A new formulation of Netto’s argument and the case of degree seven.

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We are interested in groups with the following properties:
Let $n \geq 7$ be an odd integer, and consider subgroups $G \leq S_n$ that satisfy:

1. $G = \langle s_1, s_2, s_3, s_4 \rangle$ where $s_1 s_2 s_3 s_4 = 1$.

2. $(s_1, s_2, s_3, s_4) = ((2)^m, (2)^m, (2)^m, (2)^{m-3}(4))$ where $m = \frac{n-1}{2}$. The notation $(2)^m$ means that $s_i$ is a product of $m$ disjoint 2-cycles, and similarly, the cycle decomposition of $s_4$ consists of $m - 3$ disjoint 2-cycles and one 4-cycle.

3. $G$ acts primitively on $S = \{x_1, x_2, ..., x_n\}$.

Such groups occur in [1], pg. 17 (specifically, Prop. 3.6) as the monodromy groups of certain curve covers. In this paper, we prove the following fact about these groups:

**Theorem 1.** If a group $G$ satisfies the above conditions then $G = A_n$, if $n \equiv 1 \pmod{4}$ and $G = S_n$, if $n \equiv 3 \pmod{4}$.

In our demonstration, the case $n = 7$ will be considered separately because for the case $n \geq 9$ we prove a more general result:

**Theorem 2.** If $G$ is a primitive permutation group of degree $n \geq 9$ and $G$ contains a 4-cycle or a $(2,2)$-cycle then $G \geq A_n$.

The motivation for Theorem 2 came from Netto who, in his Theory of Substitutions, gave a sketched proof of an equivalent theorem (cf. [2] pg. 133-8). Unfortunately, a new formulation of his argument was necessary because of some ambiguity in his exposition (see Remark 1 below). In our new formulation, Theorem 2 follows from Propositions 1 and 2 appearing below.

Because it is a more general result, we begin with our proof of Theorem 2 and its antecedent propositions. Following this, we consider the case of $n = 7$ and conclude with our proof of Theorem 1. Before beginning, we need to establish the following terminology:

**Definition.** Let $G$ be a group acting on a set $\Omega$, and let $T$ be a subset of $G$. We define the support of $T$ by

$$\text{supp}(T) := \{x \in \Omega \mid x^g \neq x \text{ for at least one } g \in T\}.$$
When $T$ is a singleton, i.e. $T = \{g\}$, we write $\text{supp}(g)$ instead of $\text{supp}(\{g\})$. Additionally, due to its ubiquitous application, we define the following shorthand:

$$\text{supp}(g)(h) := \text{supp}(g) \cap \text{supp}(h).$$

**Proposition 1.** Let $G$ be a primitive permutation group of degree $n \geq 9$ and let $g, b \in G$ be two $(2,2)$ cycles such that:

$$\#\text{supp}(a_1)(a_2) = 1.$$  \hspace{1cm} (1)

Then $G \geq A_n$ unless we have:

$$\#\text{supp}(b_1)(a_2) = 0, \#\text{supp}(a_1)(b_2) = 0, \text{ and } \#\text{supp}(b_1)(b_2) = 1. \hspace{1cm} (2)$$

**Proof.** Let $G$ be a primitive permutation group acting on the set $\Omega := \{1, 2, \ldots, n\}$ with $n \geq 9$ and let $g, b \in G$ be two $(2,2)$ cycles such that condition (1) holds.

Our argument concerns the following variables: $\#\text{supp}(b_1)(a_2) \in \{0, 1\}$, and $\#\text{supp}(b_2)(a_1, b_1) \in \{0, 1, 2\}$. We begin by supposing that $\#\text{supp}(b_1)(a_2) = 1$, and show that for each possible value of $\#\text{supp}(b_2)(a_1 b_1), G \geq A_n$. Then, we suppose that $\#\text{supp}(b_1)(a_2) = 0$ and show that $G \geq A_n$ unless condition (2) holds.

Claim 1: $\#\text{supp}(b_1)(a_2) = 1 \implies G \geq A_n$.

The following three subcases are organized according to the value of $\#\text{supp}(b_2)(a_1 b_1) \in \{0, 1, 2\}$ wherein we show that in each case, $G \geq A_n$.

Case 0: $\#\text{supp}(b_2)(a_1 b_1) = 0$. Up to conjugacy, we have $g_1 = a_1 b_1 = (12)(34)$ and $g_2 = a_2 b_2 = (13)(56)$. Thus, we have $g_3 := g_2 g_1 g_2 = (12)(34)(23) \in G$ and consequently $g_1, g_2 = (13)(24) \in G$. Now, let $\Gamma := \{1, 2, 3, 4\}$ and $\Delta := \Omega \setminus \Gamma$. Then $G_\Delta$ is transitive on the set $\Gamma$ because $g_1, g_3, g_1 g_3 \in G_\Delta$. Since by assumption $|\Omega| \geq 9$, we have $|\Gamma| = 4 < \frac{1}{2} |\Omega|$, and so it follows from a result of Marggraf (cf. [3] pg. 35) that $G \geq A_n$.

Case 1: $\#\text{supp}(b_2)(a_1 b_1) = 1$. There are two cases to consider here: either i) $\#\text{supp}(b_2)(a_1) = 1$ and $\#\text{supp}(b_2)(b_1) = 0$ or ii) $\#\text{supp}(b_2)(a_1) = 0$ and $\#\text{supp}(b_2)(b_1) = 1$. In either case, we claim that $G$ contains a 5-cycle. If we have i), then up to conjugacy, $g_1 = a_1 b_1 = (12)(34)$ and $g_2 = a_2 b_2 = (13)(24)$ which gives $g_1 g_2 = (13)(24) \in G$. On the other hand, if we have ii), then up to conjugacy, $g_1 = a_1 b_1 = (12)(34)$ and $g_2 = a_2 b_2 = (13)(24)$ which gives $g_1 g_2 = (13)(24) \in G$. So, if $\#\text{supp}(b_1)(a_2) = 1$ and $\#\text{supp}(b_2)(a_1 b_1) = 1$ then $G$ contains a 5-cycle. By Netto (cf. [2] pg. 93), a primitive group of degree $n$ which contains a $p$-cycle of order $p < \frac{2n}{3}$ contains the alternating group. In the case where $p = 5$ this is $15 < 2n$. Since by assumption $n \geq 9$, the inequality holds and so $G \geq A_n$.

Case 2: $\#\text{supp}(b_2)(a_1 b_1) = 2$. We cannot have $\#\text{supp}(b_2)(a_1) = 2$ because condition (1) and $b_2 = a_1$ would imply that $b_2$ and $a_2$ are not disjoint. Similarly, we cannot have $\#\text{supp}(b_2)(b_1) = 2$ because $\#\text{supp}(b_1)(a_2) = 1$ and $b_2 = b_1$ would also imply that $b_2$ and $a_2$ are not disjoint. Therefore, there is only one case to consider: $\#\text{supp}(b_2)(a_1) = 1$ and $\#\text{supp}(b_2)(b_1) = 1$. Up to conjugacy, this is $g_1 = a_1 b_1 = (12)(34)$ and $g_2 = a_2 b_2 = (13)(24)$. Thus, we have $g_3 := g_1 g_2 = (12)(34)(23) \in G$. Now, as in Case 0 above, let $\Gamma := \{1, 2, 3, 4\}$ and $\Delta := \Omega \setminus \Gamma$. Then, $g_1, g_2, g_3 \in G_\Delta$ and $G_\Delta$ is transitive on $\Gamma$. Since
\[ |\Gamma| = 4, \text{ the above cited result of Marggraf gives that } G \geq A_n. \]

Since the preceding cases exhaust the possibilities for \( \#\text{supp}(b_2)(a_1b_1) \in \{0, 1, 2\} \), Claim 1 is proved. We turn now to Claim 2.

Claim 2: \( \#\text{supp}(b_1)(a_2) = 0 \implies G \geq A_n \) unless \( (2) \).

As before, we divide the argument into subcases according to the value of \( \#\text{supp}(b_2)(a_1b_1) \in \{0, 1, 2\} \). Since \( (2) \) is included in the case \( \#\text{supp}(b_2)(a_1b_1) = 1, \) our first task is to show that if \( \#\text{supp}(b_2)(a_1b_1) \in \{0, 2\} \) then \( G \geq A_n \).

Case 0: \( \#\text{supp}(b_2)(a_1b_1) = 0 \). Up to conjugacy, we have \( g_1 = a_1b_1 = (12)(34) \) and \( g_2 = a_2b_2 = (15)(67) \). Thus, \( (g_1g_2)^2 = (125) \in G \). By a well-known result, if a primitive group contains a 3-cycle then \( G \geq A_n \) (cf. [3] pg. 34).

Case 2: \( \#\text{supp}(b_2)(a_1b_1) = 1 \). We cannot have \( \#\text{supp}(b_2)(a_1) = 2 \) because \( (1) \) \( \#\text{supp}(a_1)(a_2) = 1 \) and \( b_2 = a_1 \) would imply that \( a_2 \) and \( b_2 \) are not disjoint. Thus, there are two subcases to consider: either \( i) \) \( \#\text{supp}(b_2)(a_1) = 1 \) and \( \#\text{supp}(b_2)(b_1) = 1 \) or \( ii) \) \( \#\text{supp}(b_2)(a_1) = 0 \) and \( \#\text{supp}(b_2)(b_1) = 2 \). If we have \( i) \) then up to conjugacy, \( g_1 = a_1b_1 = (12)(34) \) and \( g_2 = a_2b_2 = (15)(23) \). Thus, \( g_1g_2 = (12)(34)(15)(23) = (15243) \in G \). While, if we have \( ii) \) then up to conjugacy, \( g_1 = a_1b_1 = (12)(34), \)

Thus, from the preceding two cases, we conclude that if \( \#\text{supp}(b_1)(a_2) = 0 \) and \( \#\text{supp}(b_2)(a_1b_1) \in \{0, 2\} \) then \( G \geq A_n \). The final case to consider is \( \#\text{supp}(b_2)(a_1b_1) = 1 \) where we show that \( G \geq A_n \) unless \( (2) \) holds.

Case 1: \( \#\text{supp}(b_2)(a_1b_1) = 1 \). Here, there are two cases to consider: either \( i) \) \( \#\text{supp}(b_2)(a_1) = 1 \) and \( \#\text{supp}(b_2)(b_1) = 0 \) or \( ii) \) \( \#\text{supp}(b_2)(a_1) = 0 \) and \( \#\text{supp}(b_2)(b_1) = 1 \). If we can show that the former implies \( G \geq A_n \) then we will have proven the proposition because \( ii) \) is precisely \( (2) \). If we have \( i) \) then up to conjugacy, \( g_1 = a_1b_1 = (12)(34) \) and \( g_2 = a_2b_2 = (15)(26) \). Also in the group is the element \( g_3 := g_1g_2 \) and consequently the element \( g_3g_2 = (25)(16)(15)(26) = (12) \). Now, let \( \Gamma := \{1, 2, 5, 6\} \) and \( \Delta := \Omega \setminus \Gamma \). Since, \( g_2, g_3, g_3g_2 \in G_\Delta, \) \( G_\Delta \) is transitive on the set \( \Gamma \). As before, \( |\Gamma| = 4 \) and the result of Marggraf cited above gives \( G \geq A_n \).

Thus, we have shown that if \( (1) \) holds then \( G \geq A_n \) unless \( (2) \) holds.

**Corollary 1.** Let \( G \) be a primitive permutation group of degree \( n \geq 9 \) which contains a \((2, 2)\) cycle \( g_1 = a_1b_1 \). Then \( G \) contains a second \((2, 2)\) cycle \( g_2 = a_2b_2 \) such that \( (1) \) holds. Moreover, \( G \geq A_n \) unless \( (2) \) holds.

**Proof.** Let \( G \) act on the set \( \Omega := \{1, \ldots, n\} \). Without loss of generality, we can assume that \( a_1 = (12) \). By a lemma of Rudio (cf. [2] pg. 78), the primitivity of \( G \) implies that there exists an \( h \in G \) such that \( 1^h \in \{1, 2\} \) while \( 2^h \notin \{1, 2\} \). Let \( a_2 := h a_1 h^{-1} = (1^h 2^{h}) \) and \( b_2 := h b_1 h^{-1} \). Then, \( g_2 := h g_1 h^{-1} \) is \( a_2b_2 \in G \) and by our choice of \( h \) we have \( (1) \) \( \#\text{supp}(a_1)(a_2) = 1 \). By Proposition 1, \( G \geq A_n \) unless \( (2) \) holds as well.

**Remark 1.** Proposition 1 roughly corresponds to cases A and B of Netto’s argument. Some specific differences in our formulation are the appeal to a result of Marggraf, and the fact that a 5-cycle in a primitive group of degree \( \geq 9 \) implies that the group.
contains the alternating group. Although new, these are only simplifications over Netto; his treatment of the third case is what in fact necessitated our new formulation.

In his case C analysis on pg. 136, Netto lists a number of possible cases for a \((2,2)\) cycle. One of which is \((x_1x_m)(x_nx_p)\) where \(m, n, p \in \{1, \ldots, n\} \setminus \{1, 2, 5\}\). On the following page, Netto claims that in every case a proper combination of this element with the other elements already determined to be in the group, \((x_1x_2)(x_3x_4), (x_1x_5)(x_3x_6)\), and \((x_2x_5)(x_4x_6)\), results in a 7-cycle. However, if \(m = 3, n = 7\) and \(p = 8\) then no such 7-cycle appears in any combination of these elements; in fact, the permutation group generated by these elements is of order 48 which cannot contain a 7-cycle. It is not entirely clear from the exposition whether Netto had a different range in mind for the indices or if this case was simply overlooked. In either case, a new argument was needed.

Since Netto’s case C originally began with \((x_1x_2)(x_3x_4)\) and \((x_1x_5)(x_3x_6)\) being in the group and these elements satisfy conditions (1) and (2) from above, we have given a new proof that such conditions imply that \(G \geq A_n\). This proof appears as:

**Proposition 2.** Let \(G\) be a primitive permutation group of degree \(n \geq 9\) acting on a set \(\Omega\). If \(G\) contains a subset \(T_k := \{g_1, g_2, \ldots, g_k\}\) with \(k \geq 2\), such that the following conditions hold:

1. \(g_i = a_ib_i\) is a \((2,2)\) cycle for \(1 \leq i \leq k\),
2. \(\exists \ t \in \Omega \ s.t. \ for \ i \neq j, 1 \leq i, j \leq k, \ supp(a_i)(a_j) = \{t\}\),
3. \(\Delta_a^k \cap \Delta_b^k = \emptyset\) where \(\Delta_a^k := \bigcup_{i=1}^k supp(a_i)\) and \(\Delta_b^k := \bigcup_{i=1}^k supp(b_i)\),

then \(G \geq A_n\).

**Proof.** Let \(G\) be a primitive permutation group of degree \(n \geq 9\) and let \(T_k\) be a subset of \(G\) that satisfies the hypotheses of the proposition.

Suppose that \(G \nless A_n\). We begin by establishing that the primitivity of \(G\) implies the existence of an element \(g_{k+1} = a_{k+1}b_{k+1} \in G\) such that \(g_{k+1}\) is a \((2,2)\) cycle and \(supp(a_i)(a_j) = \{t\}\) for \(i \neq j, 1 \leq i, j \leq k + 1\). In other words, \(T_{k+1} := T_k \cup \{g_{k+1}\}\) satisfies the first two conditions of the proposition. We then show that because we have supposed \(G \nless A_n\), \(T_{k+1}\) must also satisfy the third condition of the proposition. To complete the proof, we show that this implies a contradiction and hence we must have \(G \geq A_n\).

To begin, we have \(g_k = a_kb_k = (t u)(v w)\) where \(u, v, w \in \Omega\). Note that \(t \in supp(a_k)\) because \(g_k \in T_k\) and \(T_k\) satisfies condition 2 of the proposition. Since \(\Delta_a^k \neq \Omega\) by condition 3, the lemma of Rudio implies that there exists \(h \in G\) such that \(t^h \in \Delta_a^k\) while \(u^h \notin \Delta_a^k\). Consider \(h a_k h^{-1} = (t^h u^h)\) and \(h b_k h^{-1} = (v^h w^h)\). If \(t^h = t\) then let \(a_{k+1} := h a_k h^{-1} \) and \(b_{k+1} := h b_k h^{-1}\). Then, \(g_{k+1} := a_{k+1}b_{k+1} = h g_k h^{-1} = (t u^9)(v^9 w^9) \in G\) is a \((2,2)\) cycle, and \(supp(a_i)(a_j) = \{t\}\) for \(i \neq j, 1 \leq i, j \leq k + 1\). Thus, \(T_{k+1} := T_k \cup \{g_{k+1}\}\) satisfies the first two conditions of the proposition.

We now consider the case \(t^h \neq t\) and show that here too the set \(T_{k+1} := T_k \cup \{g_{k+1}\}\) satisfies the first two conditions of the proposition. If \(t^h \neq t\) then we still have \(t^h \in
Then, by (3), we have \( \#\text{supp}(a) \) for any \( s \in G \), and from \( \#\text{supp}(a) \) of the proposition. More precisely, from \( \Delta_k \neq 0 \) we prove presently. First of all, \( \#\text{supp}(a) \) must also satisfy the third condition; that is, \( \Delta_k \neq 0 \). Hence, in either case, \( T_{k+1} \) satisfies the first two conditions of the proposition.

The next step in the proof is to show that because of our supposition that \( G \neq A_n \), \( T_{k+1} \) must also satisfy the third condition; that is, \( \Delta_k \neq 0 \). The fact that \( T_{k+1} \) satisfies all three of the conditions will then be shown to imply a contradiction.

To show \( \Delta_k \neq 0 \), note that for distinct \( g_i, g_j \in T_{k+1}, \#\text{supp}(a_i)(a_j) = 1 \) so \( g_i \) and \( g_j \) satisfy condition (1) in Proposition 1. And since we’ve assumed that \( G \neq A_n \), Proposition 1 gives us (2): \( \#\text{supp}(b_i)(a_j) = 0 \), \( \#\text{supp}(a_i)(b_j) = 0 \), and \( \#\text{supp}(b_i)(b_j) = 1 \). Now, since this applies to any distinct \( g_i, g_j \in T_{k+1} \), we have \( \Delta_k \neq 0 \). Thus, \( T_{k+1} \) satisfies all three of the conditions in the proposition. What remains to be shown is that this leads to a contradiction.

The contradiction follows from the fact that \( \#\text{supp}(T_{k+1}) \geq \#\text{supp}(T_k) + 1 \) which we prove presently. First of all, \( \#\text{supp}(T_{k+1}) \geq \#\text{supp}(T_k \cup \{ a_{k+1} \}) \) because \( T_{k+1} = T_k \cup \{ g_{k+1} \} \). Now, we need to show that \( \#\text{supp}(T_k \cup \{ a_{k+1} \}) = \#\text{supp}(T_k) + 1 \). We do so by showing that there is precisely one element in \( \text{supp}(a_{k+1}) \) that is not in \( \text{supp}(T_k) \); that is, \( \text{supp}(a_{k+1})(T_k) = 1 \). This follows from the fact that \( T_{k+1} \) satisfies the three conditions of the proposition. More precisely, from \( \Delta_k \neq 0 \) we get \( \#\text{supp}(a_{k+1}) \cap \Delta_k = \emptyset \), and from \( \text{supp}(a_{k+1})(a_i) = \{ t \} \) for \( 1 \leq i \leq k \) we get \( \text{supp}(a_{k+1}) \cap \Delta_k = \{ t \} \). Since \( \Delta_k \neq 0 \), these imply that \( \text{supp}(a_{k+1})(T_k) = \{ t \} \). Hence,

\[
\#\text{supp}(T_{k+1}) \geq \#\text{supp}(T_k \cup \{ a_{k+1} \}) = \#\text{supp}(T_k) + 1. \tag{3}
\]

With this inequality in hand, the contradiction is derived as follows. By assuming that \( G \neq A_n \), \( T_k \subseteq G \) implies \( T_{k+1} \subseteq G \) where \( T_{k+1} \) also satisfies the hypotheses of the proposition. Thus, by iteration, what we in fact have is that \( T_k \subseteq G \) implies \( T_{k+s} \subseteq G \) for any \( s \geq 0 \) by a chain of implications. In particular, \( T_k \subseteq G \) implies \( T_{k+n} \subseteq G \). Then, by (3), we have \( \#\text{supp}(T_{k+n}) \geq \#\text{supp}(T_k) + n > n \). So, \( \#\text{supp}(T_{k+n}) > n \). But, \( T_{k+n} \subseteq G \) so we cannot have that the support of \( T_{k+n} \) is greater than the degree \( n \) of \( G \). This is a contradiction. Hence, \( G \geq A_n \).

\[1\text{In symbols, this is: } a_{k+1} := g_r h g_r h^{-1} g_r^{-1} = (t^h g_r w^h g_r) \text{ and } b_{k+1} := g_r h b g_r h^{-1} g_r^{-1} = (t^h g_r w^h g_r). \text{ Therefore, } g_{k+1} \in G. \]
Proposition 1 (more precisely, Corollary 1) and Proposition 2 immediately furnish Theorem 2:

**Proof of Theorem 2.** If \( G \) contains a 4-cycle then the square of that element will be a \((2, 2)\)-cycle so we can suppose, without loss of generality, that \( G \) contains a \((2, 2)\)-cycle, \( g_1 \).

By Corollary 1, we have another element \( g_2 \) in \( G \) such that if \( G \not\cong A_n \) then conditions (1) and (2) hold. But if (1) and (2) hold then \( T_2 := \{g_1, g_2\} \) satisfies the hypotheses of Proposition 2, and so \( G \supseteq A_n \).

With this result in hand, we now turn to the specific case of \( n = 7 \). Here, we prove a more general result than is needed to establish Theorem 1; namely,

**Proposition 3.** If \( G \leq S_7 \) is transitive and contains a 4-cycle, then \( G = S_7 \).

In doing so, we make use of the following well-known lemma:

**Lemma 1.** A group of order 84 is solvable.

**Proof.** Let \( K \) be a group of order \( 84 = 7 \cdot 4 \cdot 3 \). By the Sylow Theorems, the number, \( n_7 \), of Sylow 7-subgroups of \( K \) must satisfy: \( n_7 \equiv 1 \mod 7 \) and \( n_7 | 12 \). Thus, \( n_7 = 1 \), so the unique Sylow 7-subgroup of \( K \) is normal and by forming the quotient group \( K/P_7 \), we get \(|K/P_7| = |K|/n_7| = 12\). Hence, \( P_7 \) and \( K/P_7 \) are both solvable implying \( K \) is solvable.

Also, we remark that, by an easy argument, a transitive group of prime degree is automatically primitive (cf. [3] pg. 16).

**Proof of Proposition 3.** We begin by supposing that \( G \not\cong S_7 \). Since \( G \) contains a 4-cycle and thus \( G \not\cong A_7 \), this is equivalent to supposing \( G \not\cong A_n \). Now, since \( G \) is a permutation group of prime degree, we have from Galois (cf. [3] pg. 29) that \( G \) is solvable iff for two distinct points of \( \{1, ..., 7\} \) the only element which fixes both is the identity. The 4-cycle in \( G \), however, fixes three distinct points and so \( G \) is insolvable. By Burnside (cf. [3] pg. 29), every insolvable transitive group of prime degree is 2-transitive. Hence, \( G \) is 2-transitive.

By a result of Bochert (cf. [3] pg. 41), a primitive group \( G \not\cong A_n \) satisfies:

\[ |S_n : G| \geq \frac{(n+1)!}{2} \]

With \( n = 7 \) this implies \( |S_7 : G| \geq 4! \) and hence, \( |G| \leq 210 \). As a lower bound, we have \( 60 \leq |G| \) because \( G \) is insolvable. By Wielandt (cf. [3] pg. 20) the order of a \( k \)-fold transitive group of degree \( n \) is divisible by \( n(n-1)...(n-k+1) \). In our case, \( G \) is at least 2-transitive, so \( 7(7-1) = 42 \) divides \(|G|\). Together, these conditions give \(|G| \in \{84, 126, 168, 210\}\).

Now, by Lemma 1, \( |G| \neq 84 \) because \( G \) is insolvable. Furthermore, the presence of the 4-cycle means \( |G| \) must be divisible by 4 so \( |G| \neq 126, 210 \implies |G| = 168 \). If \( |G| = 168 \) then we claim \( G \) must be simple. Suppose \( G \) had a proper normal subgroup \( H \). If \( 3 \leq |H| \leq 56 \) then by a straight cardinality argument both \( H \) and \( G/H \) must be
solvable. On the other hand, if $|H| \in \{2, 84\}$ then by Lemma 1, both $H$ and $G/H$ are again solvable. Since $G$ is insolvable, these would imply a contradiction and so $G$ must be simple.

We now show that if $G$ is simple then $G \leq A_7$. Let $G'$ and $S_7'$ be the commutator subgroups for $G$ and $S_7$ respectively. Then, $G' \leq S_7'$ because $G \leq S_7$. Since $G$ is nonabelian simple, $G = G'$. Furthermore, given that $S_7' = A_7$, this implies $G \leq A_7$. But $G$ contains a 4-cycle, which is odd, so $G \not\leq A_7$. This being a contradiction, our original assumption that $G \neq S_7$ must have been wrong. Hence, $G = S_7$.

With Theorem 2 for $n \geq 9$ and the preceding Proposition 3 for $n = 7$, we are now in a position to prove our desired result, Theorem 1:

Proof of Theorem 1. If $n = 7$ then $s_4 \in G$ is a 4-cycle and by Proposition 3, $G = S_7$. If $n \geq 9$ then because $s_4$ has cycle decomposition $(2)^{m-3}(4)$, $s_4^2 \in G$ is a $(2,2)$ cycle and $G \geq A_n$ by Theorem 2.

Moreover, if $n \equiv 1 \pmod{4}$ then $m \equiv 0 \pmod{2}$ and $s_1, s_2, s_3$ and $s_4$ are all even. Since $G$ is generated by even elements, $G \leq A_n$. Together with $G \geq A_n$, this implies $G = A_n$ if $n \equiv 1 \pmod{4}$.

On the other hand, if $n \equiv 3 \pmod{4}$, then $m \equiv 1 \pmod{2}$ and $s_1, s_2, s_3$ and $s_4$ are all odd. Since $G$ contains odd elements, $G \neq A_n$. Together with $G \geq A_n$ this implies $G = S_n$ if $n \equiv 3 \pmod{4}$.

References

