}\, X_1$, where X_∞ is the set of cusps of X. (Note that $\#X_\infty = sl(N)/N$.)

(c) If $x \in X_\infty$, then $\mathcal{C}_x$ is an irreducible curve whose normalization is a curve of genus 1.

(d) If $x \in X_1$, then $\mathcal{C}_x = E_{x,1} \cup E_{x,2}$ is the union of two curves of genus 1 which meet transversely in a unique point P_x.

6. Calculation of Intersection Numbers

Notation: Let $f_F : \mathcal{C}_F \to E_F$ denote the generic cover over $F = \kappa(X)$, and let D and W denote the closures (in $\mathcal{C}$) of the different divisor $D_F = \mathrm{Diff}(f_F)$ and of the hyperelliptic divisor W_{C_F} on $\mathcal{C}_F$.

Theorem 7: (a) The modular height of $\mathcal{C}/X$ is $h_{\mathcal{C}/X} = \frac{1}{24}sl(N)$ and the self-intersection number of the relative dualizing sheaf $\omega^0_{\mathcal{C}/X}$ is given by

$$(\omega^0_{\mathcal{C}/X})^2 = \frac{7}{5}\#X_1 + \frac{1}{5}\#X_\infty = \frac{1}{24N}(7N-6)sl(N).$$

(b) D is an irreducible curve on $\mathcal{C}$ which represents the dualizing sheaf: $\omega^0_{\mathcal{C}/C} \sim D$.

(c) If $q_1 = pr_1 \circ f_{|D} : D \to E$ and $q_2 = pr_2 \circ f_{|D} : D \to X$, then $\pi_E \circ q_1 = \overline{\delta}_{E,N} \circ q_2$, where $\overline{\delta} : X \to \mathbb{P}^1$ is the unique extension of $\delta : H \to \mathbb{P}^1$. Thus

$$\deg(\overline{\delta}) = \deg(q_1) = (\omega^0_{\mathcal{C}/X} . f^*(P \times X)).$$

(d) We have $6D \sim 2W + f^*(E \times A)$, for some $A \in \mathrm{Div}(X)$, and hence

$$\deg(q_1) = \frac{N}{6}\deg(A) = \frac{N}{36}(9(\omega^0_{\mathcal{C}/X})^2 - W^2).$$