Curves of genus 2 with elliptic differentials and associated Hurwitz spaces

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1 Motivation

In the whole paper K is a field of finite type over its prime field K_0 of characteristic $p \geq 0$ and not equal to 2. As typical case we can take K as number field or as function field of one variable over a finite field. With K_s we denote its separable closure. With n we denote an *odd* number larger than 5 and prime to p.

We begin by stating three motivations.

1.1 Rational Fundamental Groups

Let C be a smooth projective geometrically irreducible curve over K with function field $F(C)$.

We are interested in unramified Galois extension $U(C)$ of $F(C)$ with the additional property that K is algebraically closed in $U(C)$.

The finite quotients of the Galois group $G(U(C)/F(C))$ correspond to covers

$$D \xrightarrow{f} C$$

where D is a smooth projective absolutely irreducible curve defined over K and f is Galois (and hence étale).

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Base Points Take $P \in C(K)$ and consider a connected Galois cover $D \xrightarrow{f} C$ in which P splits completely. Then D is absolutely irreducible. Moreover, normalizations of fibre products of such covers (for P fixed) satisfy the same conditions, and so there is a maximal unramified extension of $F(C)$ in which P is totally split, or equivalently, an absolutely irreducible étale (pro-)cover of C over K in which the fibre over P consists of K-rational points, which is Galois with group $\Pi_K(C, P)$, the K-rational fundamental group of C *with base point* P.

1.1.1 Questions

Let g be a non-negative integer.

Q1: Are there finitely generated fields K, curves C of genus g defined over K, prime numbers ℓ and points $P \in C(K)$ such that $\Pi_K(C, P)$ has an ℓ-adic subrepresentation with infinite image?

Q2: Are there finitely generated fields K, curves C of genus g defined over K and points $P \in C(K)$ such that $\Pi_K(C, P)$ is infinite?

Q3: Are there finitely generated fields K and curves C of genus g defined over K such that there is a finite extension K_1/K and an infinite tower of projective absolutely irreducible Galois covers of C defined over K_1?

Q4: For given $n \in \mathbf{N}$ and fixed K and g are there curves C of genus g defined over K such that there are projective absolutely irreducible Galois covers of C of degree $\geq n$?

Known Results

1. For $g = 0$ there are no unramified extensions of C and so the answer to all questions is negative.

2. The same result is true for $g = 1$, as was explained in [FKV], §2. However, in this case the negative answer to **Q4** (which implies negative answers to all others) relies on the deep result of Merel on the uniform bounds of torsion points of elliptic curves over fixed number fields.

3. In [FK1] it is shown that there are examples for positive answers for Question **Q1**. In all these examples K has positive characteristic and the genus of C is larger than 2.[1]

4. If $\mathrm{char}(K) \equiv 3 \bmod 4$ and K contains a fourth root of unity then for all $g \geq 3$ there is an explicitly given subvariety of positive dimension in the moduli space of curves of curves g and curves C such that **Q1** is true for curves with moduli point in this set ([FK1]).

5. For any field K containing the fourth roots of unity and for all $g \geq 3$ there is an explicitly given subvariety of positive dimension in the moduli space of curves of curves g and curves C such that **Q2** is true for curves with moduli point in this set ([FKV]).

6. There are examples (see [I] and [Ki])for curves of genus 2 over finite fields for which **Q1** is true.

We want to ask in this paper whether there is, as in the case of curves of higher genus, a subspace of positive dimension in the moduli space of curves of genus 2 for which some of the questions can be answered positively. As we will see we can get partial answers by restricting us to curves of genus 2 with elliptic differentials, i.e. with Jacobians isogenous to a product of elliptic curves. In particular, **Q4** has a positive answer (Proposition 6.10) for any K, and **Q3** is true in positive characteristic (Corollary 7.15), in both cases for non-constant curves.

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1.2 Isomorphic Galois Representations

One of the most efficient tools for studying the absolute Galois group G_K of K is the investigation of representations induced by the action on torsion groups of abelian varieties, or more generally, on cohomology groups attached to varieties. In fact the results mentioned in 1.1.1 are obtained in this way.

[1] We remark that if we replace C by a number field the Conjecture of Fontaine-Mazur implies the negative answer to **Q1**.

Conversely, it is natural to ask how much information about these representations is needed to characterize the isogeny class of abelian varieties. By celebrated results of Tate (for finite fields), Faltings (for number fields) and Zarhin (for fields of positive characteristic) we know that ℓ-adic representations attached to Tate modules as well as infinitely many representations on torsion points of different prime orders are enough. For a discussion and some generalizations we refer to [FJ].

This question is closely related to arithmetic properties of abelian varieties. A discussion in a rather general frame can be found in [Fr3].

In this paper we restrict ourselves to the case that we look at pairs of elliptic curves E and E' defined over K.

The representations of G_K induced on the points $E[n]$ respectively $E[n']$ of order dividing n of E respectively E' are denoted by $\rho_{E/K,n}$ respectively $\rho_{E'/K,n}$.

Conjecture 1 (Darmon[Da])
There is a number $n_0(K)$ such that for all elliptic curves E, E' over K and all $n \geq n_0(K)$ we get:

$$\text{If } \rho_{E/K,n} \cong \rho_{E'/K,n} \text{ then } E \text{ is isogenous to } E'.$$

A variant is

Conjecture 2 (Darmon/Kani)
There is a number n_0 such that for every $n \geq n_0$ and every number field K there are, up to simultaneous twists, only finitely many pairs (E, E') of elliptic curves over K which are not isogenous yet whose mod n Galois representations are isomorphic: $\rho_{E/K,n} \cong \rho_{E'/K,n}$. Moreover, for prime numbers n the bound n_0 can be chosen to be 23.

Much weaker is

Conjecture 3 (Frey)
We fix an elliptic curve E_0/K. Let $S(E_0)$ denote the set of primes ℓ for which there exists a K-rational cyclic isogeny of degree ℓ of E_0.

There is a number $n_0(E, K)$ such that for all elliptic curves E over K and all $n \geq n_0(E, K)$ and prime to elements in $S(E_0)$ we get:

$$E \text{ is isogenous to } E_0 \text{ iff } \rho_{E/K,n} \cong \rho_{E_0/K,n}.$$

There is a close relation of these conjectures with other conjectures in diophantine geometry. We shall give some diophantine motivation for Conjecture 2 in Subsection 6.4. There it is shown that the conjecture predicts properties of rational points on explicitly given surfaces of general type, and Lang's conjecture together with a conjectural modular description of curves of genus ≤ 1 (cf. Conjecture 4) on these surfaces would imply Conjecture 2.

Conjecture 3 is true if the height conjecture for elliptic curves holds; cf. [Fr2]. In particular, it is true in the case that K is a function field over a finite field.

For $K = \mathbb{Q}$ the height conjecture is equivalent with the ABC-conjecture and with the degree conjecture for modular parameterizations of elliptic curves ([Fr2]). Moreover, by switching from n to $2n$, we get an equivalence of Conjecture 3 for *even* numbers with the asymptotic Fermat conjecture.

1.3 A "Special" Hurwitz Space

Our third motivation comes from group theory. By Hurwitz spaces one parameterizes covers of a base curve with given monodromy group and given ramification type. Over $\mathbb{C}$ as ground field and (mainly) $\mathbb{P}^1$ as base curve this is classical theory due to Hurwitz[Hu]. By the work of Fulton, Fried–Völklein and many others, the theory of Hurwitz spaces is extended to a powerful tool in Arithmetic Geometry.

We want to study covers

$$\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$$

of degree n which are primitive (i.e. have no proper intermediate subcovers) and which are ramified at 5 points with a certain ramification structure (as described below). In order to classify such covers, we shall use the following terminology.

Definition 1.1 Let $f_i : C_i \to C$ be two covers of the curve C.

(a) f_1 is *isomorphic* to f_2 ($f_1 \cong f_2$) if there is an isomorphism $\alpha : C_1 \to C_2$ with $f_1 = f_2 \circ \alpha$.

(b) f_1 is *equivalent* to f_2 ($f_1 \sim f_2$)if there is an isomorphim $\alpha_1 : C_1 \to C_2$ and an automorphism $\alpha_2 : C \to C$ with $\alpha_2 \circ f_1 = f_2 \circ \alpha_1$.

Thus, the set of equivalence classes of covers $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ as above has two points which can move "freely" and hence the associated Hurwitz space is a surface.

Let us now assume in addition that the (different) points $P_1, \ldots, P_5 \in \mathbb{P}^1(K_s)$ ramify with ramification order at most 2. We want to make sure that the monodromy group of φ is S_n and so we assume that P_5 has as ramification cycle a transposition. Together with the condition that φ is primitive this implies by group theory ([Wi] or [Hup]) that the Galois closure of φ is a regular cover with Galois group S_n; cf. [Fr1] or Corollary 3.7 below for more detail. Note that this condition clearly implies that P_5 has exactly one ramified extension in the cover.

For $1 \leq i \leq 5$ let r_i denote the number of *unramified* extensions of P_i after base change to K_s. By assumption $r_5 = n - 2$. The Hurwitz genus formula for φ yields:

$$-2 = -2n + 1 + \sum_{1 \leq i \leq 4} \frac{(n - r_i)}{2} = 1 - \sum_{1 \leq i \leq 4} \frac{r_i}{2}.$$

Thus $\sum r_i = 6$, and hence, since n is odd, r_i is odd and so there are three points, say P_1, P_2, P_3, with $r_i = 1$ and one point P_4 with $r_4 = 3$. Since this numbering convention will be in force throughout the paper, we formulate it here for easy reference

Convention 1.2 The points $P_1, \ldots, P_4$ have been numbered in such a way that $r_1 = r_2 = r_3 = 1$ and $r_4 = 3$. Thus, there is a unique unramified extension above P_i for $i = 1, 2, 3$, and there are precisely 3 unramified extensions above P_4.

Since $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is defined over K, the discriminant divisor $\mathrm{disc}(\varphi) = P_1 + \ldots + P_5$ is K-rational. Since $n - 2 > 3$ the points P_5 and P_4 have to be K-rational and P_1, P_2, P_3 are permuted under the Galois action.

Summarizing, we have:

Proposition 1.3 *Let φ be a primitive K-cover of degree n of the projective line by itself ramified in the following way.*

(*) *φ is ramified in five points $P_1, \ldots, P_5$ in $\mathbb{P}^1(K_s)$ with ramification orders ≤ 2 and the ramification cycle of P_5 in the monodromy group is a transposition.*

Then the Galois closure $\overline{\varphi} : \overline{C} \to \mathbb{P}^1_K$ of φ has Galois group S_n and is regular over K. The ramification structure induces five involutions $\sigma_i \in S_n$ which *generate S_n with $\prod \sigma_i = 1$, where σ_5 is a transposition, σ_4 is a product of $\frac{n-3}{2}$ transpositions and each of σ_i, $i = 1,2,3$ is a product of $\frac{n-1}{2}$ transpositions (after a suitable numeration).*

Thus, the ramification structure of φ (or, more precisely, of $\overline{\varphi}$) is described by the tuple

$$\mathbf{C} = (cl(\sigma_1), \ldots, cl(\sigma_5)) = ((2)^{\frac{n-1}{2}}, (2)^{\frac{n-1}{2}}, (2)^{\frac{n-1}{2}}, (2)^{\frac{n-3}{2}}, (2)),$$

where $cl(\sigma_i)$ denotes the conjugacy class of σ_i in S_n (which is determined by its cycle decomposition), and the symbol $(2)^k$ means that cycle decomposition of σ_i consists of k transpositions.

Conversely we have the following result from Galois theory.

Proposition 1.4 *Suppose $\overline{\varphi} : \overline{C} \to \mathbb{P}^1$ is Galois cover with Galois group $G \simeq S_n$ which has ramification structure given by $\mathbf{C}$ as above. Then $\overline{\varphi}$ is the Galois closure of a (primitive) cover $\varphi : \mathbb{P}^1 = H_x \backslash \overline{C} \to \mathbb{P}^1$ of degree n which satisfies condition $(*)$. Here $H_x \simeq S_{n-1}$ is the stabilizer subgroup of an element $x \in \{1, \ldots, n\}$.*

We can express the above in terms of the associated *Hurwitz spaces* as follows. Let $P_n^*(K)$ denote the set of *isomorphism classes* of primitive covers $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree n which satisfy $(*)$, and let $H^{ab}(S_n, \mathbf{C})(K)$ denote the set of isomorphism classes of (regular) Galois covers $\overline{\varphi} : \overline{C} \to \mathbb{P}^1_K$ with group S_n and ramification structure $\mathbf{C}$ as above. Then the above Propositions 1.3 and 1.4 show that the map $\varphi \mapsto \overline{\varphi}$ induces a bijection $P_n^*(K) \xrightarrow{\sim} H^{ab}(S_n, \mathbf{C})(K)$. Now since S_n has no outer automorphisms (as $n \neq 6$), it follows that $H^{ab}(S_n, \mathbf{C})(K)$ can be identified with the set of K-rational points of the Hurwitz space $H^{in}(S_n, \mathbf{C})$ as defined in [V1], ch. 10.

Since S_n has trivial centre, the Hurwitz space $H^{in}(S_n, \mathbf{C})$ *finely* represents the associated Hurwitz functor $\mathcal{H}_{\mathbb{P}^1}(S_n, \mathbf{C})$ which classifies covers φ of type $\mathbf{C}$; cf. Wewers[We], p. 66, 70. However, this moduli space (which has dimension 5) is not quite the space we want because we want to study equivalence classes rather than isomorphism classes of covers. Thus, we want to consider instead the quotient space

$$H_n^* := H^{in}(S_n, \mathbf{C}) / \mathrm{Aut}(\mathbb{P}^1)$$

(which has dimension 2). It can be shown (cf. Proposition 4.8 below) that H_n^* *coarsely represents* the functor $\mathcal{H}_n^*$ which classifies equivalence classes of such covers.

Remark 1.5 (a) Since H_n^* is only a coarse moduli space for the functor $\mathcal{H}_n^*$ (which is the quotient functor $\mathcal{H}_{\mathbb{P}^1}(S_n, \mathbf{C})/\mathrm{Aut}(\mathbb{P}^1)$), it is more difficult to characterize the K-rational points of H_n^*. It is clear, however, that if $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a K-cover of type (*), then φ defines a K-rational point on $H^{in}(S_n, \mathbf{C})$ and hence also on H_n^*. We thus have a natural map $\mathcal{H}_n^*(K) \to H_n^*(K)$ (which is a bijection if K is algebraically closed).

(b) Since $H^{in}(S_n, \mathbf{C})$ is finite and étale over the smooth affine space $U_5 \subset \mathbb{P}^5$ (cf. [We], p. 7), it follows that is also smooth and affine. Thus, H_n^* is normal and affine. This argument, however, does not guarantee that $H^{in}(S_n, \mathbf{C})$ and H_n^* are connected; this will be established later by another method (cf. Theorem 4.9).

(c) The system $(\sigma_1, \cdots \sigma_5)$ is not rigid (in the sense of [V1], p. 38). Of course it is not difficult to see this by group theory but it is more difficult to compute the number of non-conjugate isomorphism classes. We shall get an answer by using geometrical arguments in Theorem 3.11.

We shall see in the next sections that points on H_n^* carry much more arithmetical information. In fact, we shall associate to such points covers of elliptic curves by curves of genus 2, and so one is naturally led to a *Hurwitz moduli functor* (in the spirit of Fulton [Fu]). A detailed discussion is given in [Ka5].

By applying this we can approach some of the problems mentioned in Remark 1.5 and see that there is a close connection between Subsection 1.2 and Subsection 1.3.

2 Genus 2 Covers of Elliptic Curves

2.1 Normalized Covers

Let C be a curve of genus 2 over K with Weierstraß points $\mathcal{W}_C = \{W_1, \ldots, W_6\}$ and *Weierstraß divisor* $W_C = W_1 + \ldots + W_6$. (Note that the divisor W_C is K-rational, whereas the individual points W_i need not be.)
We assume that C has an elliptic differential over K. This means that there is an elliptic curve E over K and a non-constant K-morphism

$$f : C \to E.$$

In addition we assume that f has odd[2] degree n.

Definition 2.1 A cover

$$f : C \to E$$

(of the above type) is called *normalized* ([Ka5]) if

(a) f is *minimal*, i.e. there is no non-trivial isogeny $\eta : E' \to E$ and a cover $f' : C \to E'$ with $f = \eta \circ f'$.

(b) The norm (direct image) of W_C has the form

$$f_* W_C = 3 \cdot 0_E + P'_1 + P'_2 + P'_3. \tag{1}$$

where P'_i are the non-trivial points of order 2 of E.

Note that every minimal cover $f_0 : C \to E$ is equivalent to a normalized cover; more precisely, we have (cf. [Ka5], Proposition 2.2):

Lemma 2.2 *Assume that $f_0 : C \to E$ is a minimal cover of an elliptic curve E by a curve of genus 2 of odd degree n defined over K. Then there is a unique translation $\tau : E \to E$ such that $f = \tau \circ f_0$ is normalized.*

For our purposes it is important to note that a normalized cover also satisfies the condition

$$f \circ \omega_C = [-1]_E \circ f, \tag{2}$$

where ω_C denotes the hyperelliptic involution of C; cf. [Ka5], Corollary 2.3. It follows that f factors over the hyperelliptic cover $\pi' : C \to C/\langle\omega\rangle = \mathbb{P}^1$.

By the Riemann-Hurwitz genus formula we see that the different of f has to be a divisor of degree 2.

2.2 Associated $\mathbb{P}^1$-Covers

We return to covers $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of type (*) and discuss their relation to curves of genus 2 with elliptic differentials. Indeed, it is this connection which motivated the consideration of this specific group theoretical situation (*); cf. [Fr1].

[2] In fact, the case that n is even is also important (see Sections 3 and 6.4), and is analyzed in [Ka5] and [Di].

To explain this connection, note first that the points $P_1, \cdots, P_4$ determine an elliptic curve E/K which is unique up to a quadratic twist. This curve is constructed as follows. Choose a coordinate x of $\mathbb{P}^1_K$ such that $(x)_\infty = P_4$, and a cubic polynomial $f_3(x) \in K[x]$ with $(f_3)_0 = P_1 + P_2 + P_3$. Then the (Weierstraß) equation

$$y^2 = f_3(x)$$

defines an elliptic curve E/K and a double cover $\pi : E \to \mathbb{P}^1$ which is ramified at $P_1, \ldots, P_4$ and which maps the zero-point $0_E \in E(K)$ to P_4. Thus, the group of 2-torsion points of E is $E[2] = \{0_E, P_1', P_2', P_3'\}$, where $P_i' \in \pi^{-1}(P_i)$ is the (unique) point above P_i. Note that P_i' is K-rational if and only if P_i is K-rational.

Let $C_0 = E \times_{\mathbb{P}^1} \mathbb{P}^1$ be the fibre product of E and $\mathbb{P}^1$ over $\mathbb{P}^1$ (with respect to the morphisms π and φ). It is immediate that C_0 is an irreducible curve with function field $F = K(X, Y)$, with X and Y satisfying a hyperelliptic equation of the form

$$Y^2 = f_3(X) \cdot g_3(X)$$

where g_3 is a polynomial of degree 3 corresponding to the 3 unramified extensions of P_4 in the cover φ (cf. Convention 1.2). Thus, the normalization $C = \widetilde{C_0}$ of C_0 is a curve of genus 2 with morphisms $f : C \to E$ and $\pi' : C \to \mathbb{P}^1$, and $f_*(W_C) = 3 \cdot P_4' + P_1' + P_2' + P_3'$.

We get the diagram of morphisms

$$\begin{array}{ccc} & & C \\ & \overset{\pi'}{\swarrow} & \\ \mathbb{P}^1 & & \downarrow f \\ \varphi \downarrow & & E \\ & \overset{\pi}{\swarrow} & \\ \mathbb{P}^1 & & \end{array}$$

The point P_5 has two distinct extensions P and $P' = [-1]_E P$ to E, and there is exactly one point Q resp. $Q' = \omega Q$ over P resp. P' which is ramified of order 2. Hence the discriminant divisor of f is equal to $\pi^*(P_5)$ and the different divisor of f is equal to $Q + Q'$.

If we assume that φ is primitive, then it follows that f is also primitive (or minimal). Thus, we have

Proposition 2.3 *Let* $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ *be a primitive cover of odd degree* n *satisfying* $(*)$ *and Convention 1.2 with respect to the points* $P_1, \ldots, P_5$*, and let* E/K *be an elliptic curve with a morphism* $\pi : E \to \mathbb{P}^1$ *of degree* 2 *which is ramified at* $P_1, \ldots, P_4$ *such that* $\pi(0_E) = P_4$*. Then the normalization* C *of the fibre product* $\mathbb{P}^1 \times_{\mathbb{P}^1} E$ *is a curve of genus* 2 *defined over* K *and the morphism*

$$f = \varphi_{(\pi)} : C = (\mathbb{P}^1 \times_{\mathbb{P}^1} E)^{\sim} \to E$$

is normalized and has discriminant $\mathrm{Disc}(f) = \pi^*(P_5)$.

Conversely, assume that $f_0 : C \to E$ is a minimal cover of an elliptic curve E by a curve of genus 2 of odd degree n defined over K. After an appropriate translation (Lemma 2.2) we get a normalized cover

$$f : C \to E.$$

Since f satisfies (2), we can pass to the respective quotients and obtain a primitive cover

$$\varphi : C/\langle\omega\rangle = \mathbb{P}^1 \to E/\langle -id_E\rangle = \mathbb{P}^1$$

of degree n such that $\varphi \circ \pi' = \pi \circ f$.

The Weierstraß points of C are mapped onto $E[2]$ and hence to the ramification points of $\pi : E \to \mathbb{P}^1$. By our hypothesis, three of the Weierstraß points lie over the zero-point of E whose image in $\mathbb{P}^1(K)$ is denoted by P_4. The three others are mapped one-to-one to the points of order 2 of E. Denote by $P_1, ..., P_3$ the images of these points. The discriminant divisor of f is the conorm (or pullback) of a divisor of $\mathbb{P}^1$ with support equal to one point $P_5 \in \mathbb{P}^1(K)$. It follows that φ is unramified outside of $\{P_1, ..., P_5\}$. Summarizing, we have

Proposition 2.4 *Let* $f_0 : C \to E$ *be a minimal genus 2 cover of odd degree* n *of an elliptic curve* E/K*. Then there is a unique translation* $\tau : E \to E$ *such that* $f = \tau \circ f_0$ *is normalized. Moreover,* $\pi \circ f$ *factors over the hyperelliptic cover* $\pi' : C \to C/\langle\omega\rangle = \mathbb{P}^1$*, i.e. there is a unique cover* $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ *such that* $\varphi \circ \pi' = \pi \circ f$*. (In particular,* φ *is primitive and has degree* n*.) Let* $P_1, \ldots, P_3$ *denote the images of the non-zero* 2*-torsion points of* E *under* π *and put* $P_4 = \pi(0_E)$*. In addition, let* $P_5 \in \mathbb{P}^1(K)$ *be the unique point such that* $\mathrm{Disc}(f) = \pi^*(P_5)$*. Then* $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ *satisfies* $(*)$ *and Convention 1.2 if and only if* $\mathrm{Disc}(f)$ *is reduced, i.e. if and only* $P_5 \notin \{P_1, \ldots, P_4\}$.

Proof. Everything except the last statement has been explained above. Since $\mathrm{Disc}(f)$ is a divisor of degree 2 and invariant under $-id_E$ it is of the form $P+(-id_E(P))$ and hence it is reduced iff P_5 is unramified, i.e. not contained in $\{P_1, \ldots, P_4\}$.

Note that the above construction guarantees that $P_1, \ldots, P_4$ are distinct points, but not necessarily $P_1, \ldots, P_5$. Assume now that P_5 is different from P_i, $i = 1, ..., 4$ (the "generic" case, see Subsection 3.2). Then φ defines a point in the Hurwitz space H_n^*.

The previous propositions suggest that the moduli space H_n^* is closely related to another moduli space H_n, which represents the moduli problem $\mathcal{H}_n$ classifying isomorphism classes of pairs (C, f) where C is a curve of genus 2 and f is a *normalized* covering map from C to an elliptic curve E of degree n. More precisely, let us look at the subproblem $\mathcal{H}'_n$ of $\mathcal{H}_n$ which classifies the covers for which $\mathrm{Disc}(f)$ is reduced. Then we shall see that $\mathcal{H}_n$ can be (coarsely) represented by a surface H_n, and $\mathcal{H}'_n$ by an open subset H'_n of H_n. Moreover, over fields K of characteristic 0, H'_n turns out to be isomorphic to the the Hurwitz space H_n^*; cf. Theorem 4.9 below.

The advantage of the spaces H'_n and H_n is that we can embed them explicitly as open subsets into an explicitly given surface.

To describe this embedding we shall use torsion structures on elliptic curves.

2.3 The "Basic Construction"

For details of the following discussion we refer to [FK].

Let (C, f) be a (normalized) genus 2 cover of E, and let J_C denote the Jacobian variety of C. Then the map f induces homomorphisms $f_* : J_C \to E$ and $f^* : E \to J_C$. Because of the minimality of f, the morphism f^* is injective, and the kernel E_* of f_* is an elliptic curve intersecting $f^*(E)$ exactly in the points of order n. It follows that there is a G_K-module isomorphism

$$\alpha : E[n] \to E_*[n]$$

which turns out to be anti-isometric with respect to the Weil-pairings on $E[n]$ and on $E_*[n]$. The abelian variety J_C together with its natural principal polarization is isomorphic to $(E \times E_*)/\Gamma_\alpha$ where $\Gamma_\alpha \leq E[n] \times E_*[n]$ is the graph of α.

Conversely, take a K-rational triple (E, E', α) with E, E' elliptic curves over K and α a K-rational isomorphism from $E[n]$ to $E'[n]$ whose graph Γ_α is isotropic in $E[n] \times E'[n]$. Then $(E \times E')/\Gamma_\alpha$ is a principally polarized abelian variety, and there is a K-rational effective divisor C in the class of $n(0_E \times E' + E \times 0_{E'})/\Gamma_\alpha$ which is either a curve of genus 2 *or* a union of two elliptic curves intersecting in one point. If C is irreducible, then the projection p_1 of $E \times E'$ to E induces a cover

$$f_\alpha : C \to E$$

of degree n and hence we get a K-rational solution of the moduli problem $\mathcal{H}_n$ defined above.

Moreover, if the discriminant divisor $\mathrm{Disc}(f_\alpha)$ is reduced, then $f_\alpha \in \mathcal{H}'_n(K)$, and the associated cover $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ gives rise to a point in the moduli space H^*_n.

The triples (E, E', α) are parameterized by an open part of the *modular diagonal quotient surface* $Z_{n,-1}/K$ (cf. [Ka7]) and Subsection 4.1). If K contains a primitive n-th root of unity $\zeta_n \in K$, then $Z_{n,-1}$ is a quotient of the product surface $X(n) \times X(n)$, where $X(n)$ is the modular curve parameterizing elliptic curves with level-n-structure (of fixed determinant); explicitly,

$$Z_{n,-1} = (X(n) \times X(n))/\Delta_{-1}(\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})),$$

where $\Delta_{-1}(G_n)$ is the "twisted diagonal subgroup" of $G_n = \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ with respect to $\tau_{-1} \in \mathrm{Aut}(G_n)$, i.e. $\Delta_{-1}(G_n) = \{(g, \tau_{-1}(g)) : g \in G_n\}$ where τ_{-1} is the automorphism of G_n given by conjugation with an element $\beta \in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ of determinant -1; cf. [KS] and [Fr2]. If $\zeta_n \notin K$, then the construction of $Z_{n,-1}$ is more complicated; in this case $Z_{N,-1}$ is still a quotient of $X(n) \times X(n)$, (where now $X(n)/K$ denotes Shimura's canonical model of $X(n)/\mathbb{C}$), but now the quotient has to be taken with respect to an etale group scheme rather than a (constant) group (scheme); cf. [Ka7] for more detail.

For us it is important that we have found a very explicit variety which is isomorphic to the Hurwitz space H^*_n (as we shall see), and this allows us to study its geometric and diophantine properties. So we want to compare the moduli spaces H^*_n and H'_n which were introduced in the previous subsections. Although the discussion in there shows that there is a relation between their sets of K-rational points, this does not suffice to show that their coarse moduli spaces are isomorphic. For this, we shall study the associated moduli functors and/or stacks in more detail. We begin to do this "fiberwise".

3 Covers with Fixed Base Curve

3.1 The Hurwitz spaces $H_{E/K,N}$ and $H'_{E/K,N}$

Before we discuss the moduli problem $\mathcal{H}_n$ (with moduli space H_n) in more detail, let us consider the simpler problem of classifying the (normalized) genus 2 covers $f : C \to E$ over a *fixed* elliptic curve E/K. Thus, in the "dictionary" of the previous subsection, this corresponds (roughly) to case that we study covers $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ in which the points $P_1, \ldots, P_4$ are fixed.

In this discussion we can drop the condition that the degree of the cover is odd, and we shall indicate this by switching from n to N. Note, however, that in the case that N is even, the normalization condition (1) has to be replaced by the condition

$$f_* W_C = 2(P'_1 + P'_2 + P'_3). \tag{3}$$

Then, as in the case of odd degree covers, for each minimal cover $f_0 : C \to E$ there is a unique translation $\tau : E \to E$ such that $f = \tau \circ f_0$ is normalized (cf. [Ka5], Proposition 2.2). Moreover, f satisfies condition (2).

For any extension field L of K, let $E_L = E \otimes L$ denote the elliptic curve E lifted to L. We now consider the set

$$\mathrm{Cov}_{E/K,N}(L) := \{f : C \to E_L : \ f \text{ is a normalized genus 2 cover defined over } L \text{ with } \deg(f) = N\}/\simeq$$

of isomorphism classes of normalized covers. As is explained in [Ka5], the assignment $L \mapsto \mathrm{Cov}_{E/K,N}(L)$ can be extended in a natural way to a functor $\mathcal{H}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets}$, and by Theorem 1.1 of [Ka5] we have:

Theorem 3.1 *If $N \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,N}/K$ with the property that $H_{E/K,N} \otimes K_s$ is an open subset of the modular curve $X(N)_{/K_s}$.*

Remark 3.2 The fact that the curve $H := H_{E/K,N}$ *finely* represents the functor $\mathcal{H}_{E/K,N}$ means that there exists a *universal* normalized genus 2 cover $f_H : \mathcal{C}_H \to E \times H$ of degree N with the property that every normalized genus 2 cover $f : C \to E \times S$ of degree N (where S is any K-scheme) is obtained uniquely from f_H by base-change. In particular, the set $\mathrm{Cov}_{E/K,N}(K)$ of covers can be identified with the set of fibres $f_x := (f_H)_x : \mathcal{C}_x \to E_x = E$ of f_H, where $x \in H(K)$.

The main idea for proving this is to use the "basic construction" of genus 2 curves which was presented in [FK], [Ka3] (and which was sketched in Subsection 2.3). In [Ka5], this construction is generalized to families of covers of elliptic curves, and this yields an (open) embedding of functors

$$\Psi = \Psi_{E/K,N} : \mathcal{H}_{E/K,N} \hookrightarrow \mathcal{X}_{E/K,N,-1}, \tag{4}$$

where $\mathcal{X}_{E/K,N,-1}$ is the functor which classifies (isomorphism classes) of pairs (E', ψ) consisting of an elliptic curve E'/K and an anti-isometry $\psi : E[N] \xrightarrow{\sim} E'[N]$ (with respect to the e_N-pairings). Since this latter functor is finely representable by an affine curve $X_{E/K,N,-1}$, it follows that $\mathcal{H}_{E/K,N}$ is represented by an open subset $H_{E/K,N} \subset X_{E/K,N,-1}$. Note that if $E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$ as a G_K-module, then the curve $X_{E/K,N,-1}$ is isomorphic to the usual affine modular curve $X'(N) = X(N) \setminus \{\text{cusps}\}$ of level N whose K-rational points correspond bijectively to isomorphism classes (E', α), where E'/K is an elliptic curve and $\alpha : E'[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$ is a level-N-structure (of fixed "determinant" ζ_N). Since this is always true over a finite separable extension of K, we see that $H_{E/K,N} \otimes K_s$ is (isomorphic to) a subset of $X'(N)_{/K_s}$.

Remark 3.3 (a) It is interesting to note that this representability result is obtained by purely algebraic techniques and hence does not use (not even implicitly) the Riemann Existence Theorem (RET).

(b) Note that while the moduli scheme $X_{E/K,N,-1}$ depends only on the structure of the K-group scheme $E[N]$, the subscheme $H_{E/K,N}$ *does depend* on the choice of E/K; cf. [Ka5]. Moreover, if we replace E/K by a non-isomorphic twist E'/K of E/K, then $E[N] \not\simeq E'[N]$ (as G_K-modules), and hence the functors $\mathcal{X}_{E/K,N,-1}$ and $\mathcal{X}_{E'/K,N,-1}$ are not isomorphic (as functors on $\underline{Sch}_{/K}$), and the same is true for $\mathcal{H}_{E/K,N}$ and $\mathcal{H}_{E'/K,N}$. This shows that the moduli problem $\mathcal{H}_{E/K,N}$ is more refined than the corresponding one for covers $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying condition $(*)$ with fixed ramification points $P_1, \ldots, P_4$.

In view of the correspondence between genus 2 covers of elliptic curves and covers of $\mathbb{P}^1$ (cf. Proposition 2.4), we shall also be interested in the subfunctor $\mathcal{H}'_{E/K,N}$ of $\mathcal{H}_{E/K,N}$ which classifies those (normalized) genus 2 covers $f : C \to E$ for which $\mathrm{Disc}(f)$ is reduced. More precisely, for any K-scheme S, let

$$\mathcal{H}'_{E/K,N}(S) = \{(C \xrightarrow{f} E_S) \in \mathcal{H}_{E/K,N}(S) : \mathrm{Disc}(f_s) \text{ is reduced }, \forall s \in S\}.$$

It is clear that this defines a subfunctor of $\mathcal{H}_{E/K,N}$, and so we should expect that it is represented by an open subscheme $H'_{E/K,N}$ of $H_{E/K,N}$. This follows from the following more precise assertion.

Proposition 3.4 *The rule* $f \mapsto \mathrm{Disc}(f)$ *defines a morphism*

$$\delta_{E/K,N} : H_{E/K,N} \to \mathbb{P}^1$$

such that $\mathrm{Disc}(f_x) = \pi^*(\delta_{E/K,N}(x))$, *for every* $x \in H_{E/K,N}(K)$. *In particular, for every* $\bar{P} \in \mathbb{P}^1(K)$ *we have*

$$\delta^{-1}_{E/K,N}(\bar{P}) = \mathrm{Cov}_{E/K,N,\bar{P}}(K) := \{f \in \mathrm{Cov}_{E/K,N}(K) : \mathrm{Disc}(f) = \pi^*(\bar{P})\},$$

and hence the functor $\mathcal{H}'_{E/K,N}$ *is represented by the open subset* $H'_{E/K,N} := \delta^{-1}_{E/K,N}(\mathbb{P}^1_K \setminus \pi(E[2]))$ *of* $H_{E/K,N}$.

Proof. The first two assertions are proved in [Ka6], Proposition 12, and the last is an immediate consequence of this and Theorem 3.1.

Remark 3.5 Let $\underline{P} = (P_1, \ldots, P_5) \in (\mathbb{P}^1_K)^5(K)$ be a tuple of 5 distinct points with $P_4 = \infty$. For an odd integer n, let $\mathrm{Cov}_{\underline{P},n}(K)$ denote the set of isomorphism classes of K-covers $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of $\deg(\varphi) = n$ satisfying condition $(*)$ and Convention 1.2. If E/K is an elliptic curve such that $\pi_E(E[2]) = \{P_1, \ldots, P_4\}$ (and $\pi_E(0_E) = P_4$), then the discussion of subsection 2.2 shows that we have a bijection

$$\mathrm{Cov}_{\underline{P},n}(K) \xrightarrow{\sim} \mathrm{Cov}_{E/K,n,P_5}(K).$$

Thus, by allowing the point P_5 to move, we obtain a bijection

$$H_{P_1,\ldots,P_4,n}(K) \xrightarrow{\sim} H'_{E/K,n}(K),$$

where $H_{P_1,\ldots,P_4,n} \subset H^{in}(S_n, \mathbf{C})$ is the Hurwitz space with fixed ramification points $P_1, \ldots, P_4$. We therefore see that the set

$$H^{\partial}_{E/K,n} := H_{E/K,n} \setminus H'_{E/K,n} = \delta^{-1}_{E/K,n}(\pi(E[2]))$$

is a "boundary" of the Hurwitz space parametrizing $\mathbb{P}^1$-covers, even though it is contained in the Hurwitz space $H_{E/K,N}$ parametrizing genus 2 covers of E.

3.2 The relative boundary $H^{\partial}_{E/K,n} = H_{E/K,n} \setminus H'_{E/K,n}$

We now describe the genus 2 covers which correspond to the points of the "relative boundary" $H^{\partial}_{E/K,n} = H_{E/K,n} \setminus H'_{E/K,n}$ (where n is odd) in more detail. By Theorem 3.1 and Proposition 3.4 we know that each such point corresponds to a unique normalized genus 2 cover $f : C \to E$ of degree n such that its discriminant has the form $\mathrm{Disc}(f) = 2P'_k$, where $P'_k \in E[2]$; such covers are classified as Type II in [Fr1], p. 85.

By Proposition 2.4 we still have an associated $\mathbb{P}^1$-cover $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, but now the ramification structure of φ no longer satisfies the "generic" condition $(*)$ of Subsection 1.3. Indeed, since these special covers may be viewed as the "limits" of the generic case when we let the point P_5 move to coincide with one of the points $P_1, \ldots, P_4$, we obtain new ramification types, i.e. new cycle decompositions (σ_i), where, as before, $\sigma_i \in G_\varphi = \mathrm{Gal}(\overline{\varphi})$ denotes a generator of a ramification group above P_i.

Here we shall work out all the ramification types for these special covers. (For comparison purposes we also include the generic case as case 0.) It turns out that the ramification type $\mathbf{C} = \mathbf{C}_\varphi$ of φ is completely determined by the *different* $D_f = \mathrm{Diff}(f)$ of f.

Proposition 3.6 *Put* $m = \frac{n-1}{2}$ *and let* $i, k \in \{1, 2, 3\}$. *Then the ramification structure of* φ *is given by the following table:*

Case	*Condition*	$(\sigma_i), i \neq k$	(σ_k)	(σ_4)	(σ_5)
0	$\mathrm{Disc}(f) \neq 2P$	$(2)^m$	–	$(2)^{m-1}$	(2)
1	$\mathrm{Disc}(f) = 2P'_k, D_f \neq 2P$	$(2)^m$	$(2)^{m-2}(4)$	$(2)^{m-1}$	–
2	$\mathrm{Disc}(f) = 2P'_k, D_f = 2P$	$(2)^m$	$(2)^{m-1}(3)$	$(2)^{m-1}$	–
3	$\mathrm{Disc}(f) = 2P'_4, D_f \neq 2P$	$(2)^m$	–	$(2)^{m-3}(4)$	–
4	$\mathrm{Disc}(f) = 2P'_4, D_f = 2P$	$(2)^m$	–	$(2)^{m-2}(3)$	–

Proof. Put $f_1 = \pi \circ f = \varphi \circ \pi' : C \to \mathbb{P}^1$. In each of the cases, we shall first work out the ramification structure of the points in $f_1^{-1}(P_j)$, for $j = 1, \ldots, 5$, and then deduce that of $\varphi^{-1}(P_j)$ from this. From this we can then read of the cycle structure of $\sigma_j \in G_\varphi$. (Here G_φ is the *monomdromy group* of φ, i.e. the Galois group of the Galois closure $\overline{\varphi}$ of φ.)

Case 0. Here $\mathrm{Disc}(f) = P'_5 + [-1]P'_5$, and $P'_5 \notin E[2] = \{P'_1, P'_2, P'_3, P'_4\}$, where $P'_4 = 0_E$. Thus, $P'_1, \ldots, P'_4$ are unramified in f, but are ramified of order 2 for π_E. It follows that there are n points in $f_1^{-1}(P_j)$, $j = 1, \ldots, 4$, and all have ramification index 2 with respect to f_1. Now for $j = 1, 2, 3$, precisely one point (the Weierstrass point) $W_j \in f_1^{-1}(P_j)$ is ramified in π', whereas the others are not. Thus, $\pi'(W_j)$ is unramified wrt. φ , whereas the other $m = \frac{n-1}{2}$ points in $\varphi^{-1}(P_j)$ are each ramified of order 2. It follows that σ_j is a product of m transpositions, as claimed. On the other hand, since $f_1^{-1}(P_4)$ contains 3 Weierstrass points, there are 3 unramified points in $\varphi^{-1}(P_4)$, and the other $m - 1 = \frac{n-3}{2}$ points are ramified of order 2. Thus, $(\sigma_4) = (2)^{m-1}$. Finally, since $f_1^{-1}(P_5)$ contains precisely two points with ramification index 2 (which are interchanged by the hyperelliptic involution), whereas the others are unramified, it follows that $\varphi^{-1}(P_5)$ has a unique point of index 2, and so σ_5 is a transposition.

Case 1. Here $D_f = P_{k1} + P_{k2}$ with $P_{k1} \neq P_{k2}$ and $f(P_{k1}) = f(P_{k2}) = P'_k$, so the ramification structure above P_j for $j = 1, \ldots, 4$, $j \neq k$ is as in case 0. Now above P'_k the ramification structure is $(2)^2$ (wrt. f), so above P_k the ramification is $(2)^{n-4}(4)^2$. Now P_{kj} cannot be a Weierstrass point, for otherwise the ramification indices wrt. π' and f are both 2, which is impossible since all ramification groups are cyclic; cf. Abhyankar's Lemma. Thus, the unique Weierstrass point $W_k \in f_1^{-1}(P_k)$ has index 2 wrt. f_1 and hence $\pi(W_k)$ is unramified wrt. φ. Thus, σ_k has type $(2)^{\frac{n-5}{2}}(4)$.

Case 2. Here $D_f = 2W_k$, the unique Weierstrass point above P'_k. Thus, above P'_k the ramification structure is (3) and hence above P_k it is $(2)^{n-3}(6)$ wrt. f_1. It follows that σ_k has type $(2)^{\frac{n-3}{2}}(3)$.

Case 3. This is similar to case 1. Here $D_f = P_{41} + P_{42}$ with $P_{41} \neq P_{42}$ and $f(P_{41}) = f(P_{42}) = P'_4 = 0_E$, so the ramification structure above P_j for $j = 1, \ldots, 3$, is as in case 0. Now above 0_E the ramification structure is $(2)^2$ (wrt. f), so above P_4 the ramification wrt. f_1 is $(2)^{n-4}(4)^2$. As in case 1, P_{4j} cannot be a Weierstrass point, so we see that σ_4 has type $(2)^{\frac{n-7}{2}}(4)$ because the images of the 3 Weierstrass points are unramified wrt. φ.

Case 4. Here $D_f = 2W'_k$, where W'_k is one of the 3 Weierstrass points above 0_E. Thus, above 0_E the ramification structure is (3) and hence above P_k it is $(2)^{n-3}(6)$ wrt. f_1. It follows (cf. case 3) that σ_4 has type $(2)^{\frac{n-5}{2}}(3)$.

Corollary 3.7 *(a) In cases* 0 *and* 2*, the monodromy group of* φ *and* f *is* $G_\varphi = G_f = S_n$.

(b) In case 4*, we have* $G_f = A_n$*. Moreover,* $G_\varphi = S_n$ *when* $n \equiv 1 \pmod 4$*, and* $G_\varphi = A_n$ *when* $n \equiv 3 \pmod 4$.

Proof. Let $\tilde{F}_0$ (respectively, $\tilde{F}$) be the Galois hull of $\kappa(\mathbb{P}^1)/\varphi^*\kappa(\mathbb{P}^1)$ (respectively, of $\kappa(C)/f^*\kappa(E)$. Since $\tilde{F} = \tilde{F}_0\kappa(E)$ by Galois theory, we see that $G_\varphi = G_f$ if and only if $\kappa(E) \not\subset \tilde{F}_0$.

Moreover, since f is minimal, we see that $G_\varphi = \mathrm{Gal}(\tilde{F}_0/\kappa(\mathbb{P}^1))$ and $G_f = \mathrm{Gal}(\tilde{F}/\kappa(E))$ are primitive permutation groups of degree n; cf. [Hup], II.1.4. In case 0, σ_5 is a 2-cycle, so $G_\varphi = S_n$ by [Hup], II.4.5b). In case 2 (respectively, case 4) we have from Proposition 3.6 that σ_4^2 (respectively, σ_k^2) is a 3-cycle, so $G_\varphi \geq A_n$ by [Hup], II.4.5c).

(a) Since Proposition 3.6 shows that $G_\varphi \not\leq A_n$, it follows that $G_\varphi = S_n$ in these cases. Moreover, suppose $\kappa(E) \subset \tilde{F}_0$. Then $\kappa(E) = F_1 := (\tilde{F}_0)^{A_n}$, because A_n is the unique subgroup of index 2 of S_n. Now in case 0 and m even we see from Proposition 3.6 that σ_4 and σ_5 are odd and the others are even permutations, so $F_1/\kappa(\mathbb{P}^1)$ is ramified precisely at P_4 and P_5. Thus, $F_1 \neq \kappa(E)$ (which is ramified at $P_1, \ldots P_4$). Similarly, we have in the other cases that $F_1 \neq \kappa(E)$, and so $G_f = G_\varphi$.

(b) If m is even (i.e. $n \equiv 1(4)$), then $G_\varphi \not\leq A_n$ by Proposition 3.6 and so $G_\varphi = S_n$. Moreover, $F_1/\kappa(\mathbb{P}^1)$ is ramified precisely at P_4, P, so $F_1 \neq \kappa(E)$ and $G_f = G_\varphi = S_n$.

If m is odd, then by Proposition 3.6 we have $\sigma_i \in A_n$, for all $i = 1, \ldots, 4$, and so $G_\varphi \leq A_n$ (because G_φ is generated by the σ_i's and their conjugates). Thus $G_\varphi = A_n$. Since A_n has no subgroup of index 2, it follows that $\kappa(E) \not\subset \tilde{F}_0$ and so $G_f = G_\varphi$.

Remark 3.8 The cases 2 and 4 have studied in detail from a different perspective by authors studying problems in *billiard dynamics* and *square tilings*; cf. [Mc1], [HL1] and the references therein. More precisely, if $K = \mathbb{C}$, and if $f : C \to E$ is a genus 2 cover of type 2 or 4, and ω_E is a holomorphic differential on E, then $(C, f^*\omega_E)$ defines a point in the moduli space $\mathcal{H}(2)$ of genus 2 curves with a differential 1-form having a double zero (cf. [Mc1]).

By [Mc1], these two cases can be distinguished by their "spin invariant": those in case 2 have spin 0, and those in case 4 have spin 1. Indeed, in case 2 (respectively, case 4) we have 1 (respectively, 3) Weierstrass points above the ramification point P_k' (respectively, P_4'), so the assertion follows from [Mc1], Theorem 6.1. (Note that [Mc1] and [HL1] assume that the ramification takes place over the origin of E, so in case 2 they consider the cover $\tau_{P_k'} \circ f$ in place of the cover f.)

3.3 The boundary of $H_{E/K,N}$

Since the moduli space $H := H_{E/K,N}$ is a smooth, irreducible (affine) curve, it has a unique normal compactification X. Here we shall study the "boundary" $\partial H = X \setminus H$ of H and how the correspondence between covers $f_x : C_x \to E$ and points of H (cf. Remark 3.2) extends to ∂H.

The first step towards this goal is to observe that $\partial H = S(C)$, the set of places of *bad reduction* of the curve C/F which is the generic fibre of the relative universal curve $\mathcal{C}_H/H$ (cf. Remark 3.2). As a consequence, ∂H decomposes as

$$\partial H \;=\; S(C) \;=\; S_0 \cup S_1$$

where S_0 is the set of points of X where the Jacobian J_C of C/F has bad reduction and where S_1 is the set of places where C has bad reduction yet J_C had good reduction.

To explain how the points of ∂H correspond to (degenerate) covers, we shall use the theory of minimal models to construct a *canonical compactification* $f : \mathcal{C} \to E \times X$ of the universal cover $f_H : \mathcal{C}_H \to E \times H$ (cf. Remark 3.2) and then study its fibres.

To explain this in more detail, assume for simplicity that $K = \overline{K}$ is algebraically closed. In this case we know that H is an open subset of the modular curve $X(N)$, which is therefore its canonical compactification, i.e. $X = X(N)$. In addition, we shall assume that $\mathrm{char}(K) \nmid N!$ (i.e. that either $\mathrm{char}(K) = 0$ or that $\mathrm{char}(K) > N$); in the contrary case the structure of the boundary (and of the corresponding covers) is much more complicated.

Let $p : \mathcal{C} \to X$ denote the *minimal model* of C/F over X. Thus, $\mathcal{C}$ is a projective, smooth surface over K, and $p : \mathcal{C} \to X$ is a genus 2 fibration whose fibres do not contain any rational (-1)-curves. Moreover, the restriction of p to $p^{-1}(H)$ is canonically isomorphic to $p_H := pr_2 \circ f_H : \mathcal{C}_H \to H$ (which is the minimal model of C/F over H) and so we can identify $\mathcal{C}_H$ with the open subscheme $p^{-1}(H)$ of $\mathcal{C}$. Thus, $\mathcal{C}$ is a canonical compactification of $\mathcal{C}_H$.

It turns out that f_H extends to a *finite* morphism $f : \mathcal{C} \to E_X = E \times X$. Moreover, the precise structure of the fibres of $\mathcal{C}/X$ is as follows:

Theorem 3.9 *If* $\mathrm{char}(K) \nmid N!$*, then* $\mathcal{C}/X(N)$ *is a stable curve which has precisely one singular point in each singular fibre* $\mathcal{C}_x$*, and the universal cover* f_H *extends to a* finite *morphism* $f : \mathcal{C} \to E_{X(N)} := E \times X(N)$*. Furthermore, a fibre* $\mathcal{C}_x$ *is singular if and only if* $x \in \partial H$*, i.e.* $S = \partial H$*, and the set* S_0 *(as* *defined above) coincides with the set $X(N)_\infty$ of cusps of $X(N)$. Moreover, the structure of the singular fibres of $\mathcal{C}/X(N)$ is as follows:*

(a) If $x \in S_0$, then the fibre $\mathcal{C}_x$ is an irreducible curve whose normalization is a curve of genus 1. Furthermore, $\mathcal{C}_x$ has a unique singular point P_x, and the restriction $f_x = f_{\mathcal{C}_x} : \mathcal{C}_x \to (E_X)_x \simeq E$ is a cover of degree N such that $f_x(P_x) = 0_E$.

(b) If $x \in S_1$, then $\mathcal{C}_x = E_{x,1} \cup E_{x,2}$ is the union of two curves of genus 1 which meet transversely in a unique point P_x. Furthermore, $f_x(P_x) \in E[2]$ and the cover $f_x : \mathcal{C}_x \to E$ has degree N in the sense that $d_{x,1} + d_{x,2} = N$, where $d_{x,i} = \deg(f_{x,i})$ is the degree of $f_{x,i} := (f_x)_{|E_{x,i}} : E_{x,i} \to E$.

Moreover, the modular height of $\mathcal{C}/X(N)$ is $h_{\mathcal{C}/X} = \frac{1}{24} sl(N)$, where $sl(N) = \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, and the self-intersection number of the relative dualizing sheaf $\omega^0_{\mathcal{C}/X}$ is given by

$$(\omega^0_{\mathcal{C}/X(N)})^2 = \frac{7}{5}\#S_1 + \frac{1}{5}\#S_0 = \frac{1}{24N}(7N-6)sl(N). \tag{5}$$

Remark 3.10 The proof of this proposition is based on the ideas and techniques of [FK], pp. 161–166. Note, however, that the conclusions obtained here are much stronger than those of [FK], for here we are able to determine the number of irreducible components of each fibre. In addition, in this situation one of the cases (i.e. the case β_2) of [FK] does not occur.

The reason for obtaining such strong results is due to two facts. First of all, by using the results of [Ka3], [Ka4] (cf. Step 8 below), we are able to determine precisely the total number of singular fibres in each case. Secondly, we can use a result of Mumford to determine certain invariants which give information about the individual degeneration types. It is interesting to note that such a formula does not seem to be available in the arithmetic case; indeed, a considerable portion of the paper [FK] was devoted to deriving a weak version of such a formula. (This was pointed out to us by Barry Mazur.)

Proof of Theorem 3.9 (Sketch). For convenience of the reader, we outline the main ideas of the proof of Theorem 3.9 and refer to [Ka6] for details. (The above theorem is contained in Corollary 16, Theorem 17, Proposition 19, Theorem 22, Corollary 24 and Corollary 25 of [Ka6].)

Step 1: *f_H extends to a morphism $f : \mathcal{C} \to E_X$.*

Since E_X is a smooth proper curve over $X = X(N)$, this follows from Zariski's Main Theorem; cf. [Ka6], Proposition 15 and Corollary 16.

Step 2: *$\mathcal{C}/X$ is semistable and $S_0 = X(N)_\infty$.*

By the basic construction, $J_C \sim E_F \times E'$, where $E_F = E \otimes F$ and E'/F is an elliptic curve. Now since $E'[N] \simeq E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$ (as G_F-modules), it follows that both E' and J_C are semi-stable abelian varieties (as E has good reduction), and so $\mathcal{C}$ is a semi-stable curve (cf. [Ka6], proof of Theorem 17).

Clearly, J_C has bad reduction at $x \in X$ if and only if E' does. Since $E'/X(N)$ is the universal elliptic curve with level-N-structure (of fixed determinant), it follows that $S_0 = X(N)_\infty$.

Step 3: $S(C) = \partial H$.

Since $p_H : \mathcal{C}_H \to H$ has smooth fibres, it is clear that $S(C) \subset \partial H$. The opposite inclusion follows from the construction of H in [Ka5]; cf. [Ka6], Proposition 19 for a more detailed explanation.

Step 4: *The preliminary structure of $\mathcal{C}_x$ for $x \in S_0$.*

Here we are in case β) of [FK], p. 163ff. By the argument given there (or by the one presented in [Ka6], proof of Proposition 19), we conclude that $\mathcal{C}_x$ has a unique component $\mathcal{C}_{x,0}$ whose normalization has genus 1, and that there is at least one singular point $P_x \in \mathcal{C}_x$ such that $\mathcal{C}_x \setminus \{P_x\}$ is connected.

Step 5: *The preliminary structure of $\mathcal{C}_x$ for $x \in S_1$.*

Here we are in case α) of [FK], p. 161ff. Thus, if δ_x denotes the number of singular points of $\mathcal{C}_x$, then $\mathcal{C}_x$ has $\delta_x + 1$ irreducible components: two elliptic curves $E_{x,1}$, $E_{x,2}$ which are connected by a chain of $\mathbb{P}^1$'s.

Step 6: *Computation of the modular height $h_{\mathcal{C}/X}$.*

Recall that if $p : \mathcal{C} \to X$ is any semi-stable of genus $g \geq 1$, then its *modular height* is defined by $h_{\mathcal{C}/X} = \deg(\lambda)$, where $\lambda = \lambda_{\mathcal{C}/X} = \wedge^g p_* \omega^0_{\mathcal{C}/X} \in \mathrm{Pic}(X)$. Note that we also have $\lambda \simeq \lambda_{\mathcal{J}/X} := s^*\Omega^1_{\mathcal{J}/X}$, where $\mathcal{J}/X$ denote the Néron model of the Jacobian J_C and $s : X \to \mathcal{J}$ denotes the zero-section of $\mathcal{J}/X$; this is a formula due to Parshin and Arakelov (cf. [Ka6], proof of Theorem 17).

Since $J_C \sim E_F \times E'_F$ (by an etale isogeny), and since E_F/F is constant, we see (by using the second description of λ) that $\lambda_{\mathcal{C}/X} \simeq \lambda_{\mathcal{E}'/X}$. Now for any (semi-stable) elliptic curve $\mathcal{E}'/X$ we have (by Noether's formula) that $h_{\mathcal{E}'/X} = \frac{1}{12}\delta_{\mathcal{E}'/X}$, where $\mathcal{E}'$ denotes the Néron model of E' over X. But $\delta_{\mathcal{E}'/X} = N\#S_0$ (because each singular fibre $\mathcal{E}'_x$ of $\mathcal{E}'/X$ lies over $X(N)_\infty = S_0$

(cf. Step 2) and is a Néron polygon of length N), and so we obtain

$$h_{\mathcal{C}/X} = h_{\mathcal{E}'/X} = \frac{N}{12}\#S_0 = \frac{1}{24}sl(N), \tag{6}$$

where the latter equation follows from the first equality of (11) below.

Step 7: *The invariants δ_0, δ_1, ω^2 and Mumford's formula.*

Let δ_1 (respectively, δ_0) denote the number of singular points of the fibres of $\mathcal{C}/X$ which disconnect (respectively, which do not disconnect) the fibre. By Steps 4 and 5 we see that

$$\delta_0 \geq \#S_0 \quad \text{and} \quad \delta_1 \geq \#S_1. \tag{7}$$

Indeed, if $x \in S_0$, then by Step 4 there is at least one singular point P_x of $\mathcal{C}_x$ which does not disconnect the fibre, and if $x \in S_1$, then by Step 5 each of the $\delta_x \geq 1$ singular points in $\mathcal{C}_x$ disconnects the fibre.

There is a remarkable relation which connects the invariants $h = h_{\mathcal{C}/X}$, δ_0, and δ_1:

$$12h = \frac{6}{5}\delta_0 + \frac{12}{5}\delta_1. \tag{8}$$

This equation is an immediate consequence of the following two relations which also involve the invariant $\omega^2 = (\omega^0_{\mathcal{C}/X})^2$, the self-intersection number of the relative dualizing sheaf $\omega^0_{\mathcal{C}/X}$ of $\mathcal{C}/X$:

$$12h = \omega^2 + \delta_0 + \delta_1, \quad \text{and} \quad \omega^2 = \frac{1}{5}\delta_0 + \frac{7}{5}\delta_1, \tag{9}$$

which are called *Noether's formula* and *Mumford's formula* respectively; cf. [Ka6], proof of Theorem 22 for the relevant references.

By combining equation (8) with the inequalities (7), we obtain the inequality

$$\#S_1 \leq \frac{1}{12}(5N-6)\#S_0. \tag{10}$$

Indeed, since $h = \frac{N}{12}\#S_0$ by (6), we thus obtain from (8) and (7) that $N\#S_0 = 12h \geq \frac{6}{5}\#S_0 + \frac{12}{5}\#S_1$, which is (10).

Step 8: *Computation of $\#S_0$ and $\#S_1$.*

Whereas the computation of $\#S_0$ is easy (because $S_0 = X(N)_\infty$ by step 2), the determination of $\#S_1$ is much harder. Indeed, as is explained in the proof of [Ka6], Theorem 21 (see also [Ka5], Theorem 6.2), this is essentially the “mass formula” of [Ka3] , [Ka4]. (The hypothesis $\mathrm{char}(K) \nmid N!$ is crucial here.) One obtains:

$$(11) \qquad \#S_0 = \frac{sl(N)}{2N} \quad \text{and} \quad \#S_1 = \frac{1}{24N}(5N-6)sl(N).$$

Step 9: $\delta_x = 1$*, for all* $x \in S = S_0 \cup S_1$.

By equation (11) we see that we have equality in (10), and so we must have equality in (7) as well. Thus, $\delta_0 = \#S_0$ and $\delta_1 = \#S_1$, which means that $\delta_x = 1$, for all $x \in S = S_0 \cup S_1$, i.e. that $\mathcal{C}_x$ has a unique singular point. From this (together with Steps 4, 5) it is clear that the structure of $\mathcal{C}_x$ for $x \in S_0 \cup S_1$ is as claimed in parts (a), (b), and so $\mathcal{C}/X$ is a stable curve.

Step 10: f *is finite.*

Since f is proper, it is enough to show that f is quasi-finite, i.e. that none of the components of a fibre $\mathcal{C}_x$ of p is mapped to a point under f. For the points $x \notin S_1$, this easy (since $\mathcal{C}_x$ is irreducible). If $x \in S_1$, it is clear that at least one of the two components is not mapped to a point. To see that both have this property is a bit more difficult, particularly if N even; see [Ka6], Proposition 20, for the proof.

Since the rest of the assertions (concerning $f_x(P_x)$) are verified easily, and since (5) follows from (9) (because $\delta_0 = \#S_0$ and $\delta_1 = \#S_1$ by step 9), this concludes the proof (sketch) of the theorem.

3.4 Application: the Rigidity Number

As an application of the above geometric ideas and constructions, we shall now compute the number $\#\mathrm{Cov}_{E/K,N,\bar{P}}(K)$ of covers with fixed discriminant $\pi^*(\bar{P})$; this number may be viewed (via the identification of Remark 3.5) as a *measure of non-rigidity* of the Hurwitz space H_n^* because for rigid systems this number is 1; cf. [V1], p. 39.

Although this “rigidity number” can be defined via group theory, it seems very difficult to compute this number directly by counting tuples of conjugacy classes (cf. Remark 3.17). Here instead we shall use the geometric results of the previous subsections to calculate this number.

Theorem 3.11 *Let* $N \geq 3$ *be an integer, and suppose that* K *is an algebraically closed field. If* $\mathrm{char}(K) \nmid N!$*, then for every* $\bar{P} \in \mathbb{P}^1(K) \setminus \pi(E[2])$ *we have*

$$\#\mathrm{Cov}_{E/K,N,\bar{P}}(K) = \frac{1}{12}(N-1)sl(N),$$

where, as before, $sl(N) := \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$*.*

We briefly sketch the main ideas involved in the proof of Theorem 3.11. (For more detail, cf. [Ka6].) The basic idea is to relate the non-rigidity number $\#\mathrm{Cov}_{E/K,N,\bar{P}}(K)$ to the degree of the discriminant map $\delta_{E/K,N}$ (cf. Proposition 3.4) which in turn can be computed by using the intersection theory on compactification $\mathcal{C}$ of the universal genus 2 cover $f : \mathcal{C} \to E \times X(N)$. The key ingredients for this latter computation are the precise knowledge of the degenerate fibres of $\mathcal{C}$ (cf. Theorem 3.9) and the formula (5) for the self-intersection number $(\omega^0_{\mathcal{C}/X})^2$ of the relative dualizing sheaf.

Step 1: $\#\mathrm{Cov}_{E/K,N,\bar{P}}(K) = \deg(\delta_{E/K,N})$. By Proposition 3.4 we know that $\#\mathrm{Cov}_{E/K,N,\bar{P}}(K) = \#\delta_{E/K,N}(\bar{P}))$. On the other hand, since $\delta_{E/K,N}$ is finite and etale over $\bar{P} \notin \pi(E[2])$ (cf. [Ka6], Theorem 13), the assertion follows.

Step 2: *Analysis of the different divisor* D *on the compactification* $\mathcal{C}$*.*

We next want to interpret $\deg(\delta)$ as an intersection number. For this, we shall study the (Kähler) different divisor $\mathrm{Diff}(f)$ of the compactification $f : \mathcal{C} \to E \times X(N)$ of the universal cover f_H; cf. subsection 3.3.

Let D_F denote the different divisor of the generic (universal) cover $f_F : C = \mathcal{C}_F \to E_F$, and let D be its closure in the compactification $\mathcal{C}$. By using the explicit structure of the fibres of $p : \mathcal{C} \to X(N)$ (cf. Theorem 3.9) together with the Riemann-Hurwitz formula (for relative curves), we obtain

Proposition 3.12 *The divisor* D *is an irreducible curve on* $\mathcal{C}$ *which is a degree* 2 *cover of* $X(N)$ *via the map* $\pi_D := pr_2 \circ f_{|D}$*. If* $\delta_D := pr_1 \circ f_{|D} : D \to E$ *denotes the projection onto the first factor, then*

$$\pi_E \circ \delta_D = \overline{\delta}_{E/K,N} \circ \pi_D.$$

In addition, $D = \mathrm{Diff}(f)$*, the different divisor of* $f : \mathcal{C} \to E_X$*, and hence we have* $\omega^0_{\mathcal{C}/X(N)} \sim D$*. Thus, for any* $P \in E(K)$ *we have*

$$\deg(\delta_{E/K,N}) = \deg(\delta_D) = (\omega^0_{\mathcal{C}/X(N)} . f^*(P \times X)). \tag{12}$$

Proof. [Ka6], Theorem 26. Note that this makes heavy use of the Structure Theorem 3.9.

Step 3: *Analysis of the Weierstraß divisor* W*.*

By using the above proposition we see that the theorem will follow once we have computed the intersection number $(\omega^0_{\mathcal{C}/X}.f^*(P \times X))$. As we shall see next, this number is closely related to the self-intersection number of the *Weierstraß divisor* W on $\mathcal{C}$. This divisor W is the closure in $\mathcal{C}$ of the (usual) Weierstraß divisor W_F on $C = \mathcal{C}_F$.

Proposition 3.13 *We have $6D \sim 2W + p^*A$, for some divisor $A \in \mathrm{Div}(X)$ of degree*

$$(13)\ \ \deg(A) = \#X(N)_\infty - \frac{4}{3}W^2 = \frac{1}{6}(9(\omega^0_{\mathcal{C}/X})^2 - W^2) = \frac{1}{2N}(N-1)sl(N),$$

and hence

$$(14)\qquad \deg(\delta_D) = \frac{N}{6}\deg(A) = \frac{1}{12}(N-1)sl(N).$$

Proof. (Sketch; cf. [Ka6], Proposition 29). Since $6D_F$ and $2W_F$ are both 6-canonical divisors on C, we have $6D_F \sim 2W_F$. Moreover, since the fibres over S_1 are the only reducible fibres of $\mathcal{C}/X$ and since for $x \in S_1$, both $6D$ and $2W$ meet each component $E_{x,i}$ of $\mathcal{C}_x$ with multiplicity 6 (cf. the diagram on p. 162 of [FK]), we can conclude that $6D - 2W \sim p^*(A)$, for some $A \in \mathrm{Div}(X)$.

To calculate the degree of A, we shall use formula (34) of [Ka6]:

$$(15)\qquad (\omega^0_{\mathcal{C}/X(N)}.W) + W^2 = \#X(N)_\infty = \frac{1}{2N}sl(N).$$

This formula is established by applying the adjunction formula to the divisor W (after base-changing to $X(2N)$).

From (15) and the equivalence $6D \sim 2W + p^*A$, the first two equalities of (13) follow readily. To deduce the third, note that the first two yield that $6\deg(A) = (9\omega^2 - W^2) = 6\#X(N)_\infty - 8W^2$, and so

$$(16)\qquad W^2 = \frac{6}{7}\#X(N)_\infty - \frac{9}{7}\omega^2 = -\frac{3}{8N}(N-2)sl(N),$$

the latter by the important formula (5). From this, the last equation of (13) is immediate.

Finally, since it easy to see that $f_*(2W) \sim 12(0_E \times X)$ (and since $\omega \sim D$ by Proposition 3.12) we have $6f_*\omega \sim 12(0_E \times X) + N(E \times A)$, and so by (12) and the projection formula we obtain

$$6\deg(\delta_D) = 6(\omega.f^*(0_E \times X)) = ((12(0_E \times X) + N(E \times A)).0_E \times X) = N\deg(A),$$

which proves the first equality of (14). The second follows immediately from (13).

Step 4: *Conclusion.*

Combining the formula of Step 1 with equations (12) and (14) yields the desired formula of Theorem 3.11.

From the above result (and its proof) we can also compute the number of points in the "interior boundary" $H^{\partial}_{E/K,N}$ (cf. Remark 3.5(b)).

Corollary 3.14 *In the above situation we have*

$$\#H^{\partial}_{E/K,N} = \frac{(4N-3)(N-2)}{24N} sl(N). \tag{17}$$

Proof. We apply the Riemann-Hurwitz formula to the cover $\overline{\delta} = \overline{\delta}_{E/K,N} : X = X(N) \to \mathbb{P}^1$. Since $\overline{\delta}$ is tamely ramified (cf. [Ka6], Proposition 31) and since $\overline{\delta}$ is unramified outside of $\pi(E[2])$ (cf. Step 1 of the proof of Theorem 3.11), we have by Riemann-Hurwitz

$$2g_X - 2 = \deg(\overline{\delta})(-2) + 4\deg(\overline{\delta}) - \#(\overline{\delta}^{-1}(\pi(E[2]))).$$

It is easy to see that $\overline{\delta}^{-1}(\pi(E[2])) = \partial H \cup H^{\partial}_{E/K,N}$. Thus, since $2g_X - 2 = \frac{N-6}{12N} sl(N)$, since $\#\partial H = \frac{1}{24N}(5N+6)sl(N)$ by (11), and since $\deg \overline{\delta} = \frac{1}{12}(N-1)sl(N)$ (cf. (14) and (12)), we obtain that $\#H^{\partial}_{E/K,N} = 2\deg(\overline{\delta}) - \#\partial H - (2g_X - 2) = \frac{1}{6}(N-1)sl(N) - \frac{1}{24N}(5N+6)sl(N) - \frac{N-6}{12N}sl(N) = \frac{4N^2-11N+6}{24N}sl(N)$, and so (17) follows.

Remark 3.15 If we compare the above formula (17) with (11), we see that there are more points in the relative boundary $H^{\partial}_{E/K,N}$ than in the boundary $\partial H_{E/K,N}$; more precisely, we have

$$\#H^{\partial}_{E/K,N} - \#\partial H_{E/K,N} \; = \; \frac{1}{6}(N-6)sl(N).$$

As yet another application of Theorem 3.11, we compute the number $c_{N,D} = \#\mathrm{Cov}^{all}_{E,N,D}$ of *all* genus 2 covers of E of fixed degree $N \geq 1$ and fixed discriminant divisor $D \in \mathrm{Div}(E)$. (Thus, here we no longer assume that the cover is normalized.) Since this number is closely related to the *weighted number* $\bar{c}_{N,D} := \sum_{f \in \mathrm{Cov}^{all}_{E,N,D}} \frac{1}{|\mathrm{Aut}(f)|}$ of such covers and since the latter leads to simpler formulae, we determine $\bar{c}_{N,D}$ as well.

Corollary 3.16 *If* $\mathrm{char}(K) \nmid N!$ *and* $D \in \mathrm{Div}(E)$ *is an effective divisor of degree* 2*, then*

$$(18)\ \bar{c}_{N,D} = \frac{N}{3\mu_D}(\sigma_3(N) - N\sigma_1(N)) - \frac{\mu_D - 1}{24}(7\sigma_3(N) - (6N+1)\sigma_1(N))$$

where $\mu_D = 1$ *if* D *is reduced and* $\mu_D = 2$ *otherwise, and where* $\sigma_k(n) = \sum_{d|n} d^k$ *denotes the sum of the* k*th powers of the divisors of* n*. Moreover, if we put* $\sigma_1(N/2) = 0$ *if* N *is odd, then the total number of genus* 2 *covers is given by*

$$c_{N,D} = \bar{c}_{N,D} + \left(\frac{N}{\mu_D} - (\mu_D - 1)\right)\sigma_1(N/2). \quad (19)$$

Proof. [Ka6], Theorem 1. As is explained in section 2 of that paper, this theorem can be deduced from Theorem 3.11.

Remark 3.17 It is interesting to note that the number $\bar{c}_{N,D}$ (for D reduced) can be computed by a method that is essentially group-theoretic (and which goes back to Hurwitz[Hu]); cf. [Dij] and also [Ma]. Thus, by reversing the reasoning of the proof of the above corollary (by using a Moebius inversion formula), one can obtain a group-theoretical proof of Theorem 3.11.

4 Comparison of the Moduli Spaces

4.1 The Moduli Spaces H_N, H'_N and $Z_{N,-1}$

We now turn to the study of the moduli problems $\mathcal{H}_N$ and $\mathcal{H}'_N$ which were (briefly) introduced at the end of Subsection 2.2. For this, we need to extend the previous Hurwitz functor $\mathcal{H}_{E/K,N}$ (cf. Subsection 3.1) for a fixed elliptic curve E/K to a functor over a *variable* elliptic curve E/K.

A convenient framework for studying such questions is the concept of a *moduli problem for elliptic curves* as introduced by Katz-Mazur[KM], p. 107: by definition, such a moduli problem is a contravariant functor

$$\mathcal{P} : \underline{Ell}_{/R} \to \underline{Sets}$$

from the category $\underline{Ell}_{/R}$ to the category of sets. Here, as in [KM], p. 122, the *moduli stack* $\underline{Ell}_{/R}$ *of all elliptic curves* over a ring R is the category whose objects are (relative) elliptic curves E/S where S is any R-scheme, and whose morphisms from E/S to E'/S' consist of cartesian diagrams (with vertices E/S and E'/S') over R.

As is explained in [KM], pp. 108 and 125, each moduli problem $\mathcal{P}$ on $\underline{Ell}_{/R}$ gives rise to a contravariant functor

$$\widetilde{\mathcal{P}} : \underline{Sch}_{/R} \to \underline{Sets}$$

which classifies isomorphism classes of $\mathcal{P}$-structures. This functor $\widetilde{\mathcal{P}}$ gives us a more naive interpretation of our moduli problem. However, although the study of $\widetilde{\mathcal{P}}$ suffices for the discussion of coarse moduli spaces, it is the study of $\mathcal{P}$ which yields the most powerful (and natural) results.

Here we shall be interested in two (or three) specific moduli problems on $\underline{Ell}_{/R_N}$, where throughout $R_N = \mathbb{Z}[1/(2N)]$; these are defined by the rules

$$\mathcal{H}_N(E/S) = \mathcal{H}_{E/S,N}(S) \quad \text{and} \quad \mathcal{Z}_{N,-1}(E/S) = \mathcal{X}_{E/S,N,-1}(S),$$

for $E/S \in ob(\underline{Ell}_{/R})$, where the functors $\mathcal{H}_N(E/S)$ and $\mathcal{Z}_{N,-1}$ are as in Subsection 3.1. Clearly, each morphism in $\underline{Ell}_{/R}$ induces a natural map (by base-change) between the relevant sets, so that we obtain a moduli problem on $\underline{Ell}_{/R}$ as claimed.

In addition, we shall be interested in the moduli problem $\mathcal{H}'_N$ which is defined by rule $\mathcal{H}'_N(E/S) = \mathcal{H}'_{E/S,N}(S)$ (where $\mathcal{H}'_{E/S,N}$ is the subfunctor of $\mathcal{H}_{E/S,N}$ defined at the end of Subsection 3.1). Since this extra condition is obviously compatible with base change, we see that $\mathcal{H}'_N$ is a subfunctor of $\mathcal{H}_N$ and hence a moduli problem on $\underline{Ell}_{/R}$.

Note that the functor $\widetilde{\mathcal{H}}_N : \underline{Sch}_{/R} \to \underline{Sets}$ associated to $\mathcal{H}_N$ has the following natural interpretation:

$$\widetilde{\mathcal{H}}_N(S) = \{(C \xrightarrow{f} E) \in \mathcal{H}_N(E/S) : E/S \text{ is a (relative) elliptic curve}\}/\sim.$$

Moreover, $\widetilde{\mathcal{H}}'_N$ and $\widetilde{\mathcal{Z}}_{N,-1}$ have similar interpretations. Thus, although the moduli problems $\mathcal{H}_N$ and $\mathcal{H}'_N$ "classify" *isomorphism classes* of covers, their associated functors $\widetilde{\mathcal{H}}_N$ and $\widetilde{\mathcal{H}}'_N$ naturally "classify" *equivalence classes* of covers (in the sense of Definition 1.1).

Theorem 4.1 *If $N \geq 3$, then the moduli problems $\mathcal{H}_N$, $\mathcal{H}'_N$ and $\mathcal{Z}_{N,-1}$ are relatively representable, (quasi-) affine, smooth and geometrically connected of relative dimension 1 over $\underline{Ell}_{/R_N}$. Furthermore, we have open embeddings (of functors)*

$$\Psi_N : \mathcal{H}_N \hookrightarrow \mathcal{Z}_{N,-1} \quad \textit{and} \quad \Psi'_N : \mathcal{H}'_N \hookrightarrow \mathcal{H}_N \hookrightarrow \mathcal{Z}_{N,-1}.$$

Proof. For a fixed elliptic curve E/S, the "fibre functor" of $\mathcal{H}_N$ at E/S is by construction just the functor $\mathcal{H}_{E/S,N}$, i.e.

$$(\mathcal{H}_N)_{E/S} = \mathcal{H}_{E/S,N} : \underline{Sch}_{/S} \to \underline{Sets}.$$

Now by Theorem 5.18 of [Ka5] the functor $\mathcal{H}_{E/S,N}$ is representable for every E/S, and this means precisely that $\mathcal{H}_N$ is relatively representable (cf. [KM], p. 108). (Actually, in [Ka5] the representability of $H_{E/S,N}$ is asserted only for an affine base S, but from this the assertion follows (by gluing) for any base S.) Moreover, since the scheme $H_{E/S,N}$ representing $\mathcal{H}_{E/S,N}$ is quasi-affine, smooth and geometrically connected of relative dimension 1 over S, this means that functor $\mathcal{H}_N$ also has these properties by definition (cf. [KM], p. 109).

By construction, the fibre functor $(\mathcal{H}'_N)_{E/S} = \mathcal{H}'_{E/S,N}$ is a subfunctor of $(\mathcal{H}_N)_{E/S}$, and it is easy to see that $(\mathcal{H}'_N)_{E/S}$ is represented by an open subscheme $H' = H'_{E/S,N}$ of $H = H_{E/S,N}$. If S is a field, then this was proven in Proposition 3.4. In the general case we have $H' = H \setminus Z$, where Z is the closed subscheme of H which is universal for the relation $\mathrm{Disc}(f_H) \leq E_H[2]^{\#}$ in the sense of [KM], Key Lemma 1.3.4. Here, $f_H : C \to E_H$ denotes the universal normalized genus 2 cover of degree N over E_H and $E[2]^{\#} = E[2] - 0_{E_H}$ (viewed as a Cartier divisor as in [Ka5]).

The proof of the assertions about $\mathcal{Z}_{N,-1}$ are entirely analogous to those for $\mathcal{H}_N$; here we use [Ka5], Corollary 4.3 in place of Theorem 5.18. (Note that $\mathcal{X}_{E/S,N,-1}$ is the fibre functor of $\mathcal{Z}_{N,-1}$ at E/S.) Finally, the open embedding Ψ_N is induced by the (open) embeddings $\Psi_{E/S,N} : \mathcal{H}_{E/S,N} \to \mathcal{X}_{E/S,N,-1}$; cf. (4) or [Ka5], Theorem 5.18.

Remark 4.2 The above moduli problems $\mathcal{H}'_N, \mathcal{H}_N$ and $\mathcal{Z}_{N,-1}$ are in fact affine and not just quasi-affine. For $\mathcal{Z}_{N,-1}$, this is immediate from the above construction, but not for $\mathcal{H}'_N$ and $\mathcal{H}_N$. Since we don't need this here, we omit the proof.

It is clear that none of the functors $\mathcal{H}_N$, $\mathcal{H}'_N$ and $\mathcal{Z}_{N,-1}$ is representable, for none is rigid (because the $[-1]$ map acts trivially on each). However, since all are (quasi-) affine and relatively representable, there exists a *coarse moduli scheme* $H_N := M(\mathcal{H}_N)$, and $H'_N := M(\mathcal{H}'_N)$ and $Z_{N,-1} = M(\mathcal{Z}_{N,-1})$ for each; cf. [KM], p. 224. Note that $Z_{N,-1}$ (and hence also H_N and H'_N) comes equipped with a natural morphism

$$p_N : Z_{N,-1} \to M_1 := \mathrm{Spec}(R_N[j]) = M([\Gamma(1)]),$$

to the j-line over R_N which is the coarse moduli space $M_1 = M([\Gamma(1)])$ attached to the moduli problem $[\Gamma(1)]$ classifying isomorphism classes of elliptic curves; cf. [KM], p. 229. Thus we have:

Corollary 4.3 *The coarse moduli scheme $H_N := M(\mathcal{H}_N)$ of $\mathcal{H}_N$ is an open subscheme of the coarse moduli scheme $Z_{N,-1} = M(\mathcal{Z}_{N,-1})$ of $\mathcal{Z}_{N,-1}$. The latter is normal and affine of relative dimension* 1 *over $M_1 = \mathrm{Spec}(R_N[j])$. In addition, the coarse moduli scheme $H'_N := M(\mathcal{H}'_N)$ of $\mathcal{H}'_N$ is an open subscheme of H_N and hence also of $Z_{N,-1}$.*

Remark 4.4 As a first approximation, we can view the previously considered moduli spaces $H_{E/K,N}$ as the fibres of the morphism $p_N : H_N \to M_1$. Indeed, since each E/K gives rise to a point $x_E \in M_1$ such that $\kappa(x_E) = K$ and $j(E) = j(x_E)$, we might expect that the fibre $(H_N)_x$ over x is "essentially" (i.e. up to twists) the curve $H_{E/K,N}$. While this is true "in general", it is not always true. For example, at a (geometric) point x where $\mathrm{Aut}(E_x) \neq \pm 1$, the fibre of p_N is an affine curve whose compactification has lower genus than those of the generic case.

By the above corollary we see that H_N and $Z_{N,-1}$ are (quasi-) affine normal surfaces over $\mathrm{Spec}(R_N)$. We next want to identify the surfaces in terms of the modular diagonal quotient surfaces introduced in [KS].

Even though these surfaces can be defined over R_N (cf. [Ka7]), we shall explain their construction only over the ring $\tilde{R}_N = R_N[\zeta_N] = \mathbb{Z}[\frac{1}{2N}, \zeta_N]$, where ζ_N is a primitive N-th root of unity. (This situation suffices for geometric purposes, but not for arithmetic applications.) Over $\tilde{R}_N$ we have the smooth affine curve $X(N)'_{/\tilde{R}_N}$ which represents the moduli problem $[\Gamma(N)]^{can}$ of [KM], p. 283. It comes equipped with a natural action of the group $G_N = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$; thus $G_N \times G_N$ acts on the product $Y_N = X(N)'_{/\tilde{R}_N} \times_{\tilde{R}_N} X(N)'_{/\tilde{R}_N}$. For any $\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times$, let

$$\Delta_\varepsilon = \{(g, \tau_\varepsilon(g)) : g \in G_N\} \leq G_N \times G_N$$

denote the "twisted diagonal subgroup", i.e. the graph of the automorphism $\tau_\varepsilon \in \mathrm{Aut}(G_N)$ defined by $\tau_\varepsilon(g) = \sigma_\varepsilon g \sigma_\varepsilon^{-1}$, where $\sigma_\varepsilon = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Then the quotient

$$Z_{N,\varepsilon} = Y_N/\Delta_\varepsilon$$

is called the *modular diagonal quotient surface* of type (N, ε).

Proposition 4.5 *The modular diagonal quotient surface $Z_{N,\varepsilon}$ is the coarse moduli space of the moduli problem $\mathcal{Z}_{N,\varepsilon}$; i.e. $M(\mathcal{Z}_{N,\varepsilon}) = Z_{N,\varepsilon}$, and hence the coarse moduli spaces H_N and H'_N are open subsets of $Z_{N,-1}$.*

Proof (Sketch). As was mentioned above, the moduli problem $[\Gamma(N)]^{can}_{/\tilde{R}_N}$ is represented by E_N/X, where $X = X(N)'_{\tilde{R}_N}$, Thus, since $\mathcal{Z}_{N,\varepsilon}$ is affine and relatively representable by Theorem 4.1, it follows that the simultaneous moduli problem $\mathcal{P}_{N,\varepsilon} := [\Gamma(N)]^{can} \times \mathcal{Z}_{N,\varepsilon}$ is represented by p^*E/Y_ε where $p : Y_\varepsilon \to X$ represents the functor $\mathcal{X}_{E_N/X,N,\varepsilon}$ on $\underline{Sch}_{/X}$; cf. [KM], p. 109. Thus, by [KM], p. 224, we see that the coarse moduli space of $\mathcal{Z}_{N,\varepsilon}$ is $M(\mathcal{P}_{N,\varepsilon})/G_N$ because $[\Gamma(N)]^{can}_{/\tilde{R}_N}$ is etale with group G_N.

Now $Y_\varepsilon \simeq X \times_{\tilde{R}_N} X$ because X also represents the functor $\mathcal{X}_{E_N/\tilde{R}_N,N,\varepsilon}$ on $\underline{Sch}_{\tilde{R}_N}$, and via this identification the group G_N is mapped to $\Delta_\varepsilon \leq G_N \times G_N$. Thus $M(\mathcal{Z}_{N,\varepsilon}) = Z_{N,\varepsilon}$, as claimed. The last assertion follows from Corollary 4.3.

While the above result gives a concrete interpretation of the coarse moduli space $Z_{N,-1}$ and hence of the subspaces H_n and H'_n, it has the disadvantage that its interpretation as a coarse moduli space has nothing to with covers of curves. As a result, it seems somewhat mysterious how the spaces H_n and H'_n are embedded in $Z_{n,-1}$. To remedy this, we now explain how $Z_{n,-1}$ is the coarse moduli space of suitable Hurwitz functor.

As a first step towards this, let $\mathcal{A}_N$ denote the moduli problem on $\underline{Ell}$ which, for a fixed elliptic curve E/S, is the set

$$\mathcal{A}_N(E/S) = \mathcal{A}_{E/S,N}(S) = \{(J, \lambda, h)\}/\simeq,$$

of isomorphism classes of tripels (J, λ, h) consisting of a principally polarized abelian surface (J, λ) over S and and injective homomorphism $h : E \to J$ of degree N in the sense of [Ka5], §5.1 (p. 24). We then have:

Proposition 4.6 *There is a natural isomorphism $\mathbf{\Psi}' : \mathcal{A}_N \xrightarrow{\sim} \mathcal{Z}_{N,-1}$ of moduli problems, and hence $Z_{N,-1}$ is also a coarse moduli scheme of $\mathcal{A}_N$.*

Proof. For each $E/S \in ob(\underline{Ell})$ we have by Theorem 5.10 of [Ka5] an isomorphism $\mathbf{\Psi}'_{E/S} : (\mathcal{A}_N)_{E/S} = \mathcal{A}_{E/S,N} \xrightarrow{\sim} (\mathcal{Z}_{N,-1})_{E/S} = \mathcal{X}_{E/S,N,-1}$ of the corresponding fibre functors, and these fit together to define the desired isomorphism. The last assertion follows from Proposition 4.5.

We next need to identify $\mathcal{A}_N$ with a Hurwitz moduli problem. This, however, is more problematic. Consider first the moduli problem H_N^{st} on $\underline{Ell}$ which, for a given E/S, classifies finite flat covers $f : C \to E$ of degree N where C/S is a *stable* curve of genus 2 (in the sense of [DM]), i.e.

$$\mathcal{H}_N^{st}(E/S) = \{f : C \to E \text{ finite, flat}, C/S \text{ stable of genus } 2, \deg(f) = N\}/\simeq .$$

Next, let $\mathcal{H}_N^{\S}$ be the subfunctor of $\mathcal{H}_N^{st}$ consisting of the those S-covers $f : C \to E$ such that $(J_{C_s}, \theta_{C_s}, f_s^*) \in \mathcal{A}_N(\mathrm{Spec}(\kappa(s)))$, for every $s \in S$; here $J_{C_s}/\kappa(s)$ is the Jacobian of the fibre C_s and θ_s its canonical polarization. Since the Jacobian exists for any stable curve C/S (cf. [BLR], Th. 9.4/1), the rule $(f : C \to E) \mapsto (J_C, \theta, f^*)$ defines a morphism of moduli problems

$$j_N^{\S} : \mathcal{H}_N^{\S} \to \mathcal{A}_N.$$

However, $j_N^{\S}$ fails to be injective since no normalization condition was imposed. To remedy this, we might like to impose (if $N = n$ is odd) the normalization condition (1), but this will not work if the cover $f_s : C_s \to E_s$ is reducible (i.e. C_s is a union of two elliptic curves) because the hyperelliptic involution σ_{C_s} now has 7 (in place of 6) fixed points. Instead, we could require that (in all cases) $f^{-1}(0_{E_s}) \cap \mathrm{Fix}(\sigma_{C_s})$ has at least 3 points, and (as will be shown elsewhere) this does lead to a moduli problem $\mathcal{H}_n^{\dagger}$ which is isomorphic to $\mathcal{A}_n$ and which contains H_n as a natural open subproblem.

4.2 Hurwitz Moduli Stacks

We now analyze the moduli space H_n^* which was introduced in Subsection 1.3. It turns out the corresponding moduli problem $\mathcal{H}_n^*$ can be studied by a formalism which is very similar to that used for the moduli problem $\mathcal{H}_n$. Here, however, we have to replace the stack $\underline{Ell}_{/R}$ of elliptic curves over R by the stack $(\underline{M}_0)_{/R}$ of curves of genus 0 over R. The latter is defined completely analogous to $\underline{Ell}_{/R}$, i.e. $(\underline{M}_0)_{/R}$ is the category fibred in groupoids over $\underline{Sch}_{/R}$ whose objects are relative (smooth, proper) curves $X \to S$ of genus 0, and whose morphisms are cartesian diagrams.

We shall now interpret $\mathcal{H}_n^*$ as a *moduli problem on* $(\underline{M}_0)_{/R}$, i.e. as a contravariant functor $\mathcal{H}_n^* : (\underline{M}_0)_{/R} \to \underline{Sets}$; here R is any ring in which $n!$ is invertible.

For this, let $X/S \in ob((\underline{M}_0)_{/R})$ be a relative genus 0 curve, and consider the set

$$\mathrm{Cov}^*_{X/S,n} := \{\varphi : Y \to X : \varphi \text{ is a finite cover of degree } n \text{ whose fibres } \varphi_t \text{ satisfy } (*)\ \}/\simeq_X .$$

Then the rule $\mathcal{H}^*_n(X/S) = \mathrm{Cov}^*_{X/S,n}$ defines a moduli problem on $(\underline{M}_0)_{/R}$, i.e. a functor $\mathcal{H}^*_n : (\underline{M}_0)_{/R} \to \underline{Sets}$.

Similar to the case of moduli problems on $\underline{Ell}_{/R}$, each moduli problem $\mathcal{P}$ on $(\underline{M}_0)_{/R}$ gives rise to a functor $\widetilde{\mathcal{P}} : \underline{Sch}_{/R} \to \underline{Sets}$ by considering isomorphism classes of $\mathcal{P}$-structures. In particular, for $\mathcal{P} = \mathcal{H}^*_n$ we obtain:

$$\widetilde{\mathcal{H}}^*_n(S) = \{(Y \to X) \in \mathrm{Cov}^*_{X/S,n} : X/S \text{ is a (relative) curve of genus } 0\}/\sim,$$

where, as before, the indicated equivalence relation is the weak equivalence of covers. Note that if $S = \mathrm{Spec}(K)$ is a field, then the set $\widetilde{\mathcal{H}}^*_n(S)$ coincides with the set $\widetilde{\mathcal{H}}^*_n(K)$ defined in Remark 1.5(a).

Remark 4.7 (a) Although we don't need this here, we observe that the moduli problem $\mathcal{H}^*_n \simeq \tilde{H}^*_n$ is *relatively representable*, i.e. for each $X/S \in ob((\underline{M}_0)_{/R})$, the fibre functor $(\mathcal{H}^*_n)_{X/S} : \underline{Sch}_{/S} \to \underline{Sets}$ defined by $T \mapsto \mathcal{H}^*_n(X_T/T)$ is representable. Indeed, if $X = \mathbb{P}^1_K$, where K is a field, then the results of Fried-Völklein and Wewers show that this functor is represented by the scheme $H^{in}(S_n, \mathbf{C}) \otimes K$,as was explained in subsection 1.3. For a general scheme X/S, this follows from the results of Wewers [We].

(b) For later usage we observe here that the concept of a moduli problem on $(\underline{M}_0)_{/R}$ can be reformulated in terms of *moduli stacks* over $(\underline{M}_0)_{/R}$: the latter are categories $p : \underline{C} \to (\underline{M}_0)_{/R}$ over $(\underline{M}_0)_{/R}$ which are fibred in groupoids. Indeed, given a moduli problem $\mathcal{P} : (\underline{M}_0)_{/R} \to \underline{Sets}$, we can define its *classifying stack* $\underline{\mathcal{P}}$ by the rule that its fibre category over $X/S \in ob((\underline{M}_0)_{/R})$ is $\underline{\mathcal{P}}_{X/S} = \mathcal{P}(X/S)$ (where the morphisms are the identities). Conversely, each moduli stack $\underline{C}$ over $(\underline{M}_0)_{/R}$ gives rise to a moduli problem $\mathcal{P}_{\underline{C}}$ on $(\underline{M}_0)_{/R}$.

(c) In the case of the moduli problem $\mathcal{H}^*_n$, it is immediate that the associated classifying stack is equivalent (in the sense of categories) to the fibred category $\underline{\mathcal{H}}^*_n$ whose objects are morphisms are triples $Y \to X \to S$ with $X/S \in ob((\underline{M}_0)_{/R})$ and $Y/X \in \mathcal{H}^*_n(X/S)$, and whose morphisms are (extended) cartesian diagrams. Thus, by the discussion of Subsection 1.3, we see that we can identify $\underline{\mathcal{H}}*_n$ with the stack $\mathcal{H}_{0,0,m}$ of [BR], Theorem 6.22, for a suitable monodromy $m = (G, H, \xi)$. (More precisely, $G = S_n$ and $H \simeq S_{n-1}$ is the stabilizer of an element of the set $\{1, \ldots, n\}$ (on which S_n acts), and ξ is the Hurwitz data given by the class $\mathbf{C}$.)

It is clear that the moduli problem $\mathcal{H}_n^*$ cannot be representable because the group scheme $\mathrm{Aut}(\mathbb{P}^1)$ acts non-trivially. However, we have:

Proposition 4.8 *The moduli problem $\mathcal{H}_n^*$ has a coarse moduli space which is given by*

$$M(\mathcal{H}_n^*) = H_n^* = H^{in}(S_n, \mathbf{C})/\mathrm{Aut}(\mathbb{P}^1).$$

Proof. This follows easily from [BR]. Indeed, via identification $\underline{\mathcal{H}}_n^* = \mathcal{H}_{0,0,m}$ of Remark 4.7(c) it follows from [BR], Théorème 6.22 that $\underline{\mathcal{H}}_n^* \simeq \mathcal{H}_{g,S_n,\xi}/\Delta(m)$ for a suitable integer g. But since here $N_G(H) = H$, we see that $\Delta(m) = 1$, and so $\underline{\mathcal{H}}_n^* \simeq \mathcal{H}_{g,S_n,\xi}$. Now by formula (6.17) of [BR] we know that $\mathcal{H}_{g,S_n,\xi} \simeq \mathcal{FH}_{g,S_n,\xi}/\mathrm{Aut}(\mathbb{P}^1)$. Since the results of Fried/Völklein/Wewers[We] show that the Fried-Völklein stack $\mathcal{FH}_{g,S_n,\xi}$ is representable by $H^{in}(S_n, \mathbf{C})$, the assertion about the coarse moduli spaces follows from [BR], Théorème 6.3 (3).

4.3 The Main Result

We now want to relate the moduli problem $\mathcal{H}'_n$ (on $\underline{Ell}_{/R}$) to the moduli problem $\mathcal{H}_n^*$ (on $(\underline{M}_0)_{/R}$). In order to be able to compare them, we shall pass to their classifying stacks $\underline{\mathcal{H}}'_n$ and $\underline{\mathcal{H}}_n^*$ (over $\underline{Ell}$ and over $\underline{M}_0$, respectively), and view both of these as fibred categories over $\underline{Sch}_{/R}$; cf. Remark 4.7(b). We then have:

Theorem 4.9 *The rule $(C \xrightarrow{f} E) \mapsto (C/\langle\omega_C\rangle \to E/\langle[-1]\rangle)$ defines a functor $q = q_n : \underline{\mathcal{H}}'_n \to \underline{\mathcal{H}}_n^*$, and the induced map*

$$M(q) : H'_n = M(\mathcal{H}'_n) \ \to \ H_n^* = M(\mathcal{H}_n^*)$$

on the coarse moduli schemes is surjective and radical. Thus, if $R = K$ is a field of characteristic 0, then $M(q)$ is an isomorphism and hence $H'_n \simeq H_n^*$ is an irreducible, normal affine surface.

Proof. Let $(C \xrightarrow{f} E) \in ob(\underline{\mathcal{H}}'_n)$, where C and E are S-curves. Then, since f is normalized, we have $f\omega_C = [-1]f$, and so we have an induced morphism

$\varphi : Y := C/\langle \omega_C \rangle \to X := E/\langle [-1] \rangle$. (The quotients exist and are smooth S-curves by Lønsted-Kleiman[LK], Theorem 4.12.) Moreover, if $s \in S$, then $\mathrm{Disc}(f_s)$ is reduced (by hypothesis), and by Proposition 2.4 we see that φ_s is a cover of type $(*)$, which means that $(Y \xrightarrow{\varphi} X) \in ob(\underline{\mathcal{H}}_n^*)$. Since this construction is compatible with base-change (because 2 is invertible in S), we see that this defines the desired functor q.

The functor q induces a natural transformation $\widetilde{q} : \widetilde{\mathcal{H}}'_n \to \widetilde{\mathcal{H}}^*_n$ between the associated functors on $\underline{Sch}_{/R}$. Thus, by the universal property of coarse moduli spaces, there is unique morphism $M(q) = M(\widetilde{q}) : H'_n = M(\mathcal{H}'_n) \to H^*_n = M(\mathcal{H}^*_n)$ such that $\mu'\widetilde{q} = h_{M(q)}\mu^*$, where $\mu' : \mathcal{H}^*_n \to h_{M(\mathcal{H}'_n)}$ and $\mu^* : \mathcal{H}^*_n \to h_{M(\mathcal{H}^*_n)}$ are the natural transformations associated to the coarse moduli spaces. (Recall that Propositions 4.5 and 4.8 show that we have the identifications $M(\mathcal{H}'_n) = H'_n$ and $M(\mathcal{H}^*_n) = H^*_n$.)

Now the discussion of Subsection 2.2 shows:

Claim: If $S = \mathrm{Spec}(k)$, where k is an algebraically closed field, then

$$\widetilde{q}_S : \widetilde{\mathcal{H}}'_n(S) \to \widetilde{\mathcal{H}}^*_n(S)$$

is a bijection.

Thus, since $\mu'_S : \widetilde{\mathcal{H}}'_n(S) \xrightarrow{\sim} h_{M(\mathcal{H}'_n)}(S) = H'_n(k)$ and $\mu^*_S : \widetilde{\mathcal{H}}^*_n(S) \xrightarrow{\sim} \widetilde{\mathcal{H}}^*_n(S) = H^*_n(k)$ are bijections, it follows that $M(q)_k : H'_n(k) \to H^*_n(k)$ is a bijection. Thus, $M(q)$ is surjective and radical; cf. [EGA], (I,3.6.3) and (I,3.7.1). Moreover, if $R = K$ is a field of characteristic 0, then H'_n is normal and irreducible (cf. Proposition 4.5), and so we see that H^*_n is also irreducible, and that hence $M(q)$ is a birational map. In addition, $M(q)$ is separated because H'_n is quasi-affine and H^*_n is affine. Thus, since H^*_n is normal (cf. Remark 1.5(b)), it follows from Zariski's Main Theorem that $M(q)$ is a local isomorphism; cf. [EGA], (Err_{IV}, 30). By [EGA], (I, 4.4.8) we thus have that $M(q)$ is an open immersion, and hence an isomorphism since $M(q)$ is surjective.

Remark 4.10 (a) Even though $M(q)$ is an isomorphism, the functor

$$q : \underline{\mathcal{H}}'_n \to \underline{\mathcal{H}}^*_n$$

is not an isomorphism (of stacks). Indeed, q cannot be an equivalence (of categories) because of the presence of twists of elliptic curves. More precisely, if $K \supset R$ is any field for which $K^*/(K^*)^2 \neq 1$, then the discussion of

Subsection 2.2 shows that $\widetilde{q}_K : \widetilde{\mathcal{H}}'_n(K) \to \widetilde{\mathcal{H}}^*_n(K)$ cannot be injective, and so in particular q cannot be an equivalence of categories.

(b) In his preprint, Fried[Fri] proves the irreducibility of $H^{in}(S_n, \mathbf{C})$ by the method of Nielsen classes. This gives a different proof of the irreducibility of H^*_n.

4.4 Connection with Humbert surfaces

If H is any Hurwitz scheme which classifies curve covers $f : C \to C_1$ with fixed genus $g = g_C$, then the forget map $(f : C \to C_1) \mapsto C$ induces a natural morphism

$$\mu_H : H \to M_g$$

to the moduli space M_g which classifies isomorphism classes of genus g curves. The image $\mu_H(H)$ is thus a subscheme of M_g which frequently carries interesting information. For example, if $H = H_{d,g}$ is the Hurwitz space of simple genus g covers of degree d of $\mathbb{P}^1$, then the $\mu_H(H_{d,g})$'s ($d \geq 2$) define cycles on M_g which are important for the study of the geometry of M_g; cf. [HM], p. 32, 175, for more detail.

Specializing this to our case $H = H_N$ which classifies normalized genus 2 covers, we thus obtain a morphism

$$\mu_N = \mu_{H_N} : H_N \to M_2.$$

It turns out that the image of μ_N is essentially a *Humbert surface*. To make this more precise, recall (cf. [Hum] or [vdG], section IX.2) that for each positive integer $D \equiv 0, 1(4)$, there is an irreducible surface Hum_D contained in the space A_2 of principally polarized abelian surfaces; Hum_D is called the *Humbert surface* with *Humbert invariant* (or discriminant) D.

Since each point of $\mu_N(H_N)$ corresponds to a genus 2 curve with an elliptic differential, it follows from Humbert[Hum], Théorème 15, [3] that $\mu_N(H_N)$ is contained in some Humbert surface Hum_D with square invariant $D = d^2$; here we view M_2 as an open subset of A_2 via the (birational) map $M_2 \to A_2$ (which take a curve to its polarized Jacobian). More precisely:

Proposition 4.11 *We have $\mu_N(H_N) = \mathrm{Hum}_{N^2} \cap M_2$. More precisely, if τ denotes the involution of $Z_{N,-1}$ induced by the automorphism of $X(N) \times$*

[3] In fact, this result was already proven by Biermann in 1883 (cf. [Kr], p. 485), long before the discovery of Humbert surfaces.

$X(N)$ which interchanges the two factors, then μ_N factors over the quotient $H_N^{sym} := H_N/\langle\tau\rangle$, and the induced map

$$\mu_N^{sym} : H_N^{sym} \to \mathrm{Hum}_{N^2} \cap M_2$$

is the normalization map of the Humbert surface $\mathrm{Hum}_{N^2} \cap M_2$.

Proof. The first assertion follows from Theorem 1.9 of [Ka2] (together with the fact that every minimal cover can be normalized; cf. [Ka5], Proposition 2.2). The second assertion follows from Corollary 1.8 of [Ka2] because H_N is precisely the (open) part of $Z_{N,-1}$ whose points correspond to genus 2 curve covers; cf. Corollary 4.3, together with the discussion of the next section.

Remark 4.12 If D is not a square, then the Humbert surface H_D is birationally isomorphic to the *symmetric Hilbert modular surface* Hilb_D^{sym}; the latter is the quotient of the *Hilbert modular surface* $\mathrm{Hilb}_D = \mathrm{SL}_2(\mathfrak{O}_D)\backslash(\mathfrak{H}\times\mathfrak{H})$ by the involution τ induced by the involution of $\mathfrak{H} \times \mathfrak{H}$ which interchanges the two factors; cf. [vdG], Proposition (IX.2.5). Thus, the Corollary 1.8 of [Ka2] may be viewed as an extension of this fact to square discriminants by viewing the modular diagonal quotient surfaces $Z_{N,\varepsilon}$ as "degenerate Hilbert modular surfaces". This is the point of view taken by Hermann, who investigated the geometry of $Z_{N,\varepsilon}$ and of $Z_{N,\varepsilon}^{sym} = Z_{N,\varepsilon}/\langle\tau\rangle$ as analogues of Hilbert modular surfaces; cf. [He1], [He2].

This point of view is also adopted by McMullen in his papers. Indeed, he uses a more general definition of Hilbert modular surfaces Hilb_D in which also square discriminants D are allowed, and then shows in all cases that Hum_D is birationally isomorphic to Hilb_D^{sym}; cf. [Mc2], Theorem 4.5.

In the case that $D = N^2$, this result may be stated as follows: if $C/\overline{K}$ is a curve of genus 2 with Jacobian J_C, then

$$\langle C\rangle \in \mathrm{Hum}_{N^2}(\overline{K}) \;\Leftrightarrow\; \mathrm{End}(J_C) \text{ has real multiplication by } \mathfrak{O}_{N^2};$$

the latter means that we have an embedding $\rho : \mathfrak{O}_{N^2} := \mathbb{Z}[x]/(x^2 - Nx) \hookrightarrow \mathrm{End}(J_C)$ which is *primitive*, i.e. $\rho(\mathfrak{O}_{N^2})\mathbb{Q} \cap \mathrm{End}(J_C) = \rho(\mathfrak{O}_{N^2})$. [Indeed, if $\langle C\rangle \in \mathrm{Hum}_{N^2}(\overline{K})$, then by Proposition 4.11 we have a (normalized) subcover $f : C \to E$ of degree N to some elliptic curve E, and then the rule $x \mapsto f^*f_*$ induces such an embedding. Conversely, if we have such an embedding, then $\varepsilon = \frac{1}{N}\rho(x) \in \mathrm{End}^0(J_C)$ is a non-trivial idempotent, and one concludes (using Lange[Lan]) that $\rho(x) = N\varepsilon = f^*f_*$, for a suitable minimal cover $f : C \to E$.]

5 The Compactification of H'_n and of H^*_n and Related Boundary Curves

5.1 Compactifications: an Overview

As the constructions of the previous section show, the moduli spaces H'_n and H^*_n are not compact. It is thus of interest to construct natural compactifications of these spaces and to investigate whether or not the boundary components have a modular interpretation in terms of covers of curves.

Since H'_n was constructed as an open subset of the affine surface $Z_{n,-1}$, the natural compactification $\overline{Z}_{n,-1}$ of $Z_{n,-1}$ (which was studied in [KS]) also serves as a compactification of H'_n. While this compactification has the advantage of being very explicit, its disadvantage is that it does not readily lead to a modular interpretation in terms of covers. However, such an interpretation can be obtained by studying the fibres of the canonical compactification $\mathcal{C}$ of the universal cover $C \to E_H$ over $H = H_{E/K,n}$ for each fixed elliptic curve E/K; cf. Subsection 3.3.

More precisely, we saw in Theorem 3.9 that the study of the fibres of $\mathcal{C}$ lead to the following four types of covers $f_x : \mathcal{C}_x \to E$ of E:

I. $\mathcal{C}_x$ is a smooth curve of genus 2 and $\mathrm{Diff}(f_x)$ is reduced;

II. $\mathcal{C}_x$ is smooth curve of genus 2 and $\mathrm{Diff}(f_x) = 2P$ with $P \in E[2]$;

III. $\mathcal{C}_x$ is the union of two elliptic curves meeting at a point;

IV. $\mathcal{C}_x$ is a singular irreducible curve of arithmetic genus 2.

These cases correspond to the stratification

$$\overline{X(N)} \;=\; H'_{E/K,N} \;\dot\cup\; H^{\partial}_{E/K,N} \;\dot\cup\; S_1 \;\dot\cup\; S_0$$

which was constructed in Section 3. By varying E, we should expect that these induce a similar stratification for $\overline{Z}_{N,-1}$. This, however, is not quite correct since we also have to allow the case that E degenerates. Thus, we have a fifth case:

V. $C \to E$ is a cover of a stable curve E of arithmetic genus 1.

The precise nature of the covers of type V will be discussed in the next section.

On the other hand, the theory of Wewers [We] gives a recipe for an (abstract) compactification of $H^{in}(S_n, \mathbf{C})$ in terms of covers of $\mathbb{P}^1$; here the boundary components correspond to covers in which the monodromy group, the ramification type and the curve are (suitable) degenerations the generic situation. This, therefore, also gives a compactification of $H_n^* = H^{in}(S_n, \mathbf{C})/\mathrm{Aut}(\mathbb{P}^1) \simeq H_n'$, and so one might expect that these boundary components match up with those of H_n' defined by the first method. This is indeed the case. Thus, in both interpretations the boundary curves of the Hurwitz spaces correspond to interesting degenerations of covers.

5.2 The Boundary Curves of H_n'

In the case of the Hurwitz space $H_n' \subset Z_{n,-1}$ which parameterizes normalized genus 2 covers of elliptic curves, the above overview (§5.1) shows that boundary curves naturally split into four main types. As we shall see, these correspond to the stratification induced by the inclusions

$$H_n' \;\subset\; H_n \;\subset\; Z_{n,-1} \;\subset\; \bar{Z}_{n,-1}^{(1)} \;\subset\; \overline{Z}_{n,-1}.$$

Here $\bar{Z}_{n,-1}^{(1)} = \psi_1^{-1}(X(1))$ is the inverse image of $X(1) \simeq \mathbb{A}^1$ of the morphism

$$\psi_1 = pr_1 \circ \psi : \overline{Z}_{n,-1} \;\to\; \overline{X(1)} \times \overline{X(1)} \;\to\; \overline{X(1)} \;\simeq\; \mathbb{P}^1,$$

where, as in [KS], $\psi : \overline{Z}_{n,-1} \to \overline{X(1)} \times \overline{X(1)}$ is the map induced by the inclusion $\Delta_{-1}(G_n) \leq G_n \times G_n$. Note that the fibres of ψ_1 are geometrically irreducible (but not necessarily reduced); cf. [KS], p. 353.

We have two tasks: (i) to interpret (if possible) these boundary curves as the (coarse) moduli curves of suitable Hurwitz functors; and (ii) to identify the components of the boundary curves and to study their geometric properties.

Type I: the generic case (the points in H_n'):
By the modular description of H_n' mentioned in subsection 4.1, the points of H_n' classify genus 2 covers of type I; i.e. normalized genus 2 covers $f : C \to E$ of some elliptic curve E with reduced discriminant; cf. Corollary 4.3. We also know that H_n' is an open subset of the irreducible projective normal surface $\overline{Z}_{N,-1}$.

Type II: The points in $\partial_{II} := H_n \setminus H_n'$.

Since H_n is a coarse moduli scheme for the moduli problem $\mathcal{H}_n$ (cf. Corollary 4.3), we would expect that ∂_{II} is the coarse moduli space for the (closed) subproblem $\partial\mathcal{H}_n$ of $\mathcal{H}_n$ which classifies normalized genus 2 covers with non-reduced discriminants. This is almost correct. It is easy to see (using Theorem 3.1 and Proposition 3.4) that $\partial\mathcal{H}_n$ is relatively representable and closed in $\mathcal{H}_n$. Thus, we have a coarse moduli space $M(\partial\mathcal{H}_n)$ (of dimension 1) and a canonical morphism $M(\partial\mathcal{H}_n) \to M(\mathcal{H}_n) = H_n$ whose image is ∂_{II}. However, without further work we only know that this map is *radical*, but not necessarily that it is a closed immersion (or that $M(\partial\mathcal{H}_n)$ is reduced).

Recall from subsection 3.2 that the covers of Type II break up into four subcases, which we shall label here as types $\mathrm{II}_1, \ldots, \mathrm{II}_4$. (Thus, for $i = 1, \ldots, 4$, type II_i corresponds to case i of subsection 3.2.) These give rise to subproblems $\partial\mathcal{H}_n^{(i)}$ of $\mathcal{H}_n$, and the corresponding coarse moduli schemes induce curves $\partial_{II_i} = \partial H_n^{(i)}$ on H_n. One might expect these curves to be irreducible, and to have large genus for large n.

Remark 5.1 The work of [Mc1] and [HL1] (and others) shows that this expectation is correct for the curves ∂_{II_2} and ∂_{II_4}. More precisely, they show that their (birational) images in M_2 with respect to $\mu_n : H_n \to M_2$ (as defined in Subsection 4.4) can be interpreted as *Teichmüller curves*. This means in particular that for $i = 0, 1$ there exist subgroups $\Gamma_{i,n} \leq \mathrm{SL}_2(\mathbb{Z})$ of finite index and a birational map

$$\Gamma_{i,n}\backslash\mathfrak{H} \;\to\; \mu_n(\partial H_n^{(2+2i)}) \subset M_2, \quad \text{for } i = 0, 1.$$

Here i is the value of the *spin* as defined in [Mc1]. As Hubert/Lelièvre show, $\Gamma_{i,n}$ has index $d_{i,n} = \frac{3}{16n}(n-1-2i)sl(n)$ in $\mathrm{SL}_2(\mathbb{Z})$, at least if n is prime; note that the case $i = 0$ corresponds to their type A. Furthermore, they show that the genus of both curves is asymptotic to $\frac{1}{64}n^3$ as $n \to \infty$; cf. [HL1], Theorem 1.2. (Moreover, one can show, using the results of [Mc1], that their genus is ≥ 3 for $n \geq 11$, n prime.) It is interesting to note that in general $\Gamma_{i,n}$ is not a congruence subgroup; cf. [HL2].

Type III: The points in $\partial_{III} := Z_{n,-1} \setminus H_n$.

By the discussion at the end of Subsection 4.1 we know that $Z_{n,-1}$ is a coarse moduli space for the functor $\mathcal{A}_n$ and so the points in ∂_{III} correspond to singular (stable) genus 2 curves whose Jacobian is smooth, i.e. to curves C that are the union of two elliptic curves meeting in a single point. In [Ka3], Theorem 2.3, the set ∂_{III} was interpreted in terms of conditions involving the functor $\mathcal{Z}_{N,-1}$, i.e. in terms of triples $(E_1, E_2, \psi) \in \mathcal{Z}_{N,-1}(K)$. Roughly speaking, the condition is that ψ should be induced by an isogeny $h : E_1 \to E_2$ of a specific type. In the next section we will see in general that the set of ψ's induced by an isogeny lie on so-called *Hecke curves* $T_{m,k} \subset Z_{n,-1}$, and that $T_{m,k}$ is birationally equivalent to $X_0(m)$, provided that $mk^2 \equiv -1 \pmod{n}$; cf. Subsection 6.2 below. Reinterpreting [Ka3], Corollary 2.4 and Remark 2.5 in terms of Hecke curves, we obtain:

Proposition 5.2 *If $m = s(n-s)/t^2$ and $k \equiv ts^*$ (n), where $1 \le s \le n-1$, $(s, n) = 1$, and $s^*s \equiv 1$ (n) then $T_{m,k}$ is an irreducible component of ∂_{III}. Furthermore, if $n = p$ is prime, then every component of ∂_{III} has this form, and hence*

$$\partial_{III}(H_p) \;=\; \bigcup_{s=1}^{\frac{p-1}{2}} \;\bigcup_{t^2|s(p-s)} T_{\frac{s(p-s)}{t^2},ts^*}.$$

Type IV: The points in $\partial_{IV} := \bar{Z}^{(1)}_{n,-1} \setminus Z_{n,-1}$.

We first work out the geometric structure of ∂_{IV}. For this we note that since $\overline{X(1)} = X(1) \cup \{P_\infty\}$, we have $(\overline{X(1)} \times \overline{X(1)}) \setminus (X(1) \times X(1)) = P_\infty \times \overline{X(1)} \cup \overline{X(1)} \times P_\infty = pr_1^{-1}(P_\infty) \cup pr_2^{-1}(P_\infty)$, and so

$$\overline{Z}_{n,-1} \setminus Z_{n,-1} = \psi^{-1}(pr_1^{-1}(P_\infty) \cup pr_2^{-1}(P_\infty))_{red} = \psi_1^{-1}(P_\infty)_{red} \cup \psi_2^{-1}(P_\infty)_{red},$$

where ψ and ψ_i are as defined at the beginning of this subsection. By the discussion preceding Proposition 2.5 of [KS] we know that $C_{\infty,i} := \psi_i^{-1}(P_\infty)_{red}$ is an irreducible curve isomorphic to $X_1(n)$. We thus see that

$$\overline{Z}_{n,-1} \setminus Z^{(1)}_{n,-1} = \psi_1^{-1}(P_\infty)_{red} = C_{\infty,1},$$

and that hence

$$\partial_{IV} = Z^{(1)}_{n,-1} \setminus Z_{n,-1} = C_{\infty,2} \setminus C_{\infty,1}$$

is an open subset of $C_{\infty,2} \simeq X_1(n)$.

In order to see how the points of ∂_{IV} give rise to curve covers, we first note that the surface $\overline{Z}_{n,-1}$ has the following modular interpretation (as is shown in [Ka7]): it is the coarse moduli space of the functor $\overline{\mathcal{Z}}_{n,-1}$ which classifies triples (E_1, E_2, ψ) where E_1 and E_2 are generalized elliptic curves "of type n" over a scheme S (i.e. the fibres of E_i/S are either smooth elliptic curves or Néron polygons of length n) and $\psi : E_1[n] \to E_2[n]$ is an isomorphism of group schemes of determinant -1.

Via this interpretation, the points of $\partial_{IV}(\overline{K})$ can be identified with triples (E_1, E_2, ψ) where $E_1/\overline{K}$ is an elliptic curve and $E_2/\overline{K}$ is the Néron polygon of length n. By an extension of the "basic construction" (cf. section 2.3) we can construct a stable curve C_ψ on $J_\psi = (E_1 \times E_2)/\mathrm{Graph}(\psi)$ together with morphism $f : C_\psi \to E_1$ (induced by the projection onto the first factor). Note that the Classification Theorem 3.9 shows that C_ψ is an irreducible singular curve whose normalization is an elliptic curve and which has a unique singular point.

Type V: The points in $\partial_V := \overline{Z}_{n,-1} \setminus \bar{Z}^{(1)}_{n,-1}$.

By definition, $\partial_V = \psi_1^{-1}(P_\infty)_{red} = C_{\infty,1} \simeq X_1(n)$. For purposes of discussing the associated covers, it is useful to write

$$\partial_V = \partial_V^* \ \dot{\cup}\ \partial_{VI}, \quad \text{where } \ \partial_{VI} = C_{\infty,1} \cap C_{\infty,2}.$$

Via the above modular description (see type IV), the points of $\partial_V^*(\overline{K})$ (respectively, of ∂_{VI}) correspond to triples (E_1, E_2, ψ) where $E_1/\overline{K}$ is the Néron polygon of length n and $E_2/\overline{K}$ is an elliptic curve (respectively, $E_2/\overline{K}$ is the Néron polygon of length n).

If $(E_1, E_2, \psi) \in \partial_V^*(\overline{K})$, then by the same method as for type IV we obtain an irreducible singular curve C_ψ on J_ψ. Here, however, the projection onto the first factor only induces a rational map $f_0 : C_\psi \cdots \to E_1$. Nevertheless, if we compose f_0 with the map $c_n : E_1 \to \bar{E}_1$ which contracts all components of E_1 (and hence $\bar{E}_1$ is the irreducible Néron polygon of length 1), then $f = c_n \circ f_0 : C_\psi \to \bar{E}_1$ is a morphism and hence defines a cover of degree n.

If $(E_1, E_2, \psi) \in \partial_{VI}$, then the situation is more complicated (and has not been fully worked out). Here we need to look at stable curves C_ψ (of genus 2) whose Jacobian is totally degenerate; thus, C_ψ is the reduction of a suitable Mumford curve (of genus 2). In addition, we should have (as above) a morphism $f : C_\psi \to \bar{E}_1$ of degree n.

Note that ∂_{VI} is a finite set; in fact, one can show that its cardinality is $\#\partial_{VI}(\overline{K}) = \frac{\phi(n)}{2}$, where ϕ is the Euler totient function. It is interesting to observe that this count agrees with the intersection formula (19) of [KS].

5.3 The Boundary Curves of H_n^*

According to Wewers[We], p. 65, the space $H_r^N(S_n) \supset H^{in}(S_n, \mathbf{C})$ has a natural compactification $\overline{H}_r^N(S_n)$ whose boundary components classify "admissible m-covers" (as defined on p. 60 of [We]). As Wewers shows on p. 68ff, these components can be further analyzed in terms of "degeneration structures" of conjugacy classes.

Here we will not directly follow this description; instead we will only use it as a motivation for the ideas leading to the compactification of $H^{in}(S_n, \mathbf{C})$ given below. In particular, we shall use the following two key ideas for the construction of the compactification: 1) We need to allow covers of stable curves (which are degenerations of smooth curves) and 2) we need to consider other ramification data (which is obtained by degeneration of the generic data).

In our case, this means that we need to analyze the different types of admissible covers (and their ramification data) that can arise as a degeneration of covers $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of type $(*)$.

For this, let us first consider the degenerations of (normalized) covers $f : C \to E$. Then we have the following possibilities:

1) E and C are smooth, but the ramification data "degenerates";

2) E is smooth and C is a stable reducible curve (of arithmetic genus 2);

3) E is smooth and C is a stable singular irreducible curve;

4) E is a stable singular curve of arithmetic genus 1.

Note that the above cases correspond to the boundary components of the compactification of H_n' of Subsection 5.2; indeed, the cases 1) to 4) correspond to the boundary components ∂_{II} to ∂_V, respectively.

Each degenerate cover $f : C \to E$ gives rise to a cover $\varphi : \overline{C} := C/\langle \omega_C \rangle \to \mathbb{P}^1$ because f is still quasi-normalized. (More precisely, C and E come equipped with involutions ω_C and $[-1]_E$, respectively, such that $[-1]_E f = f \omega_C$.)

Now one has to analyze the ramification structure of the cover φ. We shall restrict ourselves to the cases 1) - 3). So the base E is an elliptic curve, and the branch points $P_1, ..., P_4$ remain distinct (because they are also the branch points of $\pi_E : E \to \mathbb{P}^1$). Thus, in these cases P_5 will move into one of $P_1, ..., P_4$, and because of symmetry, it is enough to look at the case that P_5 moves to P_4 or to P_3.

To see how the associated tuple σ changes if we coalesce two branch points P, P' of the cover φ, we choose a homotopy basis of $\mathbb{P}^1$ minus the branch points in the following way: Take a small disc D containing P and P', but none of the other branch points, and choose the loops around P and P' such that they agree outside of D and their product is homotopic (in $\mathbb{P}^1$ minus the branch points) to a loop that winds once around D and agrees with the other two loops outside of D. If $(\sigma_1, ..., \sigma_r)$ is the tuple associated with φ relative to this homotopy basis, and we coalesce the last two branch points, then the resulting cover is associated with the tuple $(\sigma_1, ...\sigma_{r-2},\ \sigma_{r-1}\sigma_r)$. In other words, coalescing the last two branch points means replacing the last two entries of the tuple by their product.

Since we are interested in the cases that P_5 moves either to P_3 or P_4 we put $\tau = \sigma_3\sigma_5$ and $\rho = \sigma_4\sigma_5$. Define $H := \langle\sigma_1, \sigma_2, \sigma_3\rangle = \langle\sigma_1, \sigma_2, \sigma_3, \rho\rangle$ and $F = \langle\sigma_1, \sigma_2, \sigma_4\rangle = \langle\sigma_1, \sigma_2, \tau\rangle$.

Case 1: C is smooth. The ramification types occurring in this case were already considered above and are listed in the table of Proposition 3.6.

Case 2: C is reducible. A very interesting case for arithmetical investigations is that C is reducible ([FK]). It is the union of two *isogenous* elliptic curves $\hat{E}_1$ and $\hat{E}_2$ linked at one point.

The cover $f : C \to E$ of degree n induces two finite covers $f_i = f_{|\hat{E}_i} : \hat{E}_i \to E$ of degree n_i with $n_1 + n_2 = n$. We can assume that n_1 is even and n_2 is odd.

This case is classified in the world of Galois representations on torsion points of elliptic curves in [Ka4]: the isomorphism α on points of order n is induced by restriction of this isogeny and satisfies an additional condition.

Here we give a classification in terms of degeneration of ramification types.

One possible ramification type of τ is $(4)^1(1)^1(2)^{(n-3)/2}$. In this case $F = N\langle\sigma_1\rangle$ where N is abelian of order n and σ_1 acts by inversion on N. F has exactly two orbits of length n_1 and n_2 with $n_1 + n_2 = n$.

One possible ramification type of ρ is $(1)^5(2)^{(n-5)/2}$. Then H is not a dihedral group, has two orbits of length n_1 and n_2 and necessarily $4 \mid n_2$.

Case 3: C is singular and irreducible. Here C is obtained from a genus one curve $\hat{E}$ by identifying two points. The induced cover $\tilde{f} : \hat{E} \to E$ is unramified by the Riemann-Hurwitz formula. Hence it is an isogeny of degree n for suitable choice of the zero points of E and $\hat{E}$. The two points of $\hat{E}$ that coincide in C are then n-division points. This situation is well-understood, and the corresponding boundary curve of the diagonal surface is a curve which is an open subset of $X_1(n)$, as we saw above in the discussion of type IV.

The characterization in terms of covers of $\mathbb{P}^1$ is given as follows: P_5 coalesces with P_4, and the ramification type of ρ is $(1)^1(2)^{(n-1)/2}$. The monodromy group $H = D_{2n}$ is the dihedral group naturally embedded into S_n.

By using group theory and Riemann's existence theorem, Völklein[V2] proves the following.

Proposition 5.3 (Völklein) *The above list covers all possibilities for the cycle types of ρ and τ. All these cases do actually occur, for any n, n_1, n_2 satisfying the given conditions.*

6 Rational Points Related to Isogenies

As we have seen in Section 4, rational points on the Hurwitz space H_n^* correspond to rational points on H_n'. Because of the large rigidity number it may be preferable to use the modular interpretation of such points as corresponding to pairs of elliptic curves with isomorphic level-n-structures and then to exploit arithmetical properties of $Z_{n,-1}$ rather than to use group theoretical methods. At the same time this will shed some light to the conjectures in Subsection 1.2. On the other hand, the interpretation of points by covers will give information about the questions asked in Subsection 1.1 (see Section 7).

6.1 Construction of Points by Isogenies

By the above-mentioned modular interpretation we know that if E and E' are elliptic curves over K and if $\psi : E[n] \to E'[n]$ is a G_K-isomorphism which is an anti-isometry with respect to the Weil pairings, then the triple (E, E', ψ) determines a K-rational point on $Z_{n,-1}$. By imposing further restrictions, one can ensure that this point lies on the open subset $H_n' \subset Z_{n,-1}$.

One way to construct such G_K-isomorphisms is by considering K-isogenies of elliptic curves. Thus, we *assume* in the following that there exists a *cyclic* K-isogeny

$$\eta : E \to E'.$$

Its degree will be denoted by $d = \deg(\eta)$. To avoid trivial cases *we always assume that* $d > 1$ *and that* η *is separable.*

Let α_n denote the restriction of η to $E[n]$. We then get G_K-isomorphisms $z \cdot \alpha_n$ from $E[n]$ to $E'[n]$ for all n prime to d and $z \in \mathbb{Z}$ prime to n.

Of course, there may be other G_K-isomorphisms between $E[n]$ and $E'[n]$. We call the triples $(E, E', z \cdot \alpha_n)$ "generic" because of the following evident observation.

Lemma 6.1 *Assume that the centralizer of* G_K *in* $Aut(E_n)$ *is* $\mathbb{Z} \cdot id_{E[n]}$ *and that* n *is prime to* d*. Then every* G_K*-isomorphism between* $E[n]$ *and* $E'[n]$ *is of the form* $z \cdot \alpha_n$ *with* $z \in \mathbb{Z}$ *prime to* d*. Moreover, if* E *has no complex multiplication, then there is a number* c *(depending on* E *and* K*) such that this property holds for every* n *with* $(n, c) = 1$*.*

Proposition 6.2 *For all* n *prime to* d *and all* $z \in \mathbb{Z}$ *prime to* n *the abelian variety* $J_{z,n} := (E \times E')/\mathrm{Graph}(z \cdot \alpha_n)$ *is isomorphic to* $E \times E'$*.*

Proof (cf. [DiFr]). The isogeny

$$\Phi : E \times E' \longrightarrow E \times E'$$

given by the matrix $\begin{pmatrix} n \circ id_E & 0 \\ -z \cdot \eta & id_{E'} \end{pmatrix}$ has kernel $\mathrm{Graph}(z \cdot \alpha_n)$.

In order to get points on H_n via the "basic construction" we need two additional properties:

Firstly, the graph of $\psi = z \cdot \alpha_n$ has to be isotropic with respect to the Weil pairing and secondly, one has to verify that the resulting curve C_ψ is irreducible.

The first condition means that

$$\deg(z \cdot \eta) = z^2 \cdot d \equiv -1 \bmod n.$$

The second condition was analyzed in [Ka3], Theorem 2.3. In the special case $n = p$ is prime, this criterion states that C_ψ is irreducible if and only if there is no isogeny $\theta : E \to E'$ of degree $k(p-k)$ with $1 \le k \le p-1$ such that $\theta_{|E[n]} = k\psi$ $(= kz\alpha_n)$; cf. [Ka3], Theorem 3.

We state the result in the simplest case (cf. [Fr1]).

Proposition 6.3 *Assume that d and n are distinct odd primes. In addition, assume that E has no complex multiplication.*[4] *Then there is an element $z \in \mathbb{Z}$ such that $z \cdot \alpha_n$ induces a covering $C \to E$ of degree n if and only if n is split into two non-principal prime ideals in $\mathbb{Z}[\sqrt{-d}]$.*

Remark 6.4 (a) There is a similar (but more complicated) criterion in the case of a general d; cf. [Fr1], Proposition 3.1.

(b) If d is odd and if $E' \not\simeq E$, then there always exists a cover $C \to E$ of degree 2 with $J_C \simeq E \times E'$; cf. [DiFr]. In particular, $E \times E'$ is the Jacobian of a curve of genus 2.

(c) On the other hand, if $d = 1, 2, 4, 6, 10, 12, 18, 22, 28, 30, 42, 58, 60, 70,$ 78, $102, 130, 190, 210, 330$ or 462 and E does not have CM, then there is *no* (irreducible) curve $C/\overline{K}$ such that $J_C \simeq E \times E'$; cf. [Ka8]. In fact, the above list of values of d are conjectured to be *all* the values for which no such curve exists. As is explained in [Ka8], this would follow from an old *Conjecture of Gauss* on the structure of class groups of imaginary quadratic fields (which known to be true under the Generalized Riemann Hypothesis).

The following result may be viewed as a converse to Proposition 6.2.

Proposition 6.5 *Let C/K be a curve with Jacobian $J_C \simeq E \times E'$. If* $\mathrm{Hom}_K(E, E') = \mathrm{Hom}_{\overline{K}}(E, E')$, *then there is a cyclic isogeny $\eta : E \to E'$, and integers $n \geq 2$ and z with* $\deg(\eta) z^2 \equiv -1 (\mathrm{mod}\ n)$ *such that $C \simeq C_\psi$, where $\psi = z\eta_{|E[n]}$. Thus, there is a minimal cover $f : C \to E$ of degree n. such that $f_* \circ \alpha = pr_E$, for a suitable isomorphism $\alpha : E \times E' \to J_C$.*

Proof (cf. [Ka8]). Fix an isomorphism $\alpha : A := E \times E' \xrightarrow{\sim} J_C$ and let $\lambda' = \widehat{\alpha} \lambda_C \alpha : A \xrightarrow{\sim} \hat{A}$ be the principal polarization induced by $\lambda_C : J_C \xrightarrow{\sim} \hat{J}_C$. The given hypothesis implies that the Néron-Severi group $\mathrm{NS}(A \otimes \overline{K})$ of $A \otimes \overline{K}$ is K-rational, i.e. $\mathrm{NS}(A \otimes \overline{K}) = \mathrm{NS}(A)$, and so there exists a divisor $D \in \mathrm{Div}(A)$ which defines the polarization λ', i.e. we have $\lambda' = \phi_D$. By Riemann-Roch we can choose D to be effective, and then we have that $C \otimes \overline{K} \simeq D \times \overline{K}$. By a Galois descent argument similar to that for Torelli's theorem (cf. Milne[Mi], p. 203) one shows that in fact $C \simeq D$. Thus, we have an embedding $j : C \to J_C$ such that $j(C) = \alpha(D)$.

Now every $D \in \mathrm{Div}(A)$ has the form $D \equiv a\theta_1 + b\theta_2 + \Gamma_h$, where $a, b \in \mathbb{Z}$ and $h \in \mathrm{Hom}(E, E')$; here $\theta_i = pr_i^*(0_{E_i})$ (where $E_1 = E$, $E_2 = E'$),

[4] For the case that E has CM see [Fr1].

and Γ_h denotes the graph of $h \in \mathrm{Hom}(E, E')$. Since D defines a principal polarization (so D is ample and $D^2 = 2$), it follows that there are positive integers n and r such that $b = n - 1$, $a = r - \deg(h)$ and $nr - \deg(h) = 1$. Thus, $\deg(h) \neq 0$ for else $D \equiv \theta_1 + \theta_2$, which is impossible since D is is irreducible. Thus, $h = k\eta$, for some cyclic isogeny $\eta : E \to E'$ and $k \neq 0$. We thus have

$$(20)\ D \equiv (r - k^2 d)\theta_1 + (n-1)\theta_2 + \Gamma_{k\eta}, \quad \text{where } d = \deg(\eta),\ rn - k^2 d = 1.$$

Let $\psi = -k\eta_{|E[n]} : E[n] \to E'[n]$. Since $d(-k)^2 \equiv -1 \,(\mathrm{mod}\, n)$, we see that ψ is an anti-isometry. Let $\Phi : A \to A$ be as defined in the proof of Proposition 6.2 (with $z = -k$). We thus obtain an isomorphism $\beta : J_\psi \xrightarrow{\sim} A$ such that $\beta\pi_\psi = \Phi$. Moreover, one checks that $\Phi^*(D) \equiv n^2(\theta_1 + \theta_2)$, and so $C_\psi \equiv \beta^* D \equiv \beta^*\alpha^*(j(C))$.

Now the basic construction gives a normalized cover $f_\psi : C_\psi \to E$ of degree n such that $(f_\psi)_* \circ \pi_\psi = [n]_E \circ pr_E$. Put $\gamma = (f_\psi)_* \circ \beta^{-1} \circ \alpha^{-1} : J_C \to E$, and let $f = \gamma \circ j : C \to E$. Then $f_* = \gamma$ and $\deg(f) = \deg(f_\psi) = n$ because $\beta^*\alpha^* j(C) \equiv C_\psi$.

From the definition of Φ we see that $pr_E \circ \Phi = [n] \circ pr_E = (f_\psi)_* \circ \pi_\psi = f_* \circ \alpha \circ \beta \circ \pi_\psi = f_* \circ \alpha \circ \Phi$, and hence $f_* \circ \alpha = pr_E$ because Φ is an isogeny.

Corollary 6.6 *In the above situation put $d = \deg(\eta)$ and $r = \frac{dz^2+1}{n}$. Then for any pair of integers $x, y \in \mathbb{Z}$ with $(dx, y) = 1$, there is a minimal elliptic subcover $f_{x,y} : C \to E$ such that*

$$(21) \quad (f_{x,y})_* \circ \alpha = y pr_E + x\widehat{\eta} pr_{E'} \quad \textit{and} \quad \deg(f_{x,y}) = drx^2 - 2dzxy + ny^2,$$

and hence there exist infinitely many minimal subcovers $C \to E$. Furthermore, if E does not have complex multiplication, then every minimal cover $f : C \to E$ is of this form (up to translates), and hence $(d, \deg(f)) = 1$.

Proof. Choose integers $a, c \in \mathbb{Z}$ such that $ay - cdx = 1$, so $g := \left(\begin{smallmatrix} a & x \\ cd & y \end{smallmatrix}\right) \in \Gamma_0(d)$. Put $\alpha_g = \left(\begin{smallmatrix} a & -x\widehat{\eta} \\ -c\eta & y \end{smallmatrix}\right) \in \mathrm{End}(A)$, where $A = E \times E'$. Note that $\alpha \in \mathrm{Aut}(A)$ because $\left(\begin{smallmatrix} y & x\widehat{\eta} \\ c\eta & x \end{smallmatrix}\right)$ is the inverse of α_g. Now $g^t \left(\begin{smallmatrix} dr & dk \\ dk & n \end{smallmatrix}\right) g = \left(\begin{smallmatrix} dr_g & dk_g \\ dz_g & n_g \end{smallmatrix}\right)$, for some integers $r_g, n_g, k_g \in \mathbb{Z}$ with $n_g r_g - k_g^2 d = 1$ and $n_g, r_g > 0$. Moreover, $n_g = drx^2 + 2dkxy + ny^2 = drx^2 - 2dzxy + ny^2$. Define $D_g \in \mathrm{Div}(A)$ by (20), using (r_g, n_g, k_g) in place of (r, n, k). If $D = \alpha^*(j(C))$ is as in Proposition 6.5, then we have $\alpha_g^*(D) \equiv D_g$ (cf. [Ka8]), so $\alpha\alpha_g(D_g) \equiv j(C)$. By (the proof of) Proposition 6.5 we thus have a minimal cover $f_g : C \to E$ of degree n_g such that $(f_g)_* \circ \alpha \circ \alpha_g = pr_E$, and so $(f_g)_* \circ \alpha = pr_E \circ \alpha_g^{-1} = pr_E \circ \binom{y\ x\widehat{\eta}}{c\eta\ x} = ypr_E + x\widehat{\eta}pr_{E'}$. Thus, $f_{x,y} := f_g$ satisfies (21).

Now suppose that E has no complex multiplication. Then $\mathrm{End}(E) = \mathbb{Z}1_E$ and $\mathrm{Hom}(E', E) = \mathbb{Z}\widehat{\eta}$, so $\mathrm{Hom}(A, E) = \mathbb{Z}pr_E + \mathbb{Z}\widehat{\eta}pr_{E'}$. Let $f : C \to E$ be a minimal cover. Then $h := f_* \circ \alpha \in \mathrm{Hom}(A, E)$, so $\exists x, y \in \mathbb{Z}$ such that $h = ypr_E + x\widehat{\eta}pr_{E'}$. We have $(x, y) = 1$ for otherwise h factors over the isogeny $[g]_E$, where $g = (x, y)$, and then f cannot be minimal. Moreover, we have $q := (d, y) = 1$ for otherwise we can factor η as $\eta = \eta_2\eta_1$, where $\ker(\eta_1) = \ker(\eta)[q]$, and then h factors over the isogeny $\widehat{\eta}_1$ of degree q because $h := \widehat{\eta}_1(\frac{y}{q}\eta_1 pr_E + x\widehat{\eta}_2 pr_{E'})$. Thus, $(dx, y) = 1$, and so f is the same as $f_{x,y}$ up to a translation because $f_* = (f_{x,y})_*$. Note that the formula (21) for the degree shows that $(\deg(f), d) = 1$ because $(n, d) = 1$ (as $dz^2 \equiv -1 \pmod{n}$).

6.2 The Universal Construction

Let j be transcendental over the prime field K_0 of K and let $F_d = K_0(j, j_d)$, where j_d is the j-invariant of an elliptic curve $E_{j,d} = E_j/\mathrm{Ker}(\eta_d)$ obtained by applying a cyclic isogeny η_d of degree d to E_j, the elliptic curve defined over $K_0(j)$ (let's say with Hasse invariant -1/2). Hence F_d is the function field of $X_0(d)/K_0$, the modular curve which parameterizes elliptic curves with cyclic isogenies of degree d. For any n and z such that $dz^2 \equiv -1 \pmod{n}$ we thus obtain an F_0-rational point $P_{d,z} = (E_j, E_{j,d}, (z\eta_d)_{|E[n]}) \in Z_{n,-1}(F_0)$ or, equivalently, a rational map $\tau_{d,z} : X_0(d) \to Z_{n,-1}$ (which is defined over K_0).

The above construction has the following natural modular interpretation. Let $\mathcal{X}_0(d)$ denote the moduli functor which classifies triples (E, E', η), where $\eta : E \to E'$ is a *cyclic* isogeny of degree d; thus, as was mentioned above, $\mathcal{X}_0(m)$ is coarsely represented by the modular curve $X_0(d)$. Then the rule $(E, E', \eta) \mapsto (E, E', z\eta_{|E[n]})$ defines a morphism of functors and hence of moduli schemes

$$\tau_{d,z} = \tau_{d,z}^{(n)} : \mathcal{X}_0(d) \to \mathcal{Z}_{n,-1} \quad \text{and} \quad \tau_{d,z} = \tau_{d,k}^{(n)} : X_0(d) \to Z_{n,-1},$$

Thus, the image $T_{d,z} = T_{d,z}^{(n)} := \tau_{d,k}^{(n)}(X_0(d))$ is a curve lying on $Z_{n,-1}$ with generic point $P_{d,z} \in Z_{n-1}(F_d)$. We call $T_{d,z}$ a *Hecke curve* because it is induced by a certain Hecke correspondence on $X(n)$; cf. [KS], p. 364 or [Ka7]. Note that

$$T_{d,z} = T_{d',z'} \quad \Leftrightarrow \quad d = d' \text{ and } z \equiv \pm z' \pmod{n},$$

as is easy to see. (In fact, if $T_{d,k} \neq T_{d',k'}$, then these curves can intersect only at *CM-points*, i.e. at points $(E, E', \psi) \in Z_{n,-1}(\overline{K})$, where E and E' are elliptic curves with complex multiplication (or are supersingular).) Note also that $\tau_{d,z}$ is easily seen to be a birational equivalence (so $T_{d,z}$ is birationally equivalent to $X_0(d)$), but in general $\tau_{d,z}$ cannot be an isomorphism because $T_{d,z}$ has singularities (at certain CM-points) whereas $X_0(d)$ does not.

We now want to find conditions to ensure that $P_{d,z}$ lies on H_n and on H'_n or, equivalently, that the curve $T_{d,z}$ meets H_n and H'_n. Note that there are values of d and z such that this is not the case, as we saw above in Proposition 5.2.

For this, assume that $\mathbb{Q}(\sqrt{-d})$ does not have class number 1 and that d is prime. Then by Chebotartev there exist infinitely many prime ideals which are not principal, and so by Proposition 6.3 there are infinitely many prime numbers n such that there is a curve $C_{j,n}$ of genus 2 defined over $F_d = K_0(j, j_d)$ covering E_j of degree n. Hence we constructed a non-constant point on $H_n(F_d)$.

However, it seems more difficult to decide whether or not such a point lies in H'_n. Here we prove:

Proposition 6.7 *Let d and n be odd integers such that $dz^2 \equiv -1 \bmod n$ for some $z \in \mathbb{Z}$. If $T_{d,z} \cap H_n \neq \emptyset$, then either $T_{d,z} \cap H'_n \neq \emptyset$ or $T_{d,z}$ is a component of $\partial_{II_3}(H'_n)$.*

Proof. By the basic construction, the hypothesis means that $(z \cdot \eta_d)_{|E_j[n]}$ induces a cover f_n of $E = E_j$ by an absolutely irreducible curve C of genus 2 over $F_d = K_0(j, j_d)$.

Suppose that $T_{d,z} \cap H'_n = \emptyset$. Then the ramification type of f_n is one of the cases (1)–(4) listed in Proposition 3.6. Note that if we are in case (3), then we are done, for then $T_{d,z}$ is a component of $\partial_{II_3}(H'_n)$.

In cases (1) and (2) we see that E_j has an F_d-rational point of order 2 because the discriminant divisor of f_n is rational over F_d. This, however, contradicts the fact E_j has no such point over F_d (because otherwise $F_{2d} \subset F_d$, which is impossible).

Finally, suppose that we are in case (4). Since the different divisor $\mathrm{Diff}(f_n) = 2W'_k$ is rational, it follows that the Weierstraß point W'_k is rational. Let $f'_n : C \to E' = E_{j,d}$ be the complementary cover. Since $f_n^{-1}(0_E) \cap (f'_n)^{-1}(0_E)$ is a subgroup of odd order, we see that $f'_n(W'_k) = P'$, for some $P' \in E'[2] \setminus \{0_{E'}\}$. Thus, E' has an F_d-rational point of order 2, and hence so does E_j (because $d = \deg(\eta)$ is odd). Thus, we obtain the same contradiction.

Remark 6.8 In Corollary 7.8 below we shall show that for a given d with $(d, 30) = 1$, there are only finitely many n's such that $T^{(n)}_{d,z_n}$ is a component of $\partial_{II_3}(H'_n)$. This is in marked contrast to the components of $\partial_{III}(H_n)$: if there is one (prime) n such that $T^{(n)}_{d,z}$ is a component of $\partial_{III}(H_n)$, then there are infinitely many primes n_i such that $T^{(n_i)}_{d,z_i}$ is a component of $\partial_{III}(H_{n_i})$. Indeed, if d is prime, then this follows from Proposition 6.3 and Chebotarev, and the general case is similar.

Example 6.9 Take $d = 13$. Then $X_0(d)$ has genus 0 and $\mathbb{Q}(\sqrt{-d})$ has class number 2, so for infinitely many n's we get a rational curve in H_n which is defined over K_0. Moreover, by Corollary 7.8 below we see that infinitely many of these also lie in H'_n (and hence also in H^*_n).

6.3 The Answer to Q4 is Yes

Recall Question **Q4** which asks in our context: For given $n_0 \in \mathbf{N}$ and finitely generated field K, are there curves C of genus 2 defined over K such that there are projective absolutely irreducible Galois covers of C of degree $\geq n_0$?

By results of [Fr1] and [Ki] we know already that the answer is positive for finite fields K. If K is infinite we can do even better.

By Example 6.9 we find a prime $n \geq n_0$ and a K-cover $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n which is ramified in points $P_1, \ldots, P_5$ (say) of order ≤ 2 with monodromy group S_n.

Now choose $P_6 \in \mathbb{P}^1(K)$ different from $P_1, \ldots, P_5$ and let $\pi : C_\varphi \to \mathbb{P}^1$ be a genus 2 cover of degree 2 which is ramified at $P_1, \ldots, P_5, P_6$.

Then the normalization of the fibre product of $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ and $\pi : C_\varphi \to \mathbb{P}^1$ over $\mathbb{P}^1$ is unramified over C_φ (Abhyankar's Lemma) and its Galois closure $\tilde{C}$ has Galois group S_n. Since P_6 is unramified in φ and ramified in P_6 it follows that $\tilde{C}$ is regular.

So we get

Proposition 6.10 *For all K and all n_0 there is a curve C of genus 2 defined over K which has a regular unramified Galois cover of degree $> n_0$.*

Moreover, if K is not a finite field, there are infinitely many such curves C which are not a twist of a curve defined over a finite field.

6.4 Finiteness Conjectures

We now discuss the Diophantine background of the conjectures presented in Subsection 1.2.

In view of (the reverse of) the basic construction, Conjecture 3 predicts that for given elliptic curve E and for large enough natural numbers N, the twisted modular curves $H_{E/K,N}$ have only "obvious" K-rational points coming from isogenies. This is a conjecture of the type of Fermat's Last Theorem: We have an infinite family of curves of growing genus depending on a fixed "parameter" (here the fixed ramification points $P_1, \ldots, P_4$) and we look at the union of all solutions over K of these curves. So it is not surprising that a proof of Conjecture 3 requires not only Faltings' theorem about the finiteness of rational solutions on *each* of the curves but also additional arithmetical information about these solutions. As was mentioned in Subsection 1.2, the height conjecture for elliptic curves (see [Fr1] delivers such an information.

If one looks at small N (less or equal to 5) the related modular curves are of genus 0 and so one can explain why there are so many examples for non-isogenous elliptic curves with the isomorphic torsion structures. But by experiments one finds easily many examples for $N = 7$ (and not so easily) examples for $N = 11$ for $K = \mathbb{Q}$. The examples become very rare for $N = 13$ and just recently Cremona has found examples for $N = 17$. As far as we know this is the record for the ground field $\mathbb{Q}$. These phenomena cannot be explained by properties of modular curves, and in fact, a question of Mazur concerning these examples was one of the main motivations for the investigations described in this paper (and in [Ka7]).

In Mazur's question, as in Conjectures 1 and 2, the elliptic curve E/K is no longer fixed, and so this leads to the study of *pairs* (E, E') of elliptic curves with G_K-isomorphic N-torsion structures. Any such isomorphism $\psi : E[N] \to E'[N]$ has a "determinant" $\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times$, and the triple (E, E', ψ) determines a unique K-rational point on the modular diagonal quotient surface $Z_{N,\varepsilon}$ (which is a natural generalization of the surface $Z_{N,-1}$ which was considered above).

By the work of Hermann, Kani and Schanz [KS], one knows that for $N \leq 10$, the (desingularization of the) compactification $\overline{Z}_{N,\varepsilon}$ of $Z_{N,\varepsilon}$ is either a rational surfaces, a K3-surface or an elliptic surface. Thus, we can find infinitely many curves of genus ≤ 1 lying on $Z_{N,\varepsilon}$, and each of these will have infinitely many points which are rational over suitable a finite extension of K. But for $N \geq 13$ (or for $N = 11, 12$ and for some ε's such as $\varepsilon = -1$), it turns out that these surfaces are of *general type*, so one expects that there are only finitely many such curves on these surfaces.

Indeed, *Lang's conjecture* predicts that for surfaces S/K of general type almost all of the K-rational points lie on the curves of genus ≤ 1 on S, and that there are only finitely many such curves (even over $\overline{K}$). This conjecture plays the role of Faltings' theorem (Mordell conjecture) for curves of genus ≥ 2.

Even if we believe this conjecture, the Conjectures 1 and 2 would not be immediate consequences. As in the case of families of curves, we would need to have additional information to find all curves of low genus on the surfaces $Z_{n,\varepsilon}$.

In Subsection 6.2 we saw that the Hecke curves $T_{m,k}$ (for m small) give rise to such curves on $Z_{N,-1}$, and the same construction yields curves on the surfaces $Z_{N,\varepsilon}$ whenever $mk^2 \equiv \varepsilon \pmod N$. One might hope that every curve is of this form:

Conjecture 4 (Kani) *For prime numbers $n \geq 23$, all curves on $\mathbb{Z}_{n,\varepsilon}$ of genus ≤ 1 are Hecke curves.*

One might expect that Lang's Conjecture and Conjecture 4 imply Conjecture 2. This is indeed the case, but is not completely obvious, due to the fact that different (non-isomorphic) triples (E_i, E_i', ψ_i) give rise to the same K-rational point on $Z_{N,\varepsilon}$. However, if $j(E_i) \neq 0, 1728$, then (as shown in [Ka7]) this can happen if and only if the points are *simultaneous twists* of each other, i.e. if there is a (quadratic) character χ on G_K such that $(E_2, E_2', \psi) \simeq ((E_1)_\chi, (E_1')_\chi, \psi_\chi)$ where $(E_1)_\chi$ and $(E_1')_\chi$ denote the χ-twists of E_1 and E_1', respectively and $\psi_\chi(E_1)_\chi[N] \to (E_1')_\chi[N]$ is the isomorphism induced by ψ. (In particular, we see that if K isn't finite, then there are infinitely many non-isomorphic triples which give rise to the same point on $Z_{N,\varepsilon}$.) We thus obtain:

Proposition 6.11 *Lang's Conjecture and Conjecture 4 imply Conjecture 2.*

7 Towers

We now want to discuss the questions concerning the existence of towers of unramified regular Galois covers of curves of genus 2. Like in the proof of Proposition 6.10 we want to use Abhyankar's Lemma to construct composites of covers of a given curve C ramified at the same places to get unramified covers. To get an infinite tower by this strategy, we have to find infinitely many covers $f_n : C \to E$ with the same discriminant divisor. Hence, we have to analyze ramification points (i.e. the discriminant) of covers resulting from the universal construction in Subsection 6.2.

7.1 Ramification points

We continue to assume that E and E' are isogenous curves with a K-rational isogeny η of minimal degree d. From now on we shall assume in addition that this degree is odd and that the j-invariant of E is not contained in an algebraic extension of the prime field K_0 of K. Thus, E does not have complex multiplication and we have $\mathrm{Hom}_K(E, E') = \mathrm{Hom}_{\overline{K}}(E, E') = \mathbb{Z}\eta$.

Let C be a curve over defined over K with $J_C \simeq E \times E'$. Then by Proposition 6.5 we know that C has a subcover $f : C \to E$ which is induced by an isogeny; in fact, C has infinitely many such subcovers.

Let $I = I(C, E, E')$ denote the set of integers $n > 1$ for which there exists a minimal subcover

$$f_n : C \to E$$

of degree n which is induced by an isogeny. (Fix one for each $n \in I$.) Let $\Delta_n = P_1^n + P_2^n$ denote the discriminant divisor (branch locus) of f_n and let $\mathrm{Diff}(f_n) = Q_1^n + Q_2^n$ its different divisor (ramification locus). Then by Proposition 6.7 we know that either $P_1^n \neq P_2^n$ or that $P_1^n = P_2^n = 0_E$ and $Q_1^n \neq Q_2^n$ (Case (3)). We denote the set of n's for which the first case holds by I_1 and the rest by I_0.

We remark that if $P_1^n \neq 0_E$ is K-rational then Q_1^n is K-rational.

Lemma 7.1 *If the set of points $\{P_j^n \in E(\overline{K}) : n \in I_1\}$ is finite, then the set $\{Q_j^n \in C(\overline{K}) : n \in I_1\}$ is finite.*

Proof. Assume that $\{P_j^n : n \in I_1\}$ is finite. Thus all points in this set are K'-rational, for a finite extension K'/K, and hence $\{Q_j^n : n \in I_1\}$ is finite. Here we use the result of Faltings/Grauert/Samuel that the set of K'-rational points on C is finite. Note that C is not isotrivial over K_0 because $J_C \simeq E \times E'$ isn't isotrivial (as $j(E) \notin \overline{K}_0$).

In the sequel we shall use a criterion of [DiFr] about equality of ramification divisors. This is based on the following criterion for ramification points.

Lemma 7.2 *Let $\iota : C \to J_C$ be an embedding, and let*

$$c : C \to E$$

be a minimal cover which maps Q to 0_E. Then c is unramified in Q if and only if $\ker(c_*)$ *intersects transversally with $\iota(C)$ in $\iota(Q)$.*

Proof. The subscheme $\iota(c^{-1}(O_E))$ of $\iota(C)$ has degree $\deg c$, contains $\iota(Q)$ and is equal to $\ker(c_*) \cap \iota(C)$. Since Q isn't a ramification point if and only if Q occurs with multiplicity 1 in this scheme, the lemma follows.

Since translations on E are étale we can apply the above criterion to arbitrary minimal covers. In [DiFr] the following result is deduced from this:

Proposition 7.3 ([DiFr]) *Assume that*

$$c_i : C \to E_i, \quad \text{for } i = 1, 2$$

are two minimal covers such that $\ker(c_{1*}) \cap \ker(c_{2*})$ *a finite group scheme. Then the ramification locus of c_1 is different (and hence disjoint) from the ramification locus of c_2 if and only if* $\ker(c_{1*}) \cap \ker(c_{2*})$ *is a reduced group scheme.*

Corollary 7.4 *Let K be a field of characteristic* 0*, and let c_1 and c_2 be minimal covers from C to E with $c_1 \neq \pm c_2$. Then the ramification loci of c_1 and c_2 are disjoint.*

Thus, if $\{f_n;\, n \in I_1\}$ is an infinite set of minimal covers from C to E, then the set of ramification loci on C is infinite.

The case of positive characteristic behaves totally different.

Proposition 7.5 *Assume that char(K)= $p > 0$. Then the ramification points of all minimal covers from C to E lie in a finite set of order $\leq p^{\dim_{\mathbb{Z}} Hom_K(J_C,E)}$.*

Proof. Let c_1 and c_2 be two minimal covers with $c_{1*} \equiv c_{2*}$ modulo $p \cdot \mathrm{Hom}_K(J_C, E)$. We claim that then $\mathrm{Diff}(c_1) = \mathrm{Diff}(c_2)$. Indeed, if $\ker(c_{1*}) = \ker(c_{2*})$, then this is clear since then $c_2 = g \circ c_1$ for some automorphism g, so assume that $\ker(c_{1*}) \neq \ker(c_{2*})$. Now the group schemes $\ker(c_{i*})[p]$ (of rank p^2) are equal as closed subschemes of J_C and hence are contained in the finite group scheme $\ker(c_{1*}) \cap \ker(c_{2*})$. Thus, the latter is not étale hence $\mathrm{Diff}(c_1) = \mathrm{Diff}(c_2)$ by Proposition 7.3.

Note that the above two propositions apply to arbitrary genus 2 curve covers $C \to E$. We now return to the situation of the curve C as at the beginning of this subsection and prove:

Proposition 7.6 *If $(d, 15) = 1$, then the set I_0 is finite.*

Proof. Suppose first that $\mathrm{char}(K) = p > 0$. If I_0 were infinite, then by Proposition 7.5 there exists an infinite subset $I_0' \subset I_0$ such that $D := \mathrm{Diff}(f_n) = \mathrm{Diff}(f_m)$ for all $n, m \in I_0'$. Note that since C, η and the f_n's are all defined over $F_d = \mathbb{F}_p(j_E, j_{E,d})$, we can and will assume henceforth that $K = F_d$.

Write $D = Q_1 + Q_2$, and put $D' = Q_1 - Q_2$. Note that D' is not necessarily rational over $K = F_d$, but that $\sigma(D') = \pm D'$, for all $\sigma \in G_K$ (because D is K-rational). Then $cl(D') \in \ker(f_{n*}) \cap \ker(f_{m*})$ for all $n, m \in I_0'$ and hence $cl(D')$ has finite order. Thus, $H := \langle cl(D') \rangle$ is a finite, K-rational subgroup on $\ker(f_{n*}) \simeq E'$, and so $H \leq \ker(\eta^t)$ because all finite etale K-rational subgroup schemes of E' have this property. Let $q|r := \#H$ be a prime. Since $q|d$, we have that $H[q] = \ker(\eta^t)[q]$, and so $\frac{\phi(q)}{2} | [K(H[q]) : K]$. But by construction, $[K(H[q]) : K] | 2$, so $\phi(q)|4$, i.e. $q \leq 5$. But since $(d, 30) = 1$, it follows that $H = 1$. This is impossible because $Q_1 \neq Q_2$, and so we conclude that I_0 must be finite.

Now suppose that $\mathrm{char}(K) = 0$, so $K_0 = \mathbb{Q}$. As above we can assume that $K = F_d = \mathbb{Q}(j_E, j_{E,d})$. Choose a prime $p \nmid 2d$. Since $X_0(d)$ has good reduction at p (Igusa), we can reduce K to $K_p = \mathbb{F}_p(j_p, j_{p,d})$ (where j_p is transcendental over $\mathbb{F}_p$). Then E, E' reduce to elliptic curves E_p, E_p'/K_p, and η reduces to a cyclic isogeny between E_p and E_p'. Thus, the G_K-isomorphisms $\psi_n : E[n] \to E'[n]$ reduce to G_{K_p}-isomorphisms $\psi_{n,p} : E_p[n] \to E_p'[n]$, and hence each cover $f_n : C \to E$ reduces to a cover $f_{n,p} : C_p \to E_p$ of the same degree. Note that C_p is again an irreducible curve because if not, there exists an isogeny diamond configuration (f, H_1, H_2) for $\psi_{n,p}$ (in the sense of [Ka3]), and this could be lifted to one of ψ_n because $\mathrm{Hom}(E, E') \to \mathrm{Hom}(E_p, E_p')$ is an isomorphism.

Thus, $I(C, E, E') \subset I(C_p, E_p, E_p')$, and hence we also have $I_0(C, E, E') \subset I_0(C_p, E_p, E_p')$ since the discriminants specialize properly. Now by the first part of the proof we know that $I_0(C_p, E_p, E_p')$ and hence so is $I_0(C, E, E')$.

Combining the last proposition with Corollary 7.4 and Lemma 7.1 yields

Corollary 7.7 *If char$(K) = 0$, then the set $\{Q_j^n \in C(\overline{K}) : n \in I\}$ and the set $\{P_j^n \in E(\overline{K}) : n \in I\}$ are infinite, provided that $(d, 30) = 1$.*

We can now finally prove the following fact which was mentioned in Remark 6.8.

Corollary 7.8 *If $(d, 30) = 1$, then there are only finitely many n's such that $T_{d,z_n}^{(n)}$ is a component of $\partial_{II}(H'_n)$, for a suitable $z_n \in \mathbb{Z}$.*

Proof. Let $\tilde{I}$ denote the set of integers $n \geq 2$ such that $T_{d,z_n}^{(n)}$ is a component of $\partial_{II}(H'_n)$. Then for each n there exists an irreducible curve C_n and a minimal cover $f_n : C_n \to E$ defined over $K = F_d = K_0(X_0(d))$ with $J_{C_n} \simeq E \times E'$, where E, E' are as at the beginning of Subsection 6.2. (In particular, E and E' satisfy the conditions of this subsection.)

By the proof of Proposition 6.5 we know that every C_n is isomorphic to an effective divisor D_n on $A = E \times E'$ which defines a principal polarization of A. Since there are only finitely many such (up to automorphisms), we conclude that there are finitely many curves $C_{n_1}, \ldots, C_{n_t}$ such that every curve C_n is K-isomorphic to one of these.

By Proposition 6.7 we know that $\tilde{I} \subset \cup_i I_0(C_{n_i}, E, E')$. Since the latter is a finite set by Proposition 7.6, we see that $\tilde{I}$ is finite as well.

The above discussion shows that our strategy for building infinite unramified towers breaks down over fields of characteristic 0 because by Corollary 7.7 the ramification and branch loci $f_n : C \to E$ are not contained in finite sets. Nevertheless, it turns out that the different divisors of these covers cannot be too "big" in the sense that they lie in a set of *bounded height*. Note that this gives another proof of the finiteness result of Proposition 7.5.

Proposition 7.9 *The divisors $\{\mathrm{Diff}(f_n) : n \in I\}$ have bounded height over $K = F_d$.*

Proof. Let F'_d be the function field of $X := X(d)/K_0$, and let $\mathcal{C}/X$ be the minimal model of $C \otimes F'_d/F'_d$. We will prove that there is an effective divisor B_0 on $\mathcal{C}$ such that for all $n \in I$ we have

$$(22) \qquad (\omega^0_{\mathcal{C}/X}.\overline{\mathrm{Diff}(f_n)}) \;\leq\; (\omega^0_{\mathcal{C}/X})^2 - \frac{1}{12}sl(d) - B_0^2 + 4(B_0.\mathcal{L}),$$

where $\overline{D}$ denote the closure of the effective divisor $D = \mathrm{Diff}(f_n)$ in $\mathcal{C}$, and $\mathcal{L}$ is any ample invertible sheaf on $\mathcal{C}/X$.

To see that this implies the statement of the proposition, recall (cf. Lang[La], p. 58) that if $K = k(T)$ is a function field of a normal projective curve T/k, then the *height* of a point $P \in \mathbb{P}^n_K(K)$ is given by $h_K(P) = \deg_T(s_P^*(L))$, where $s_P : T \to \mathbb{P}^n_T$ is the section defined by P and $L = \mathcal{O}(1)$ is the usual very ample line bundle on $\mathbb{P}^n_T$. We can extend h_K to a function on $\mathbb{P}^n_K(\overline{K})$ by setting $h_K(P') := \frac{1}{[K':K]} h_{K'}$, for $P' \in \mathbb{P}^n_{K'}$, where K'/K is a finite extension. Thus, if V/K is any projective variety and $\varphi : V \hookrightarrow \mathbb{P}^n_K$ is a projective embedding, then $h_{K,\varphi} := h_K \circ \varphi$ is the height function on $V(\overline{K})$ determined by φ. A basic result is that, up to a bounded function, $h_{K,\varphi}$ depends only on $D = \varphi^*(\mathcal{O}_{P^n_K}(1))$; cf. [La], p. 85. Note that if $\mathcal{V}/T$ is a projective model of V/K, and if φ extends to morphism $\tilde{\varphi} : \mathcal{V} \to \mathbb{P}^n_T$, then with $D = \tilde{\varphi}^*\mathcal{O}(1)$ we have

$$h_K, \varphi(P) = \tfrac{1}{\deg(\varphi(P))} \deg_{T_P}(s_P^* D) = \tfrac{1}{\deg(\varphi(P))} (\overline{P}.D)_{\mathcal{V}}, \quad \text{for all } P \in V(\overline{K});$$

here T_P is the normalization of T in the extenion $\kappa(P)$ of K, $s_P : T_P \to \mathcal{V}$ the induced T-morphism, and $\overline{P} = s_P(T_P)$ is the closure of P in $\mathcal{V}$ (if we view P as a closed point on V).

If $V = C$ is a curve, then the usual properties of height functions ([La], p. 94) show that if $S \subset C(\overline{K})$ has bounded height with respect to one height function h_{K,φ_0}, then it has bounded height with respect to every height function $h_{K,\varphi}$.

We now specialize the discussion to C/K as in the proposition (except that F_d has been replaced by F'_d). Since $C \sim E \times E$ and $E/k(X)$ has semi-stable reduction, the same is true for C; i.e. $\mathcal{C}/X$ is a semi-stable (cf. proof of Step 2 of Theorem 3.9). Now although $\omega^0_{\mathcal{C}/X}$ isn't necessarily relatively ample over T, we do have a sufficient large multiple $(\omega^0_{\mathcal{C}|X})^{\otimes m}$ defines a T-morphism $\varphi_m : \mathcal{C} \to \mathbb{P}^N_T$ for some $N = N_m$. (Indeed, if $\mathcal{C}_s/X$ denotes the associated stable model with contraction morphism $c : \mathcal{C} \to C_s$, then $\omega^0_{\mathcal{C}_s/X}$ is relatively ample (cf. [DM]) and so the assertion follows since $\omega^0_{\mathcal{C}/X} \simeq c^*\omega^0_{C_s/X}$.) Thus, by the above formula we see that (a multiple of) the left hand side of (22) can be interpreted as a height function, provided we restrict to those f_n's for which Diff(f_n) is irreducible. (The reducible ones can be dealt with in a similar manner, but they are anyways finite in number by Faltings/Grauert.)

Thus, the assertion of the proposition follows once we have proved (22). For this, let $q : \mathcal{E} \to X$ denote the minimal model of E/K. Since $\mathcal{E}/X$ is semi-stable, we have (cf. [DR], p. 175)

$$\omega^0_{\mathcal{E}/X} \simeq q^*\mathfrak{A}, \quad \text{where } \mathfrak{A} = q_*\omega^0_{\mathcal{E}/X} \in \mathrm{Pic}(X). \tag{23}$$

Note that by (6) we know that $h := \deg(\mathfrak{A}) = h_{E/X} = \frac{1}{24}sl(d)$.

Now consider $f_n : C \to E$. By Proposition 8.1 below there exists an effective divisor $B_n \geq 0$ consisting entirely of components of the (reducible) fibres of the structure map $p : \mathcal{C} \to X$ such that

$$\omega^0_{\mathcal{C}/X} \sim p^*\mathfrak{A} + \overline{D}_n + B_n, \tag{24}$$

where $D_n = \mathrm{Diff}(f_n)$. Here we have used the fact that $(f_n^*(\omega^0_{E/X}))_K \simeq p^*\mathfrak{A}$ which follows from (23) because $p_K = q_K \circ f_n$.

Fix one $n_0 \in I$ and write $D_0 = D_{n_0}$ and $B_0 = B_{n_0}$. Then from (24) (for n and n_0) we obtain

$$B_n - B_0 \sim \overline{D}_0 - D_n. \tag{25}$$

Now for any component Γ of a fibre we have $0 \leq (\Gamma.\overline{D}_n) \leq (p^*(p(\Gamma)).\overline{D}_n) = 2$, and so $|(\Gamma.(D_0 - D_n))| \leq 4$. Thus, since $D_0 =: \sum b_\Gamma \Gamma \geq 0$, we obtain from (25) that $|(B_0.(B_n - B_0))| = \sum b_\Gamma |(\Gamma.(B_n - B_0))| \leq 4 \sum b_\Gamma \leq 4(B_0.\mathcal{L})$, the latter since $(\Gamma.\mathcal{L}) \geq 1$ for all Γ because $\mathcal{L}$ is ample. Thus

$$-4(B_0.\mathcal{L}) + B_0^2 \leq (B_0.B_n) \leq B_0^2 + 4(B_0.\mathcal{L}). \tag{26}$$

Next, since $\overline{D}_0$ and B_n are effective and without common components we see that $\overline{D}_0.B_n \geq 0$. Thus, by (24) we obtain

$$(\omega.B_n) = (p^*\mathfrak{A} + \overline{D}_0 + B_0).B_n = \overline{D}_0.B_n + B_0.B_n \geq B_0.B_n,$$

where $\omega := \omega^0_{\mathcal{C}/X}$. From this, together with (24) and (26) we obtain

$$(\omega.\overline{D}_n) = \omega.(\omega - p^*\mathfrak{A} - B_n) \leq \omega^2 - 2h - B_0.B_n \leq \omega^2 - 2h - B_0^2 + 4(B_0.\mathcal{L});$$

here we have also used the fact $(\overline{D}_n.p^*\mathfrak{A}) = \deg(D_n)\deg(\mathfrak{A}) = 2h$. Since $h = \frac{1}{24}sl(d)$, this proves (22).

7.2 Towers of unramified extensions

In this subsection we discuss the question whether we can use different covers of C to E to compose them to towers of regular unramified Galois covers. First of all we will restrict ourselves to fields K of positive characteristic p, and we can assume that K is of transcendence degree 1 over K_0. Then we know by Proposition 7.5 that there are infinitely many covers

$$f_i : C \to E$$

with the same *different divisor*. In [DiFr] one finds examples of curves C which admit infinitely many subcovers $f_i : C \to E$ with the same *discriminant divisor*. We now want analyze the condition that (certain) genus 2 curve C admit subcovers with this property.

Recall the situation: We assume that we have an isogeny $\eta : E \to E'$ of degree d and that J_C is isomorphic to $E \times E'$. Moreover, we shall impose here the restriction that $d > 5$ is prime. Then we have that every elliptic curve E'' which is separably isogenous to E over $K = F_d = \mathbb{F}_p(j_E, j_{E'})$ is isomorphic to E or to E', and hence does not have any non-trivial torsion points in any quadratic extension of K. In particular, the ramification points of the f_i's on C are mapped to non-torsion points on E (or to 0_E).

In the following we denote by I the set of integers n such that there is a minimal cover $f_n : C \to E$ of degree n with $\mathrm{Disc}(f_n)$ reduced. For each n, fix such a cover f_n. Note that it follows from Corollary 6.6 and Proposition 7.6 that the set I is infinite. We also assume that the covers are compatible in the sense that there is an embedding $\iota : C \to J_C$ such that $(f_n)_* \circ \iota = f_n$, for all n.

As was mentioned above, it follows from Proposition 7.5 that there is an infinite subset I' of I such that the covers f_n with $n \in I'$ all have the same ramification points on C. We now want to find conditions which ensure that they also have the same branch point on E.

Lemma 7.10 *Assume that Q is a ramification point on C for f_{n_1} and f_{n_2} with $n_1 \neq n_2$ and that $f_{n_1}(Q) = f_{n_2}(Q)$. Then there is a unique elliptic subgroup E_0 on J_C which contains $\iota(Q)$.*

Proof. Let $h = f_{n_1,*} - f_{n_2,*} : J_C \to E$. Since $n_1 \neq n_2$, its kernel is a group scheme of dimension 1 and so the connected component of $\ker(h)$ is an elliptic curve E_0 defined over K. If $h_0 : J_C \to \overline{E}_0 := J_C/E_0$ denotes the quotient map, then $h = h_1 h_0$, for some isogeny $h_1 : \overline{E}_0 \to E$.

By hypothesis we have $\iota(Q) \in \ker(h)$, so $h_0(\iota(Q)) \in \ker(h_1)$ has finite order. Since $\overline{E}_0$ is an elliptic curve isogenous to E, it has no non-torsion points in quadratic extensions (by our hypothesis on K), and thus we have $h_0(\iota(Q)) = 0$. This means that $\iota(Q) \in E_0$, so $\iota(Q)$ lies on the elliptic subgroup E_0. Note that E_0 is the only elliptic subgroup containing $\iota(Q)$, for if $E_0' \neq E_0$ were another, then $\iota(Q)$ is in the finite subgroup scheme $E_0 \cap E_0'$, which contradicts the fact that $\iota(Q)$ has infinite order.

Lemma 7.11 *If $E_1 \leq A := E \times E'$ is an elliptic subgroup, then there is another elliptic subgroup E_2 on A such that $(E_1.E_2) = 1$. In particular, if $i_{E_k} : E_k \to A$ denote the inclusion maps, them the map $\alpha_{E_1,E_2} = i_{E_1} + i_{E_2} : E_1 \times E_2 \xrightarrow{\sim} A$ is an isomorphism.*

Proof. Write $\theta_1 = pr_E^*(0_E)$, $\theta_2 = pr_{E'}^*(0_{E'})$ and $\Gamma^* = -d\theta_1 - \theta_2 + \Gamma_{-\eta}$. Then $\theta_1, \theta_2, \Gamma^*$ is a basis of NS(A) (cf. [Ka8]), so $E_1 \equiv m_1\theta_1 + m_2\theta_2 + m_3\Gamma^*$, for some $m_i \in \mathbb{Z}$. Now since $(m_1\theta_1 + m_2\theta_2 + m_3\Gamma^*)^2 = 2(m_1m_2 - m_3^2 d)$, it follows that $m_1m_2 = m_3^2 d$ as $E_1^2 = 0$. Moreover, since E_1 is primitive (cf. [Ka2], Theorem 2.8) we have that $(m_1, m_2, m_3) = 1$. Since d is square free, it follows that $(m_1, m_2) = 1$. Thus $d = d_1d_2$ with $(d_1, d_2) = 1$ and $d_i|m_i$, and similarly $m_3^2 = (n_1n_2)^2$ with $(n_1, n_2) = 1$ and $n_i|m_i$. Thus $m_i = n_i^2 d_i$, for $i = 1, 2$, and $(n_1d_1, n_2d_d) = 1$, and hence there exist $k_1, k_2 \in \mathbb{Z}$ such that $k_1m_1d_1 - k_2m_2d_2 = 1$. Put $D = k_2^2 d_2\theta_1 + k_1^2 d_1\theta_2 + k_1k_2\Gamma^*$. Then $D^2 = 0$, and so there is an elliptic subgroup E_2 with $D \equiv mE_2$ for some $m \in \mathbb{Z}$ (cf. [Ka2], Prop. 2.3). But D is primitive since $(k_2^2d_2, k_1^2d_1) = 1$, so $m = \pm 1$. Moreover, since $D.\theta_1 = k_1^2 d_1 > 0$, we must have $m = 1$, i.e. $E_2 \equiv D$. Now $(E_1.E_2) = \frac{1}{2}(E_1 + E_2)^2 = (n_1^2d_1 + k_2^2d_2)(n_2^2d_2 + k_1^2d_1) - (n_1n_2 + k_1k_2)^2 d = (n_1k_1d_1 - n_2k_2d_2)^2 = 1^2 = 1$, and so E_2 has the desired properties. The second assertion is an immediate consequence of the first.

Corollary 7.12 *If $E_0 \leq E \times E'$ is an elliptic subgroup, then either $pr_E \circ \alpha \circ i_{E_0} = 0$, or $pr_{E'} \circ \alpha \circ i_{E_0} = 0$, for some automorphism $\alpha \in \mathrm{Aut}(E \times E')$.*

Proof. By Lemma 7.11 there is an elliptic subgroup $E_0' \leq A$ and an isomorphism $\beta : A_0 := E_0 \times E_0' \xrightarrow{\sim} A$ such that $\beta \circ i_{E_0\times 0}^{A_0} = i_{E_0}^A$.

Suppose first that $E_0 \simeq E$. Then $E_0' \simeq A/E_0 \simeq A/E \simeq E'$, so we have isomorphisms $\alpha_0 : E_0 \xrightarrow{\sim} E$ and $\alpha_1 : E_0' \xrightarrow{\sim} E'$. Thus, $\alpha = \beta \circ (\alpha_0^{-1} \times \alpha_1^{-1}) \in \mathrm{Aut}(A)$ satisfies $\alpha \circ i_{E_0} = i_{E\times 0} \circ \alpha_0$. Thus, $pr_{E'} \circ \alpha \circ i_{E_0} = pr_{E'} \circ i_{E\times 0} \circ \alpha_0 = 0$.

Next, suppose $E_0 \not\simeq E$. Then (since d is prime) $E_0 \simeq E'$, and one has by a similar argument as above that $E_0' \simeq E$, so we have isomorphisms $\alpha_0 : E_0 \xrightarrow{\sim} E'$ and $\alpha_1 : E_0' \xrightarrow{\sim} E$. If $\tau : E \times E' \xrightarrow{\sim} E' \times E$ denotes the natural map, then $\alpha = \beta \circ (\alpha_0^{-1} \times \alpha_1^{-1}) \circ \tau \in \mathrm{Aut}(A)$ satisfies $\alpha \circ i_{E_0} = i_{0\times E'} \circ \alpha_0$, and so $pr_E \circ \alpha \circ i_{E_0} = pr_E \circ i_{0\times E'} \circ \alpha_0 = 0$.

Proposition 7.13 *Let $d > 5$ be a prime and let E/K be an elliptic curve defined over $K = F_d$ with j-invariant $j = j_E$ transcendental over $\mathbb{F}_p \subset F_d$. Let $\eta : E \to E'$ be an isogeny of degree d to some curve E'/K, and let C/K*

be a genus 2 curve such that $J_C \simeq E \times E'$. Then the following conditions are equivalent.

(i) There exist infinitely many minimal covers $f_n : C \to E$ with the same branch locus.

(ii) There exist two minimal covers $f_{n_k} : C \to E$ of different degrees with the same ramification locus and the same branch locus.

(iii) There is a minimal subcover $f : C \to E$ such that its ramification locus Q lies on some elliptic subgroup $E_0 \leq J_C$.

Proof. (i) $\Rightarrow$ (ii): Clear by by Proposition 7.5.

(ii) $\Rightarrow$ (iii): Lemma 7.10.

(iii) $\Rightarrow$ (i): Fix an isomorphism $\beta : J_C \xrightarrow{\sim} E \times E'$. Applying Corollary 6.6 to $\beta(E_0) \leq E \times E'$, we have two possibilities:

Case 1: $pr_E \circ \alpha \circ i_{\beta(E_0)} = 0$, for some $\alpha \in \mathrm{Aut}(E \times E')$.

Then as in the proof of Corollary 6.6 there are integers $x_0, y_0 \in \mathbb{Z}$ with $(dx_0, y_0) = 1$ such that $f_* \circ \beta^{-1} \circ \alpha^{-1} = y_0 pr_E + x_0 \widehat{\eta} pr_{E'}$. Take $y \in \mathbb{Z}$ such that $y \equiv y_0 \pmod p$ and $(y, x_0 d) = 1$, and put $f_y := (y pr_E + x_0 \widehat{\eta} pr_{E'}) \circ \alpha \circ \beta_{|\iota(C)}$. By (the proof of) Corollary 6.6 we know that $f_y : C \to E$ is minimal. Moreover, since $(f_y)_* = f_* + (y - y_0) pr_E \circ \alpha \circ \beta \equiv f_* \,(\mathrm{mod}\, p\mathrm{End}(J_C))$, the proof of Proposition 7.5 shows that f and f_y have the same ramification point Q. Moreover, we have $(f_* - (f_y)_*)(Q) = (y - y_0) pr_E(\alpha(\beta(Q))) = 0$, and so f and f_y have the same branch point, and hence (i) holds in this case.

Case 2: $pr_{E'} \circ \alpha \circ i_{\beta(E_0)} = 0$, for some $\alpha \in \mathrm{Aut}(E \times E')$.

Again by Corollary 6.6 there are integers $x_0, y_0 \in \mathbb{Z}$ with $(dx_0, y_0) = 1$ such that $f_* \circ \beta^{-1} \circ \alpha^{-1} = y_0 pr_E + x_0 \widehat{\eta} pr_{E'}$. Take $x \in \mathbb{Z}$ such that $x \equiv x_0 \pmod p$ and $(y_0, x) = 1$, and put $f_x := (y_0 pr_E + x \widehat{\eta} pr_{E'}) \circ \alpha \circ \beta_{|\iota(C)}$. By (the proof of) Corollary 6.6 we know that $f_x : C \to E$ is minimal. Here $(f_x)_* = f_* + (x - x_0) \widehat{\eta} \circ pr_{E'} \alpha \circ \beta \equiv f_* \,(\mathrm{mod}\, p\mathrm{End}(J_C))$, so as before f and f_x have the same ramification point Q. Moreover, we have $(f_* - (f_y)_*)(Q) = (x - x_0)\widehat{\eta}(pr_{E'}(\alpha(\beta(Q)))) = 0$, and so f and f_x have the same branch point, and hence (i) holds here as well.

In [DiFr] it is shown that there always exists a cover $f : C \to E$ satisfying the condition (iii) of Proposition 7.13, the degree of f being 2. In addition, there is an explicit description of an infinite family of minimal covers $f_n : C \to E$ which have the same ramification and branch points.

Now we are nearly done. To end we have to overcome a technical difficulty. The monodromy groups occurring for the covers are the full symmetric groups S_{n_i}. These are only "nearly simple" groups and so when we take composites of covers (with normalization) it could happen that there occur constant field extensions with 2-elementary Galois groups. Hence we have to generalize the discussion and replace the ground field K by an appropriate affine scheme $\mathrm{Spec}(S)$ with function field K such that $\mathrm{Pic}(S)/(\mathrm{Pic}(S)^2$ is finite, and then show that there are infinitely many minimal covers with the same *branch divisor* over S.

This implies that there is a finite extension K_1 of K such that the curve $C \times K_1$ has an infinite tower of K_1-rational regular unramified Galois extensions with Galois group being a product of alternating groups A_n. Hence we get

Theorem 7.14 *Let K be a field of finite type of odd positive characteristic. Let E be an elliptic curve over K which has a K-rational isogeny η of prime degree $d > 5$. (For instance, take E belonging to a K-rational point on $X_0(13)$.)*

Then there is a curve C of genus 2 over K with $J_C \simeq E \times \eta(E)$ and a finite extension K_1 of K such that C has an infinite unramified regular Galois pro-cover defined over K_1.

If K is not a finite field, then the curve C can be chosen such that it is not a twist of constant curves, i.e. such that it is not isotrivial.

Note that in the above theorem we can take for the curve C/K a twist of the curve discussed in Proposition 7.13 and constructed in [DiFr].

Corollary 7.15 *Let K be a finitely generated field over $\mathbf{F}_p$. There is an algebraic subset of positive dimension in the moduli space of curves of genus 2 over K such that Question **Q3** has a positive answer for all curves in this subset.*

Remark 7.16 It is easy to see that in the infinite towers constructed in Theorem 7.14 over the non-constant curve C, there is no point of C which splits totally. So we get no answer to Question **Q2** by this construction.

Question: Can one find a curve C/F_d which satisfies the condition (iii) of Proposition 7.13 which is not isomorphic to the one used in [DiFr]? (For a given d, this curve is uniquely characterized by the property that $J_C \simeq E \times E'$ and that it has a subcover $f : C \to E$ of degree 2.)

8 Appendix: The Riemann-Hurwitz Relation for Relative Curves

Let k be a field and let T/k be a smooth projective irreducible curve with function field $K = \kappa(T)$. Let X/T be a relative curve, i.e. X/k is a smooth projective irreducible surface with a surjective morphism $p_X : X \to T$. We denote by $X_K = X \otimes K$ the generic fibre of p_X (which is a smooth projective curve over K).

Now suppose Y/T is another relative curve and that we have a non-constant, separable morphism

$$f_K : X_K = X \otimes K \to Y_K = Y \otimes K$$

of their generic fibres. Then the usual Riemann-Hurwitz relation states that we have an isomorphism

$$\omega_{X_K/K} \simeq f_K^*(\omega_{Y_K/K}) \otimes \mathcal{L}(D_{f_K}), \tag{27}$$

where $D_{f_K} \geq 0$ is an *effective* divisor (the *different divisor* of f_K). We now want to extend this relation to similar one involving the *relative dualizing sheaves* of X/T and Y/T, which are given by

$$\omega^0_{X/T} = \omega_{X/k} \otimes p_X^* \omega_{T/k}^{-1} \quad \text{and} \quad \omega^0_{Y/T} = \omega_{Y/k} \otimes p_Y^* \omega_{T/k}^{-1},$$

where $\omega_{X/k} = \wedge^2 \Omega^1_{X/k}$ and $\omega_{Y/k} = \wedge^2 \Omega^1_{Y/k}$; cf. Kleiman[Kl].

In the case that f_K extends to a T-morphism $f : X \to Y$, such a formula was established in the Appendix of [Ka6]. Here we want to show that a similar formula holds without this assumption.

For this, note first that the map f_K naturally extends to a morphism $f : U \to Y$, where $U \subset X$ is an open set such that $X \setminus U$ has codimension 2, i.e. $X \setminus U$ consists of finitely many closed points. Thus, every divisor of U extends uniquely to a divisor on X, and hence every invertible sheaf $\mathcal{L}$ on U extends uniquely to an invertible sheaf $\mathcal{L}_X$ on X.

Proposition 8.1 *Let $\overline{D}$ denote the schematic closure in X of the different divisor D_{f_K} of f_K. Then there exists an effective divisor $B = \sum b_\Gamma \Gamma \geq 0$ consisting entirely of fibre components of p_X such that if we write $D_f = \overline{D} + B$ then we have*

$$\omega^0_{X/T} \simeq (f^* \omega^0_{Y/T})_X \otimes \mathcal{L}(D_f). \tag{28}$$

Proof. First note that since $f^*\Omega^1_{Y/k}$ and $\Omega^1_{U/k}$ are locally free sheaves of rank 2 on U, we have the exact sequence

$$0 \to f^*\Omega^1_{Y/k} \to \Omega^1_{U/k} \to \Omega^1_{U/Y} \to 0.$$

Let $\mathcal{D}(U/Y) = \mathcal{F}^0(\Omega^1_{U/Y})$ denote the Kähler different of f which is by definition (cf. Kunz[Ku], p. 159) the 0-th Fitting ideal sheaf of the sheaf of differentials $\Omega^1_{U/Y}$. From the above exact sequence it follows that

$$f^*\omega_{Y/k} = \mathcal{D}(U/Y)\omega_{U/k}$$

(viewed as subsheaves of $\omega_{U/k} = \wedge^2\Omega^1_{U/k}$), and hence $\mathcal{D}(U/Y)$ is invertible. Thus, if D_f denotes the associated effective divisor defined by the invertible ideal sheaf $\mathcal{D}(U/Y)$, then we have

$$\omega_{U/k} \simeq f^*\omega_{Y/k} \otimes \mathcal{L}(D_f),$$

from which equation (28) follows immediately.

Finally we show that the divisor D_f has the form asserted in the proposition. Indeed, since $\mathcal{D}(U/Y)_{|X_K} = \mathcal{D}(X_K/Y_K)$ by general properties of the different (cf. [Ku], p. 160, property a)), we see that $D_f = \overline{D} + B$, where B is an effective divisor consisting entirely of fibre components.

References

[BR] J. Bertin, M. Romagny, Champs de Hurwitz. Preprint, 151pp.

[BLR] S. Bosch, W. Lütkebohmert, M.Raynaud, *Néron Models.* Springer-Verlag, Berlin, 1990.

[Da] H. Darmon, Serre's Conjectures. In: *Seminar on Fermat's Last Theorem* (Toronto, 1993-4) CMS Conf. Proc. **17** (1995), pp. 135-153.

[DM] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus. *Publ. Math. I.H.E.S.* **36** (1969), 75–109.

[DR] P. Deligne, M. Rapoport, Les schémas de modules de courbes elliptiques. In: *Modular functions of one variable, II,* Lecture Notes in Math. **349**, Springer-Verlag, Berlin, 1973, pp. 143–316.

[Di] C. Diem, Families of elliptic curves with genus 2 covers of degree 2. *Collect. Math.* **57** (2006), 1-25.

[DiFr] C. Diem, G. Frey, Non-constant curves of genus 2 with infinite pro-Galois covers. *Israel Journal of Mathematics*, **164** (2008), No. 1, 193–220.

[Dij] R. Dijkgraaf, Mirrow symmetry and elliptic curves. In: *The Moduli Space of Curves* (Dijkgraaf, Faber, van der Geer, eds.), Birkhäuser, Boston, 1995, pp. 149–164.

[EGA] A. Grothendieck, J. Dieudonné, Eléments de Géométrie Algébrique. Chapter I: Springer-Verlag, 1971; Chapter II-IV: Publ. IHES 8, 11, 17, 20, 24, 28, 32 (1961-1967).

[Fr1] G. Frey, On elliptic curves with isomorphic torsion structures and corresponding curves of genus 2. In: *Conference on Elliptic Curves, Modular Forms and Fermat's Last Theorem* (J. Coates, S.T. Yau, eds.), International Press, Boston, 1995, pp. 79-98.

[Fr2] G. Frey, On ternary equations of Fermat type and relations with elliptic curves. In: *Modular Forms and Fermat's Last Theorem, Boston, 1995* (G. Cornell, J. Silverman, G. Stevens, eds.), Springer-Verlag, New York, 1997, pp. 527–548.

[Fr3] G. Frey, Galois representations attached to elliptic curves and diophantine problems. In: *Number Theory* (M. Jutila, T. Metsänkylä, eds.), De Gruyter, Berlin, 2001, pp. 71-104.

[FJ] G. Frey, M. Jarden Horizontal isogeny theorems. *Forum Math.* **14** (2002), 931-952.

[FK1] G.Frey, E. Kani, Projective p-adic representations of the K-rational geometric fundamental group. *Arch. Math.* **77** (2001), 32-46.

[FK] G. Frey, E. Kani, Curves of genus 2 covering elliptic curves and an arithmetical application. In: *Arithmetic Algebraic Geometry* (G. van der Geer, F. Oort, J. Steenbrink, eds.), Progress In Math. vol. 89, Birkhäuser, Boston, 1991, pp. 153-176.

[FKV] G. Frey, E. Kani, H. Völklein, Curves with infinite K-rational geometric fundamental group. In: *Aspects of Galois Theory*, (H. Völklein et al., eds), LMS Lect. notes Ser. 256 (1999), 85-118.

[Fri] M. Fried, Frey-Kani covers. Preprint.

[FV] M. Fried and H. Völklein, The inverse Galois problem and rational points on moduli spaces. *Math. Annalen* **290** (1991), 771-800.

[Fu] W. Fulton, Hurwitz schemes and irreducibility of moduli of algebraic curves. *Ann. Math.* (2) **90** (1969), 542–575.

[HM] J. Harris, I. Morrison, *Moduli of Curves.* Springer Graduate Text **187**, Springer-Verlag, New York, 1998.

[He1] C.F. Hermann, Modulflächen quadratischer Diskriminante. *Manus. math.* **77** (1991), 95–110.

[He2] C.F. Hermann, Symmetrische Modulflächen der Diskriminante p^2. *Manus. math.* **76** (1992), 169–179.

[HL1] P. Hubert, S. Lelièvre, Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$. *Israel J. Math.* **151** (2006), 281-321.

[HL2] P. Hubert, S. Lelièvre, Noncongruence subgroups in $\mathcal{H}(2)$. *Int. Math. Res. Notes* 2005, no. 1, 47–64.

[Hum] G. Humbert, Sur les fonctions abéliennes singulières. I. *J. de Math.* (ser. 5) **5** (1899), 233–350 = Œuvres, Gauthier-Villars er Cie., Paris, 1929, pp. 297–401.

[Hu] A. Hurwitz, Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten. *Math. Ann.* **39** (1891), 1–61.

[Hup] B. Huppert, *Endliche Gruppen I.* Springer-Verlag, Berlin, 1967.

[I] Y. Ihara, On unramified extensions of function fields over finite fields. In: Adv. Stud. Pure Math. **2** (1983), 89-97.

[Ka1] E. Kani, Bounds on the number of non-rational subfields of a function field. *Invent. math.* **85** (1986), 185-198.

[Ka2] E. Kani, Elliptic curves on abelian surfaces. *Manuscr. Math.* **84** (1994), 199-223.

[Ka3] E. Kani, The number of curves of genus 2 with elliptic differentials. *J. reine angew. Math.* **485** (1997), 93-121.

[Ka4] E. Kani, The existence of curves of genus 2 with elliptic differentials. *J. Number Theory* **64** (1997), 130-161.

[Ka5] E. Kani, The Hurwitz space of genus 2 covers of an elliptic curve. *Collect. Math.* **54** (2003), 1–51.

[Ka6] E. Kani, The number of genus 2 covers of an elliptic curve. *Manusc. math.* **121** (2006), 51–80.

[Ka7] E. Kani, Mazur's question and modular diagonal quotient surfaces. In preparation.

[Ka8] E. Kani, Jacobians isomorphic to a product of two elliptic curves. In preparation.

[KS] E. Kani, W. Schanz Modular diagonal quotient surfaces. *Math. Z.* **227** (1998), 337–366.

[Ki] I. Kiming, On certain problems in the analytical arithmetic of quadratic forms arising from the theory of curves of genus 2 with elliptic differentials. *Manusc. math.* **87** (1995), 101-129.

[KM] N. Katz, B. Mazur, *Arithmetic Moduli of Elliptic Curves.* Princeton U Press, Princeton, 1985.

[Kl] S. Kleiman, Relative duality for quasi-coherent sheaves. *Comp. Math.* 41 (1980), 39–60.

[Kr] A. Krazer, *Lehrbuch der Thetafunktionen.* Leipzig, 1903; Chelsea Reprint, New York, 1970.

[Ku] E. Kunz, *Kähler Differentials.* Vieweg, Wiesbaden, 1986.

[La] S. Lang, *Fundamentals of Diophantine Geometry.* Springer Verlag, New York, 1983.

[Lan] H. Lange, Normenendomorphismen abelscher Varietäten. *J. reine angew. Math.* 290 (1977), 203–213.

[LK] S. Lønsted, S. Kleiman, Basics on families of hyperelliptic curves. *Compositio Math.* **38** (1979), 83–111.

[Ma] B. Mazur, Perturbations, deformations, and variations (and "near-misses") in geometry, physics and number theory. *Bull. AMS* **41** (2004), 307–336.

[Mc1] C. McMullen, Teichmüller curves in genus 2: discriminant and spin. *Math. Ann.* **333** (2005), 87–130.

[Mc2] C. McMullen, Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two. *Ann. Math.* **165** (2007), 397–456.

[Mi] J.S. Milne, Jacobian Varieties. In: *Arithmetic Geometry.* (G. Cornell, J. Siverman, eds.), Springer-Verlag, New York, 1986; pp. 165–212.

[Ta] G. Tamme, Teilkörper höheren Geschlechts eines algebraischen Funktionenkörpers. *Arch. Math.* **23** (1972), 257–259.

[vdG] G. van der Geer, *Hilbert Modular Surfaces.* Springer-Verlag, Berlin, 1988.

[V1] H. Völklein, *Groups as Galois Groups – an Introduction.* Cambr. Studies in Adv. Math. 53, Cambridge Univ. Press, 1996.

[V2] H. Völklein, Private communication.

[We] S. Wewers, Construction of Hurwitz spaces. IEM Preprint **21** (1998), U Essen.

[Wi] H. Wielandt, Finite permutation groups, 2nd ed.; New York - London, 1968.