Hurwitz Spaces of Genus 2 Covers of an Elliptic Curve

1. Introduction

Hurwitz spaces: – introduced by Hurwitz in 1891.

– classify covers $f : Y \to X$ of degree N over a fixed curve X/K with w ramification points $R_f \subset X$.

Problem (Hurwitz) Fix X/K, N, w and $R \subset X$.

1) Investigate the totality $H(X/K, N, w)$ of all covers $f : Y \to X$ with $\deg(f) = N$ and $\#R_f = w$.

2) Calculate the number $\#H(X/K, N, R)$ of such covers with $R_f = R$.

Remarks: 1) Throughout, we always consider equivalence classes of covers: $(Y_1 \xrightarrow{f_1} X) \sim (Y_2 \xrightarrow{f_2} X) \Leftrightarrow \exists \phi : Y_1 \xrightarrow{\sim} Y_2$ with $f_2 \circ \phi = f_1$.

2) A cover $f : Y \to X$ is called simple if

$$f^{-1}(x) \geq \deg(f) - 1, \quad \text{for all } x \in X.$$

Such covers constitute an important subclass of all covers.

Theorem A (Hurwitz, 1891): If $K = \mathbb{C}$, then

(a) $H(X/\mathbb{C}, N, w)$ is a "Riemannian space".

(b) $H^{simple}(\mathbb{P}^1/\mathbb{C}, N, w)$ is a connected manifold of dimension w (provided that $w \geq 2N-2$ and $2|w$).

(c) The discriminant map

$$\delta : H^{simple}(\mathbb{P}^1/\mathbb{C}, N, w) \to (\mathbb{P}^1)^{(w)} \setminus \Delta_w$$

is finite and etale. Thus, $\#H^{simple}(\mathbb{P}^1/\mathbb{C}, N, R)$ depends only on $w = \#R$.

Observation (Hurwitz): RET $\Rightarrow$ the calculation of $\#H(X/\mathbb{C}, N, R)$ is a purely group-theoretic problem, albeit one that is "highly complicated" (Hurwitz).

Hurwitz (1891/1901) found a "satisfactory solution" for calculating $n_{N,w} := \#H^{simple}(\mathbb{P}^1/\mathbb{C}, N, R)$:

$$\begin{aligned} n_{3,w} &= \tfrac{1}{3!}(3^{w-1} - 3), \quad (n_{2,w} = 1) \\ n_{4,w} &= \tfrac{1}{4!}(2^{w-2} - 4)(3^{w-1} - 3), \text{ etc.} \end{aligned}$$

Question: What about other ground fields?

Hurwitz (1891): partial results for $K = \mathbb{R}$.

Fulton (1969): consider covers of curve families/S:

$$f : \mathcal{Y} \to \mathbb{P}^1_S = \mathbb{P}^1 \times S, \ \ S \text{ any scheme.}$$

Thus: 1) The fibres $f_s : \mathcal{Y}_s \to (\mathbb{P}^1_S)_s = \mathbb{P}^1_{\kappa(s)}$ of f are covers of curves in the previous sense.

2) The assignment $S \mapsto H^{simple}(\mathbb{P}^1_S/S, N, w)$ defines a functor $\mathcal{H}_{N,w} : \underline{Sch} \to \underline{Sets}$.

Theorem C (Fulton, 1969): If $N \geq 3$, then the functor $\mathcal{H}_{N,w}$ is (finely) representable by a scheme $H_{N,w}/\mathbb{Z}$ of finite type. In particular, for any field K we have $H_{N,w}(K) = H^{simple}(\mathbb{P}^1/K, N, w)$.

In addition, the restriction of the discriminant map to $H_{N,w} \otimes \mathbb{Z}[1/N!] \subset H_{N,w}$,

$$\delta : H_{N,w} \otimes \mathbb{Z}[1/N!] \to (\mathbb{P}^1_{\mathbb{Z}[1/N!]})^{(w)} \setminus \Delta_w,$$

is finite and etale.

Remark: Little seems to be known about the geometric structure of $H_{N,w}$.

Aim: Study analogues of these results in the case that $X = E$ is an elliptic curve (and $w = 2$).

2. The Case $X = E$ and $w = 2$ ($\Rightarrow g_Y = 2$).

Reference: IEM Preprint No. 9 (2001), IEM Essen. (See also www.mast.queensu.ca/~kani.)

Let E/K be an elliptic curve over a field K (char $\neq 2$).

1) $E(K)$ acts on E and hence on $H(E/K, N, 2)$ via translation: $f \mapsto T_x \circ f$. Thus $H \sim H' \times E$.

2) Each $Y \xrightarrow{f} E$ factors as $Y \xrightarrow{f'} E' \xrightarrow{u_f} E$, where $u_f : E' \to E$ is the max. unramified subcover of f.

Thus: $H(\ldots)$ is a union of components which are indexed by subgroups $G \leq E$ with $\#G|N$ (and $\#G \neq N$); explictly, $G = \mathrm{Ker}(\hat{u}_f)$.

Definition: a) f is called minimal if $\deg(u_f) = 1$.

b) A cover $f : Y \to E$ with $g_Y = 2$ is called normalized if it is minimal and if

$$f(W) \subset E[2] \text{ and } \#(f^{-1}(0_E) \cap W) = \begin{cases} 3 & N \text{ odd} \\ 0 & \text{else} \end{cases}$$

where $W = \mathrm{Fix}(\sigma_Y)$ denotes the set of 6 Weierstrass points of Y. (Here: σ_Y is the hyperelliptic involution of Y.)

Notes: 1) If $f : Y \to E$ is minimal, then $\exists!\, x \in E(K)$ such that $T_x \circ f : Y \to E$ is normalized.

2) If f is normalized, then $f \circ \sigma_Y = [-1]_E \circ f$. Thus $\mathrm{Disc}(f)$ is symmetric with respect to $[-1]_E$, i.e. $[-1]_E^* \mathrm{Disc}(f) = \mathrm{Disc}(f)$.

Example: Let

$$\begin{aligned} E: \quad y^2 &= (x-a)(x-b)(x-c), \quad abc \neq 0 \\ Y: \quad s^2 &= (t^2-a)(t^2-b)(t^2-c). \end{aligned}$$

Then the cover $f : Y \to E$, given by $f^*x = t^2$, $f^*y = s$, is normalized and of degree 2.

Remark: The notion of normalized curve covers can be extended naturally to families of curve covers $f : \mathcal{Y} \to E_S = E \times S$ so as to obtain a functor

$$\mathcal{H} = \mathcal{H}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets}$$

given by $S \mapsto \mathcal{H}(S) = \{\mathcal{Y} \xrightarrow{f} E_S : f \text{ is normalized}, \deg(f) = N\}$.

More generally, if E/S is a family of elliptic curves over a base S, then one obtains in a similar way a functor

$$\mathcal{H}_{E/S,N} : \underline{Sch}_{/S} \to \underline{Sets}.$$

Theorem 1: If $N \geq 3$ and $\mathrm{char}(K) \nmid N$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,N}/K$. More precisely, $H_{E/K,N}$ is an open subscheme of a certain twist $X_{E/K,N,-1}$ of the modular curve $X(N)$ of level N. In particular,

$$H_{E/K,N} \otimes \overline{K} \overset{open}{\subset} X'(N)_{/\overline{K}} \subset X(N)_{/\overline{K}}.$$

Remarks: 1) If $K = \mathbb{C}$, then $X'(N) = \Gamma(N)\backslash\mathfrak{H}$ and $X(N) = \Gamma(N)\backslash\mathfrak{H}^*$, which is a Galois cover of $X(1) \simeq \mathbb{P}^1$ of degree

$$\overline{sl}(N) := |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}|.$$

2) Theorem 1 extends to families of elliptic curves E/S, where S any scheme (with $\frac{1}{2N} \in S$).

Example: Let $S = X'(N)_{/\mathbb{Z}[\zeta_N, 1/N]}$ and $E = E(N)$, the universal elliptic curve over $X'(N)$ (with level N structure). Then $\mathcal{H}_{E(N)/X'(N),N}$ is represented by an open affine subscheme

$$H_{E(N)/X'(N),N} \subset X'(N) \times X'(N).$$

Theorem 2: Let

$$D_{E/K,N} = X(N)_{/\overline{K}} \setminus (H_{E/K,N} \otimes \overline{K})$$

denote the degeneracy locus. Then

$$\# D_{E/K,N} \le \tfrac{1}{12N}(5N+6)\overline{sl}(N),$$

and equality holds if and only if $\mathrm{char}(K) \nmid N!$.

– reinterpretation of results of Crelle J. 485 (1997) + J. No. Th. 64 (1997).

Theorem 3: The assignment $(Y \xrightarrow{f} E) \mapsto \mathrm{Disc}(f)$ is represented by a quasi-finite morphism

$$\delta = \delta_{E/K,N} : H_{E/K,N} \to \mathbb{P}^1_K \simeq (E^{(2)})^{sym}.$$

Furthermore, if $\mathrm{char}(K) \nmid N!$, then δ is finite and etale outside of $\pi_E(E[2]) \subset \mathbb{P}^1$.

Theorem 4: If $\mathrm{char}(K) \nmid N!$, then

$$\deg(\delta_{E/K,N}) = \tfrac{1}{6}(N-1)\overline{sl}(N).$$

Remarks: 1) This degree can be viewed as a measure of non-rigidity of coverings ($\to$ Völklein).

2) H. Völklein proved Theorem 4 for $N = 3, 5, 7$ by using group theory (and a computer).

3. Some applications

(a) Rationality Questions (K a number field)

Since $g_{X(N)} \geq 2$ for $N \geq 7$, we have by Faltings' theorem (= Mordell's Conjecture):

Corollary 1: $\#\mathcal{H}_{E/K,N}(K) < \infty$, if $N \geq 7$.

Question: Is $\mathcal{H}_{E/K,N}(K) = \emptyset$, for $N >> 0$?

This is false (even for N prime), for there exist curves Y/K with ∞'ly many $f_N : Y \to E$ for which $N = \deg(f_N)$ is prime.

Conjecture (*) For each E/K there exist only finitely many genus 2 curves Y/K which have a (minimal) morphism $f : Y \to E$ of degree $N \geq 7$.

Remark: ABC conj. $\Rightarrow$ Asym. Fermat $\Rightarrow$ Conj. (*).

Moreover, the converse: Conj. (*) $\Rightarrow$ Asym. Fermat is "almost true": it implies a slightly weaker version of Frey's Conjecture 5 (which by Frey and Wiles is equivalent to the Asymptotic Fermat Conjecture (for $K = \mathbb{Q}$).)

(b) Moduli

Question: For which curves Y/K does there exist a (minimal) morphism $f : Y \to E$ of degree N?

Corollary 2: For every N there exists a morphism

$$\mu_N : H_{E(N)/X'(N),N} \to M_2$$

to the moduli space of curves of genus 2. Moreover:

a) $\mathrm{Im}(\mu_N) =$ Humbert surface with Inv. $\Delta = N^2$;

b) $\deg(\mu_N) = 2\overline{sl}(N)$; more precisely,

$\mathrm{Im}(\mu_N) \sim Z^{sym}_{N,-1} := (X(N) \times X(N))/\langle \Delta_{N,-1}, \tau \rangle$,

where $\tau(x,y) = (y,x)$ and

$$\Delta_{N,-1} = \{(g, \alpha_{-1}(g)) : g \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}\},$$

where $\alpha_{-1}(g) = Q_{-1} g Q_{-1}^{-1}$ with $Q_{-1} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

In particular, the normalization (and compactification) of the Humbert surface $\mathrm{Im}(\mu_N)$ is the symmetric Diagonal Quotient Surface $Z^{sym}_{N,-1}$.

(c) Counting Covers: $(K = \overline{K})$

Corollary 3: If $N \geq 2$ and $\mathrm{char}(K) \nmid N$, then for every $R \subset E$ with $\#R = 2$ we have

$$c_N := \sum_{f \in H^s(E/K,N,R)} \frac{1}{|\mathrm{Aut}(f)|} = \tfrac{1}{3}(N\sigma_3(N) - N^2\sigma_1(N)),$$

where $\sigma_k(N) = \sum_{d|N} d^k$. Thus, if $\mathrm{char}(K) = 0$, then $F_2(q) := \sum c_N q^N$ is a quasi-modular form of weight 6; explicitly we have

$$(1) \qquad F_2(q) = \frac{1}{51840}(10E_2^3 - 6E_2E_4 - 4E_6),$$

where $E_k = 1 + b_k \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ with $b_2 = -24, b_4 = 240$ and $b_k = -504$.

Remarks: 1) The identity (1) was first proven by R. Dijkgraaf (1995) by using the methods of mirror symmetry.

2) Theorem 4 $\Rightarrow$ Corollary 3 by using the identities

$$\sum_{n|N} \sigma_1(n)\mathrm{sl}(N/n) = \sigma_3(N),$$

$$\sum_{n|N} n\sigma_1(n)\mathrm{sl}(N/n) = N^2\sigma_1(N).$$

(d) Curves with minimal degeneration:

Let $H = H_{E/K,N} \subset X = X(N)$ be the moduli space,

$f : Y_N \to E_H = E \times H$ the universal cover,

$p : \overline{Y}_N \to X$ the minimal model of Y_N over X,

$h_{\overline{Y}_N/X} = \deg_X(p_*\omega^0_{Y_N/X})$ its modular height.

Corollary 4: The curve $\overline{Y}_N/X(N)$ is semi-stable and has bad reduction at $X \setminus H$. Furthermore, its Jacobian $J = J_N$ has bad reduction at $X(N)_\infty := X(N) \setminus X'(N)$, and its modular height is

$$h_{\overline{Y}_N/X} = h_{J/X} = \tfrac{1}{2}(2g_{X(N)} - 2 + \#X(N)_\infty).$$

In particular, for $N = 3$ one thus obtains a semi-stable family $p : \overline{Y}_3 \to \mathbb{P}^1$ whose Jacobian has precisely 4 places of bad reduction.

Remarks: 1) By a theorem of Faltings it follows (in $\mathrm{char} = 0$) that for any such curve we have the inequality

$$h_{\overline{Y}_N/X} = h_{J/X} \le \tfrac{1}{2}(2g_{X(N)} - 2 + \#X(N)_\infty).$$

2) In a recent preprint E. Viehweg and K. Zuo study the structure of families of abelian varieties with such "minimal degeneration".

4. The Basic Construction

Reference: Frey/K., Curves of genus 2 covering elliptic curves . . . (Texel Conference, 1989)

Given:

$$\begin{array}{c} Y \\ f\downarrow \\ E \end{array} \rightsquigarrow \begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ E & & E^{\perp} \end{array} \rightsquigarrow \psi : E[N] \xrightarrow{\sim} E^{\perp}[N].$$

(via the duality theory of J_Y.)

Conversely: given anti-isometry $\psi : E[N] \to E'[N]$, one can recover a (normalized) genus 2 cover

$$f_\psi : Y_\psi \to E.$$

However: the curve Y_ψ may be reducible!
$\Rightarrow H_{E/K,N} \subset X_{E/K,N,-1}$.

Note: 1) The moduli space $X_{E/K,N,-1}$ classifies pairs (E', ψ), where $\psi : E[N] \to E'[N]$ is an anti-isometry.

2) This construction also works for families! (Cf. IEM Preprint, op. cit.): $\Rightarrow$ Theorem 1.

5. Study of Degenerations

Let $H = H_{E/\overline{K},N}$ denote the moduli space,
$f_H : Y_H \to E_H = E \times_{\overline{K}} H$ the universal cover,
$X = X(N) \supset H$ the natural compactification,
$\overline{Y}/X$ the minimal model of the generic fibre of Y_H.

Facts. 1) The fibres of $\overline{Y}/X$ are semi-stable.

2) f_H extends to a morphism $f = f_X : \overline{Y} \to E_X$ which is finite if and only if $\mathrm{char}(K) \nmid N!$.

Theorem 5: Suppose $\mathrm{char}(K) \nmid N!$. Then:

(a) The fibres $\overline{Y}_x$ of $\overline{Y}/X$ are stable curves with at most one singularity.

(b) $\overline{Y}_x$ is singular if and only if $x \in D_{E/\overline{K},N} = X_\infty \,\dot\cup\, X_1$, where X_∞ is the set of cusps of X. (Note that $\#X_\infty = \overline{sl}(N)/N$.)

(c) If $x \in X_\infty$, then $\overline{Y}_x$ is an irreducible curve whose normalization is a curve of genus 1.

(d) If $x \in X_1$, then $\overline{Y}_x = E_{x,1} \cup E_{x,2}$ is the union of two curves of genus 1 which meet transversely in a unique point P_x.

6. Calculation of Intersection Numbers

Let $F = \kappa(X)$ denote the function field of $X = X(N)$,

$f_F : Y_F \to E_F$ the generic cover over F,

$D_F = \mathrm{Diff}(f_F)$ the different divisor of f_F,

$W_{C_F} \in \mathrm{Div}(Y_F)$ the hyperelliptic divisor of Y_F,

D and W their respective closures in $\overline{Y}$,

$\omega^0_{\overline{Y}/X}$ the relative dualizing sheaf of $p_{\overline{Y}} : \overline{Y} \to X$.

Theorem 6: The modular height of $\overline{Y}/X$ is

$$h_{\overline{Y}/X} := \deg((p_{\overline{Y}})_*(\omega^0_{\overline{Y}/X})) = \frac{1}{12}\overline{sl}(N),$$

and the self-intersection number of $\omega^0_{\overline{Y}/X}$ is

$$(\omega^0_{\overline{Y}/X})^2 = \frac{7}{5}\#X_1 + \frac{1}{5}\#X_\infty = \frac{1}{12N}(7N-6)\overline{sl}(N).$$

Remark: The proof uses Theorem 2, the Noether formula and Mumford's formula (which holds if $g = 2$):

$$h = \omega^2 + \delta_0 + \delta_1 \quad \text{and} \quad 5\omega^2 = \delta_0 + 7\delta_1,$$

where $h = h_{\overline{Y}/X}, \omega = \omega^0_{\overline{Y}/X}$), and δ_0 (respect. δ_1) is the number of singular points of all fibres which do not (respect. do) disconnect the fibre.

Theorem 7: (a) D is an irreducible curve on $\overline{Y}$ which represents the dualizing sheaf: $\omega^0_{\overline{Y}/X} \sim D$.

(b) If $q_1 = pr_1 \circ f_{|D} : D \to E$ and $q_2 = pr_2 \circ f_{|D} : D \to X$, then $\pi_E \circ q_1 = \overline{\delta}_{E,N} \circ q_2$, where $\overline{\delta} : X \to \mathbb{P}^1$ is the unique extension of $\delta : H \to \mathbb{P}^1$. Thus

$$\deg(\overline{\delta}) = \deg(q_1) = (\omega^0_{\overline{Y}/X}.f^*(P \times X)).$$

(c) We have $6D \sim 2W + f^*(E \times A)$, for some $A \in \mathrm{Div}(X)$, and hence

$$\deg(q_1) = \frac{N}{6}\deg(A) = \frac{N}{36}(9(\omega^0_{\overline{Y}/X})^2 - W^2).$$

(d) The self-intersection number of W is

$$W^2 = \frac{6}{7}\#X_1 - \frac{9}{7}\omega^2 = -\frac{3}{4N}(N-2)\overline{sl}(N).$$

Remark: To compute W^2, consider the pullback W^* of W to (the desingularization of) $\overline{Y} \times_X X(2N)$, and observe that $W^* = W_1 + \ldots + W_6 + B$, where the W_i's are 6 disjoint sections and B is a fibral divisor supported on the fibres over $X(2N)_\infty$.