Hurwitz spaces of genus 2 covers of an elliptic curve

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1 Introduction

Let E be an elliptic curve over a field K of characteristic $\neq 2$ and let $N > 1$ be an integer prime to char(K). The purpose of this paper is to study the family of genus 2 covers of E of fixed degree N, i.e. those covers $f : C \to E$ for which C/K is a curve of genus 2 and $\deg(f) = N$. Since we can (without loss of generality) restrict our attention those covers that are *normalized* in the sense of section 2, this investigation is essentially equivalent to the study of the set

$$\mathrm{Cov}_{E/K,N}(K) := \{f : C \to E : g_C = 2, \deg(f) = N \text{ and } f \text{ is normalized}\}/\simeq$$

of isomorphism classes of normalized genus 2 K-covers of E of degree N, where, as usual, two covers $f_i : C_i \to E$ are called isomorphic ($f_1 \simeq f_2$) if there is an isomorphism $\varphi : C_1 \to C_2$ such that $f_1 = f_2 \circ \varphi$.

The main result here is that this set can be naturally identified with the set $H_{E/K,N}(K)$ of K-rational points of an *explicit* affine curve $H_{E/K,N}/K$, and that this identification also holds for *families* of such covers (over an arbitrary base). More precisely, we first show in section 3 that the assignment $L \mapsto \mathrm{Cov}_{E/K,N}(L)$ extends in a natural way to a *Hurwitz functor* $\mathcal{H}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets}$ which is analogous to the Hurwitz functors considered by Fulton[Fu] (where the base is $\mathbb{P}^1$). The main result of this paper can then be stated as follows.

Theorem 1.1 *If $N \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,N}/K$ which is an open subset of a certain twist $X_{E/K,N,-1}$ of the modular curve $X(N)$ of level N; in particular, $H_{E,N} \otimes \overline{K}$ is isomorphic to an open subset of $X(N)_{/\overline{K}}$, where K denotes the algebraic closure of K.*

Remarks. 1) As the proof shows, exactly the same assertion holds for elliptic curves E/K over an arbitrary ring (or scheme) K; cf. Theorem 5.18.

2) The "twisted modular curve" $X_{E/K,N,-1}/K$ is defined and constructed in section 4 as the moduli space of the functor $\mathcal{X}_{E/K,N,-1}$ which assigns to any extension field L/K the set $\mathcal{X}_{E/K,N,-1}(L)$ of isomorphism classes of pairs (E', ψ) where E'/L is an elliptic curve and $\psi : E[N] \xrightarrow{\sim} E'[N]$ is an L-rational anti-isometry of the N-torsion subgroups of E and E'.

3) The above representability results are obtained by purely algebraic techniques and hence do not use (not even implicitly) the Riemann Existence Theorem.

The above theorem constitutes a refinement and extension of the "basic construction" of genus 2 curves (with elliptic differentials) which was presented in [FK], [Ka2]. Indeed, this construction shows that each genus 2 cover $f : C \to E$ of degree N determines a unique pair $(E'_f, \psi) \in \mathcal{X}_{E/K,N,-1}(K)$ and that conversely for each such pair (satisfying a suitable additional hypothesis) one can reconstruct the cover $f : C \to E$. Thus, the "basic construction" defines for each extension field L/K an injection

$$\Psi_L : \mathcal{H}_{E/K,N}(L) \hookrightarrow \mathcal{X}_{E/K,N,-1}(L) = X_{E/K,N,-1}(L)$$

and identifies the image. Here we shall see in Theorem 5.18 that this idea can be refined to obtain an open embedding of functors

$$\Psi : \mathcal{H}_{E/K,N} \hookrightarrow \mathcal{X}_{E/K,N,-1}.$$

However, instead of proving this directly, it is more convenient to divide the above construction into two steps, which amounts to a factorization of Ψ as

$$\Psi = \Psi' \circ \tau : \mathcal{H}_{E/K,N} \hookrightarrow \mathcal{A}_{E/K,N} \xrightarrow{\sim} \mathcal{X}_{E/K,N,-1},$$

in which τ is (essentially) the Torelli map which associates to a curve its (polarized) Jacobian and $\mathcal{A}_{E/K,N}$ is the functor that classifies principally polarized abelian surfaces with an embedding "of degree N" of E (cf. subsection 5.1). The fact that Ψ' is an isomorphism is proved in Theorem 5.10 and that τ is a monomorphism is shown in Proposition 5.12. Finally, the image of τ is analyzed in Proposition 5.17: it is the subfunctor $\mathcal{J}_{E/K,N}$ of $\mathcal{A}_{E/K,N}$ consisting of the "theta-smooth" elements (cf. subsection 5.3).

The fact that the Hurwitz functor $\mathcal{H}_{E/K,N}$ is representable by an open subset of the modular curve $X_{E/K,N,-1}$ has a number of interesting consequences. For example, it shows that over any number field K there are only finitely many genus 2 covers $f : C \to E$ of fixed degree $N \geq 7$ because the genus of (the compactification of) $X_{E/K,N,-1}$ is ≥ 2 for $N \geq 7$ and so $\#X_{E/K,N,-1}(K) < \infty$ by Faltings' Theorem. This means in particular that one cannot write down parametric families of such covers when $N \geq 7$, which is in sharp contrast to the fact that for $N \leq 5$ (in which case $X_{E/K,N,-1}$ is a rational curve) such families are known to exist and can be written down explicitly; cf. Krazer[Kr], p. 477ff, Kuhn[Ku], p. 48, and Frey[Fr2], p. 96ff. Another application of the explicit description of the Hurwitz space $H_{E/K,N}$ is given in [Ka4], where it is used to compute the number of genus 2 covers of E/K with a given discriminant divisor.

In order to complete the description of the Hurwitz space $H_{E/K,N}$, one should also describe the complement

$$D_{E/K,N} = X_{E/K,N,-1} \setminus H_{E/K,N},$$

which can be viewed as the "degeneracy locus" of the basic construction. (Alternately, one can view $D_{E/K,N}$ as the boundary of the Hurwitz space $H_{E/K,N}$; cf. [FKV].) However, since a description of $D_{E/K,N}$ was already given in essence in [Ka2], we merely need to cite and/or translate the relevant result; cf. Theorem 6.1 below. Similarly, the number of points in $D_{E/K,N}$ was computed in [Ka2] and [Ka3] (cf. Theorem 6.2); a slightly weaker version of this result may be stated as follows:

Theorem 1.2 *If K is algebraically closed, then the number of points in the degeneracy locus satisfies the inequality*

$$\#D_{E/K,N} \leq \frac{1}{24N}(5N-6)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}). \qquad (1)$$

Furthermore, equality holds in (1) if and only if $\mathrm{char}(K) \nmid N!$, *i.e. if and only if either* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > N$.

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Contents

2 Normalized genus 2 covers

As in the introduction, let E/K be an elliptic curve over a field K. It will be convenient (but not absolutely necessary) to assume in the sequel that $\mathrm{char}(K) \neq 2$ so as to avoid case distinctions.

A covering $f : C \to E$ is called *minimal* (or *optimal* ([Ku]) or *maximal* ([Se])) if the induced map $f^* : J_E \to J_C$ on the Jacobians is a closed immersion. Note that if C is a curve of genus 2, then this is equivalent to the condition that f does not factor over an isogeny of E of degree ≥ 2 (use Kuhn[Ku], Corollary on p. 45). Thus, by replacing E/K by an isogenous curve if necessary, we may always assume "without loss of generality" that f is minimal.

Minimal genus 2 covers of elliptic curves are partially analogous to the simple covers of $\mathbb{P}^1$ studied by Fulton[Fu]; for example, they do not have any internal automorphisms, as the following result shows.

Proposition 2.1 *Let $f : C \to E$ be a minimal genus 2 cover of E/K of degree $N \geq 3$. If $\alpha \in \mathrm{Aut}_K(C)$ is an automorphism such that $f \circ \alpha = f$, then $\alpha = id_C$.*

Proof. Without loss of generality, we may assume that K is algebraically closed. Let $G = \langle \alpha \rangle$ be the (finite) group generated by α. If $\alpha \neq id_C$, then the quotient map $\pi : C \to \overline{C} = G\backslash C$ has degree $\deg(\pi) \geq 2$ and f factors over π, i.e. $f = f' \circ \pi$. By Riemann-Hurwitz we then have $2 > g_{\overline{C}} \geq g_E = 1$, so $g_{\overline{C}} = 1$. Since f is minimal, this forces f' to be an isomorphism, i.e. f is a (ramified) Galois cover with group G. Furthermore, if we replace α by a power α^n, then the same argument shows that G cannot have any proper subgroup, and hence $N = |G|$ is prime.

Since f is Galois, the degree of its different satisfies $\deg(\mathrm{Diff}(f)) \geq N \sum_{P \in R_f}(1 - \frac{1}{e_P(f)})$ where $R_f = \{P \in E(K) : e_P(f) > 1\}$. (Note that equality holds if $\mathrm{char}(K) \nmid N$.) Thus, by the Riemann-Hurwitz relation we obtain $2 = \deg(\mathrm{Diff}(f)) \geq \frac{N}{2}\#R_f \geq \frac{N}{2}$, and hence $N \leq 4$.

Since N is prime, this leaves only the case $N = 3$. In this case there are three fibres of f which have isolated Weierstrass points in them (see the proof of Prop. 2.2 below). Thus, each of these points is fixed by α, and so we obtain $\deg(\mathrm{Diff}(f)) \geq 3$, which contradicts the Riemann-Hurwitz relation.

It is immediate that if $f : C \to E$ is a minimal genus 2 cover of E/K, then so is $T_x \circ f$, for any $x \in E(K)$, where $T_x : E \to E$ denotes the translation map.

Since $T_x \circ f$ is essentially the same cover as f, it is useful to "normalize" the genus 2 covers of E/K in a certain way so as to avoid redundant translates. This will be done by means of the Weierstrass (or hyperelliptic) divisor of C.

Notation. If C/K is a curve of genus 2, then σ_C denotes its *hyperelliptic involution* and W_C its *hyperelliptic divisor*. Recall that W_C is the divisor of fixed points of σ_C and is thus an effective divisor of degree 6. Furthermore, W_C is reduced if and only if $\mathrm{char}(K) \neq 2$; cf. Lønsted-Kleiman[LK].

Similarly, if E/K is an elliptic curve with zero 0_E, then the minus map $[-1]_E$ has the divisor $E[2]$ of 2-torsion points as its fixed point divisor. Note that the divisor $[0_E]$ defined by the point 0_E is contained in $E[2]$, and so $E[2]^\# := E[2] - [0_E]$ is an effective divisor of degree 3 on E.

Definition. A morphism $f : C \to E$ is called *normalized* if it is minimal and if the norm (or direct image) f_*W_C of the hyperelliptic divisor has the form

$$f_*W_C = 3\varepsilon[0_E] + (\varepsilon + 1)E[2]^\# \tag{2}$$

where $\varepsilon = 0$ if $\deg(f)$ is even and $\varepsilon = 1$ if $\deg(f)$ is odd. Thus, $\deg(f) \equiv \varepsilon(2)$, and we have more explicitly $f_*W_C = 3[0_E] + E[2]^\#$, if $\deg(f)$ is odd, and $f_*W_C = 2E[2]^\#$, if $\deg(f)$ is even.

This terminology is justified by the following result:

Proposition 2.2 *If $f : C \to E$ is a minimal genus 2 cover, then there is a unique point $x \in E(K)$ such that $f_{norm} := T_x \circ f$ is normalized. Furthermore, f_{norm} is "pseudo-normalized" in the sense that it satisfies the relation*

$$f_{norm} \circ \sigma_C = [-1]_E \circ f_{norm}. \tag{3}$$

Proof. It is well-known (cf. e.g. [Ku], p. 42) that there is a (unique) involution $\sigma_E \in \mathrm{Aut}(E)$ such that $f \circ \sigma_C = \sigma_E \circ f$. If $\overline{K}$ denotes the algebraic closure of K, then the unique extension $\sigma_C \otimes \overline{K}$ of σ_C to $C \otimes \overline{K}$ has four distinct fixed points $P_0, P_1, P_2, P_3 \in E(\overline{K})$, which can be numbered in such a way that

$$f_{\overline{K}*}W_{C_{\overline{K}}} = 3\varepsilon P_0 + (\varepsilon + 1)(P_1 + P_2 + P_3), \tag{4}$$

where $f_{\overline{K}} : C_{\overline{K}} = C \otimes \overline{K} \to E_{\overline{K}} = E \otimes \overline{K}$ denotes the induced cover over $\overline{K}$ (and where, as above, $\deg(f) \equiv \varepsilon(2)$). Indeed, if $\varepsilon = 1$ then this follows from [FK], Lemma 2.1, or from Kuhn[Ku], p. 44, and if $\varepsilon = 0$, then this follows from the discussion of [Ku] on the top of p. 48.

From the above description is clear that P_0 is $\mathrm{Gal}(\overline{K}/K)$-stable and hence is rational over K. Thus, the first assertion holds with $x = P_0$ (and only for this point). Furthermore, in [FK] and/or [Ku] it was already noted the second assertion follows, for we have $f_{norm} \circ \sigma_C = \sigma'_E \circ f_{norm}$, where $\sigma'_E = T_x \circ \sigma_E \circ T_x^{-1}$, and $\sigma'_E = [-1]_E$ because the fixed point set of σ'_E is $E[2]$.

Corollary 2.3 *Let $f : C \to E$ be a minimal genus 2 cover of E/K. Then f is normalized if and only if f satisfies (3) and*

$$\#(f^{-1}(0_E) \cap W_C) = 3\varepsilon. \tag{5}$$

Proof. If f is normalized, then clearly (5) holds by definition (cf. equation (2). Furthermore, Proposition 2.2 shows that f satisfies (3).

Conversely, suppose f satisfies (3). Then the proof of Prop. 2.2 shows that the points $P_0, \ldots, P_3$ of (4) are fixed under $[-1]_E$ and hence are 2-torsion points. Furthermore, equation (5) guarantees that $P_0 = 0_E$, and so (2) holds.

As we shall see below in Theorem 2.6, normalized covers $f : C \to E$ are intimately connected to the induced homomorphisms $f_* : J_C \to J_E$ and/or $f^* : J_E \to J_C$ on the Jacobians. For this we first show:

Proposition 2.4 *Let $f : C \to E$ be a normalized genus 2 cover of degree N, and let $\lambda_E : E \to \hat{E} = J_E$ be the canonical polarization on E which is defined by $\lambda_E(P) = cl(\mathcal{O}_E([P] - [0_E]))$.*

(a) If N is odd, then there is a unique effective divisor $W'_0 \leq W_C$ of degree 3 such that $f_*W'_0 = E[2]^\#$.

(b) If N is even, and the Weierstrass points of C are K-rational, then there is a divisor $W'_0 \leq W_C$ with $f_*W'_0 = E[2]^\#$. Furthermore, if $\tilde{W}'_0$ is another divisor with this property, then $cl(\mathcal{O}_C(W'_0 - \tilde{W}'_0)) \in \mathrm{Ker}(f_*)[2]$.

(c) Let $W'_0 \leq W_C$ be a divisor such that $f_*W'_0 = E[2]^\#$, and put $\mathcal{L} = \mathcal{O}_C(W'_0) \otimes \omega_C^{-1}$, where ω_C denotes the canonical sheaf of C/K. Then the map $x \mapsto cl(\mathcal{O}_C(x - W'_0) \otimes \omega_C) = cl(\mathcal{O}_C(x) \otimes \mathcal{L}^{-1})$ defines an embedding $j = j_\mathcal{L} : C \hookrightarrow J_C$ which satisfies

$$0_J \notin j(C) \quad \textit{and} \quad j \circ \sigma_C = [-1]_J \circ j \quad \textit{and} \quad \lambda_E \circ f = f_* \circ j. \tag{6}$$

Proof. (a) Let $W_0 = f^*(0_E) \cap W_C$ and $W'_0 = W_C - W_0$. Then W_0 and W'_0 are both effective divisors and $f_*W_0 = 3(0_E)$ by construction (and (2)) and hence $\deg(W_0) = \deg(W'_0) = 3$ and also $f_*W'_0 = E[2]^\#$, as claimed.

(b) The existence of W_0' is clear. If $\tilde{W}_0'$ is another divisor, then we have $cl(2(W_0' - \tilde{W}_0')) = \omega_C^{\otimes 3} \otimes (\omega_C^{\otimes 3})^{-1} \simeq \mathcal{O}_C$ (cf. part(c)), so $cl(W_0' - \tilde{W}_0') \in J_C[2]$. Moreover, $f_*(cl(\mathcal{O}_C(W_0' - \tilde{W}_0'))) = cl(\mathcal{O}(f_*W_0' - f_*\tilde{W}_0')) = 0$, so $cl(W_0' - \tilde{W}_0') \in \mathrm{Ker}(f_*)[2]$.

(c) Since $\deg(\mathcal{O}_C(W_0') \otimes \omega_C^{-1}) = 1$, the rule $x \mapsto cl(\mathcal{O}_C(x - W_0') \otimes \omega_C)$ is represented by a closed immersion $j : C \hookrightarrow J_C$.

We now verify that j satisfies (6). For this we may assume that K is algebraically closed, so $W_0' = W_1 + W_2 + W_3$ with $W_i \in C(K)$.

To prove the first equation of (6) we shall use the fact that for each $P \in C(K)$, the divisor $P + \sigma_C(P) \sim \omega_C$ is a canonical divisor on C. In particular, for each Weierstrass point $W_i \in C(K)$ we have $2W_i \sim \omega_C$, and so $2W_0' \sim \omega_C^{\otimes 3}$. Thus $j(P) + j(\sigma_C(P)) = cl(\mathcal{O}_C(P + \sigma_C(P) - 2W_0') \otimes \omega_C^{\otimes 2}) = 0$, which proves the first equality of (6).

To prove the second equality, we first note that the points $P_i := f(W_i)$, for $i = 1, 2, 3$ are (by construction) precisely the non-trivial 2-torsion points of E, and so $cl(\mathcal{O}_C(P_2 - P_3)) = cl(\mathcal{O}_C(P_1 - 0_E)) = \lambda_E(P_1)$. Moreover, by the Albanese property of J_C we know that $f_* \circ j = T_x \circ \lambda_E \circ f$, for some $x \in J_E(K)$. Thus, since $2W_2 \sim \omega_C$, we see that $f_*(j(W_1)) = f_*(cl(\mathcal{O}_C(W_1 - W_1 + W_2 - W_3))) = cl(\mathcal{O}_C(P_2 - P_3)) = \lambda_E(P_1) = \lambda_E(f(W_1))$, and so $T_x(\lambda_E f(W_1)) = f_*(j(W_1)) = \lambda_E(f(W_1))$. This forces $x = 0$ and so $f_* \circ j = \lambda_E \circ f$.

Next we observe that $0_J \notin j(C)$. Indeed, if $0_J \in j(C)$, then there is a point $P \in C(K)$ such that $P \sim \mathcal{O}_C(W_0') \otimes \omega_C^{-1} \sim W_1 - W_2 + W_3$, which is impossible as $h^0(W_1 - W_2 + W_3) = 0$ (because $h^0(W_1 + W_2) = 1$).

Corollary 2.5 *The curve $\theta := j(C) \subset J_C$ of Proposition 2.4(c) is a symmetric theta-divisor of J_C, i.e. θ is an effective divisor on J_C such that $[-1]_{J_C}\theta = \theta$ and such that $\lambda_{\mathcal{O}(\theta)} = \lambda_C : J_C \xrightarrow{\sim} \hat{J}_C$ is the canonical principal polarization of J_C. Furthermore:*

(a) If N is odd, then θ is the unique symmetric theta divisor such that

$$\theta \cap \mathrm{Ker}(f_*)[2] = \mathrm{Ker}(f_*)[2]^{\#} := \mathrm{Ker}(f_*)[2] - [0]. \tag{7}$$

(b) If N is even, then θ is a symmetric theta divisor satisfying

$$\theta \cap \mathrm{Ker}(f_*)[2] = \emptyset. \tag{8}$$

Moreover, if θ' is any symmetric theta-divisor on J_C satisfying (8), then $\theta' = T_x(\theta)$, for some $x \in \mathrm{Ker}(f_*)[2]$.

Proof. Since j is of the form $P \mapsto cl(\mathcal{O}(P) \otimes \mathcal{L}^{-1})$, where $\mathcal{L}$ is a suitable invertible sheaf of degree 1, it is clear that θ is an effective theta-divisor. Furthermore, θ is symmetric by the first equation of (6).

(a) To prove that θ satisfies (7) we may assume without loss of generality that K is algebraically closed. We first observe the first equation of (6) shows that $j(P) \in J_C[2]$ if and only if P is a Weierstrass point.

We now claim that $\#\mathrm{Ker}(f_*)[2] = 4$. This follows either by observing that $\mathrm{Ker}(f_*)$ is an elliptic curve (cf. Prop. 2.7 below) or by noting that (6) shows that $f_*(J_C[2]) = J_E[2]$ and hence it follows that $\#\mathrm{Ker}(f_*)[2] = \#(\mathrm{Ker}(f_*) \cap J_C[2]) = \#J_C[2]/\#J_E[2] = 4$.

Thus, if $W_{01}, W_{02}, W_{03} \in C(\overline{K})$ are the three Weierstrass points of C such that $f(W_{0i}) = 0_E$, then by (6) we have $\mathrm{Ker}(f_*)[2] = \{0_J, j(W_{01}), \ldots, j(W_{03})\}$, and so (7) follows.

Now let θ' be another theta-divisor of J_C satisfying (7). Then $\theta' = T_x\theta$, for some x, and so θ' is also irreducible. Moreover, by (7) we have $\theta' \cap \theta \supset \mathrm{Ker}(f_*)[2]^\#$, which has 3 distinct points (over $\overline{K}$), and so $(\theta'.\theta) \geq 3$, if $\theta' \neq \theta$. But $(\theta'.\theta) = (\theta)^2 = 2$ (by Riemann-Roch), contradiction. Thus $\theta' = \theta$.

(b) A slight modification of the proof of part (a) shows that θ satisfies (8) and that $(f_*)(\theta \cap J[2]) = E[2]^\#$ (as sets). Now suppose θ' is another symmetric theta-divisor satisfying (8). Then $\theta' = T_x\theta$, for some $x \in J_C[2]$ (because θ and θ' are both symmetric). If $x \notin \mathrm{Ker}(f_*)$, then $f_*(x) \in E[2]^\#$, and so there exists $y \in \theta \cap J[2]$ such that $f_*(y) = f_*(x)$. But then $x + y \in T_x\theta \cap \mathrm{Ker}(f_*)[2] = \theta' \cap \mathrm{Ker}(f_*)[2]$, contradiction. Thus $x \in \mathrm{Ker}(f_*)$, as claimed.

The above proposition and its corollary lead to the fundamental fact that normalized covers can be characterized by their induced homomorphisms on the Jacobians. To state this more precisely, it is useful to introduce some equivalence relations on covers and on homomorphisms.

Definition. Two (normalized) K-covers $f_i : C_i \to E$ of genus 2 are called *equivalent* if there exists an a K-isomorphism $\alpha : C_1 \xrightarrow{\sim} C_2$ such that $f_1 = f_2 \circ \alpha$. If such an α exists, then we write $f_1 \simeq f_2$.

Two injective homomorphims $h_i : J_E \to J_i := J_{C_i}$ are called *equivalent* if there exists a K-isomorphism $\alpha : J_1 \xrightarrow{\sim} J_2$ such that $h_2 = \alpha \circ h_1$ and $\hat{\alpha} \circ \lambda_2 \circ \alpha = \lambda_1$, where $\lambda_i = \lambda_{C_i} : J_i \xrightarrow{\sim} \hat{J}_i$ is the canonical polarization of the Jacobian J_i, for $i = 1, 2$. We write $h_1 \simeq h_2$ if such an isomorphism α exists.

Finally, we say that an injective homomorphism $h : E \to J_C$ has *degree*

N if $\hat{h} \circ \lambda_C \circ h = \lambda_{J_E} \circ [N]_{J_E}$; we then write $N = \deg_{\lambda_C}(h)$.

Theorem 2.6 *Let C/K be a curve of genus 2 and let $N \geq 2$ be an integer. Then the assignment $f \mapsto f^*$ induces a surjection*

$$\rho = \rho_{C,E,N} : \mathrm{Cov}_K(C, E, N) \to \mathrm{InjHom}_K(J_E, J_C, N)$$

from the set $\mathrm{Cov}_K(C, E, N)$ of equivalence classes of normalized K-covers $f : C \to E$ of degree N to the set $\mathrm{InjHom}_K(J_E, J_C, N)$ of equivalence classes of injective K-homomorphisms $h : J_E \hookrightarrow J_C$ of degree N. Furthermore, ρ is a bijection if $N > 2$ or if the Weierstrass points of C are K-rational.

Proof. First note that if $f : C \to E$ is a normalized cover, then f^* defines an equivalence class $cl(f^*)$ in the set $\mathrm{InjHom}_K(J_E, J_C, N)$. Indeed, f^* is injective because f is minimal, and f^* has degree N because $f_* = \lambda_{J_E}^{-1} \circ (f^*)\hat{} \circ \lambda_C$, and so $(f^*)\hat{} \circ \lambda_C \circ f^* = \lambda_{J_E} \circ (f_* \circ f^*) = \lambda_{J_E} \circ [N]_{J_E}$. Furthermore, the rule $f \mapsto cl(f^*)$ is compatible with the equivalence relation on $\mathrm{Cov}_K(C, E, N)$ because if $f_1 \simeq f_2$, then $f_1 = f_2 \circ \alpha$ for some $\alpha \in \mathrm{Aut}_K(C)$, and so $f_1^* = (f_2 \circ \alpha)^* = \alpha^* \circ f_2^*$. Clearly, $\alpha^* \in \mathrm{Aut}_K(J_C)$ and we have $(\alpha^*)\hat{} \circ \lambda_C \circ \alpha^* = \lambda_C \circ (\alpha_* \circ \alpha^*) = \lambda_C \circ [\deg(\alpha)] = \lambda_C$. This means that $f_1^* \simeq f_2^*$ and so the rule $f \mapsto cl(f^*)$ defines a map $\rho : \mathrm{Cov}_K(C, E, N) \to \mathrm{InjHom}_K(J_E, J_C, N)$.

We first show that ρ is surjective. Thus, let $h : E \hookrightarrow J_C$ be an injective homomorphism of degree N. Assume temporarily that C/K satisfies:

$$(\dagger) \qquad W_C \subset C(K) \text{ and } J_C[2] \subset J_C(K).$$

Since the Weierstrass points of C are now assumed to be rational, there exists an embedding $j = j_P : C \hookrightarrow J_C$ such that $\theta := j(C)$ is a symmetric theta-divisor on J_C; cf. Prop. 2.4 and Cor. 2.5. Then $[-1]_J\theta = \theta$, and so $[-1]_J$ induces a unique automorphism σ on C such that $j \circ \sigma = [-1]_J \circ j$. It is then immediate that σ has the Weierstrass points as fixed points and hence $\sigma = \sigma_C$ is the hyperelliptic involution on C.

Define $f := f_h : C \to E$ by $f = h^* \circ j$ where $h^* = \hat{h} \circ \lambda_C : J_C \to \hat{J}_E = E$. Thus, since h^* is a homomorphism we have $[-1]_E \circ f = [-1]_E \circ h^* \circ j = h^* \circ [-1]_J \circ j = h^* \circ j \circ \sigma_C = f \circ \sigma_C$, so f is pseudo-normalized, i.e. satisfies condition (3). By Prop. 2.2 we know that there exists a (unique) point $x \in E(K)$ such that $f_{norm} := T_x \circ f$ is normalized. Since f and f_{norm} both satisfy (3), it follows that $[-1]_E x = x$, i.e. $x \in E[2]$. Now since $h^* : J[2] \to E[2]$ is surjective (because h^* is surjective and the fibres of h^* are connected), there is a point $x' \in J[2]$ such that $h^*(x') = x$, and then $f_{norm} = h^* \circ j'$, where $j' = T_{x'} \circ j$. Using the autoduality property of the Jacobian, i.e., the fact that $(j')^* = j^* = -\lambda_C^{-1} : \hat{J}_C \to J_C$, we obtain by dualizing the above relation that $f^*_{norm} = (j')^* \circ (h^*)^* = -h$. Thus, $f_{norm} : C \to E$ is a normalized cover such that $f^*_{norm} = -h \simeq h$, i.e. τ is surjective (provided that condition (†) holds).

To finish the proof of the surjectivity of ρ, we now remove the hypothesis (†) and hence consider an arbitrary curve C/K. Then there exists a finite Galois cover K'/K such that $C_{K'} = C \otimes K'$ satisfies (†) and so, by what was just shown, there exists a normalized map $f = f_h : C_{K'} \to E_{K'}$ such that $f^* = -h$. Furthermore, $f = \lambda_E^{-1} \circ h^* \circ j$, where $j = j_{\mathcal{L}}$ is the embedding defined by a suitable $\mathcal{L} \in \mathrm{Pic}(C'_K)$. Consider the Galois twist f^g of f by $g \in \mathrm{Gal}(K'/K)$. Since $j^g = T_{x(g)} \circ j$ with $x(g) = cl(\mathcal{L}^g \otimes \mathcal{L}^{-1}) \in \mathrm{Pic}^0(C'_K) = J(K')$, we obtain $f^g = \lambda_E^{-1} \circ h^* \circ T_{x(g)} \circ j$. Now $\theta' := T_{x(g)}(j(C)) = j(C)^g$ is again a symmetric theta-divisor of λ_C which satisfies $\theta' \supset \mathrm{Ker}(h^*)[2]^\#$ (resp., $\theta \cap \mathrm{Ker}(h^*)[2] = \emptyset$) if N is odd (resp. if N is even) because $\mathrm{Ker}(h^*)^g = \mathrm{Ker}(h^*)$. Thus, $\theta' = j(C)$ (resp. $\theta' = T_{x'} j(C)$ with $x' \in \mathrm{Ker}(h^*)[2]$) and so $x(g) = 0$ (resp. $x(g) = x' \in \mathrm{Ker}(h^*)[2]$). Thus, in both cases $h^* \circ T_{x(g)} = h^*$, and so $f^g = f$, for all $g \in \mathrm{Gal}(K'/K)$. This means that f is defined over K, and so τ is surjective in general.

It remains to show that ρ is injective (under the stated hypotheses). For this, suppose that $f_i : C \to E$ are two normalized covers such that $f_1^* \simeq f_2^*$; we then want to show that also $f_1 \simeq f_2$. The hypothesis $f_1^* \simeq f_2^*$ means that there exists an $\alpha \in \mathrm{Aut}_K(J_C)$ with property that $f_1^* = \alpha \circ f_2^*$ and that $\hat{\alpha} \circ \lambda_C \circ \alpha = \lambda_C$. We then also have $(f_1)_* \circ \alpha = (f_2)_*$, for by dualizing the first relation we obtain $(f_1)_* = \lambda_{J_E}^{-1} \circ (f_1^*)\hat{} \circ \lambda_C = \lambda_{J_E}^{-1} \circ (f_2^*)\hat{} \circ \hat{\alpha} \circ \lambda_C = \lambda_{J_E}^{-1} \circ (f_2^*)\hat{} \circ \lambda_C \circ \alpha^{-1} = (f_2)_* \circ \alpha^{-1}$, as claimed.

To show that $f_1 \simeq f_2$, suppose first that N is odd. Then by Prop. 2.4 there exists a K-embedding $j_i : C \hookrightarrow J_C$ such that $\lambda_E \circ f_i = (f_i)_* \circ j_i$ and such that $\theta_i = j_i(C)$ is theta divisor associated to λ_C, for $i = 1, 2$. Since f_i is normalized, we have by Cor. 2.5(a) that $\theta_i \supset \mathrm{Ker}((f_i)_*)[2]^\#$. Now $\alpha^{-1}(\theta_1)$ is again a symmetric theta-divisor (associated to λ_C) and $\alpha^{-1}(\theta_1) \supset \alpha^{-1}(\mathrm{Ker}((f_1)_*)[2]^\#) = \mathrm{Ker}((f_1)_* \circ \alpha)[2]^\# = \mathrm{Ker}((f_2)_*)[2]^\#$, so $\theta_2 = \alpha^{-1}(\theta_1)$ because θ_2 is uniquely characterized by this property. Thus $\varphi := ((j_2)^{-1} \circ \alpha^{-1} \circ j_1 \in \mathrm{Aut}_K(C)$ satisfies $j_2 \circ \varphi = \alpha^{-1} \circ j_1$, and so by the above identity we have $\lambda_E \circ f_1 = (f_1)_* \circ j_1 = (f_2)_* \circ \alpha^{-1} \circ j_1 = (f_2)_* \circ j_2 \circ \varphi = \lambda_E \circ f_2 \circ \varphi$, or $f_1 = f_2 \circ \varphi$ with $\varphi \in \mathrm{Aut}_K(C)$, i.e. $f_1 \simeq f_2$.

Next assume that N is even. Then by Prop. 2.4 there exists a finite Galois extension K'/K such that over K' we have embeddings $j_i : C' = C \otimes K' \hookrightarrow J' = J_{C'}$ such that $\lambda_E \circ f_i = (f_i)_* \circ j_i$ and such that $\theta_i = j_i(C)$ are theta divisors associated to λ_C (defined over K'). Then by Cor. 2.5(b) we have $\theta_i \cap \mathrm{Ker}((f_i)_*)[2] = \emptyset$, and so also $\alpha^{-1}(\theta_1) \cap \mathrm{Ker}((f_2)_*)[2] = \alpha^{-1}(\theta_1) \cap \mathrm{Ker}((f_1)_* \circ \alpha)[2] = \alpha^{-1}(\theta_1 \cap \mathrm{Ker}((f_1)_*)[2]) = \emptyset$. Thus, by the uniqueness property of θ_2 it follows that $\theta_2 = T_x(\alpha^{-1}(\theta_1))$ for some $x \in \mathrm{Ker}((f_2)_*)[2]$, and so $\varphi := (j_2)^{-1} \circ T_x \circ \alpha^{-1} \circ j_1 \in \mathrm{Aut}_{K'}(C')$ satisfies $j_2 \circ \varphi = T_x \circ \alpha^{-1} \circ j_1$. Now since $x \in \mathrm{Ker}((f_2)_*)$, we have $(f_2)_* \circ T_x = (f_2)_*$, and so $(f_2)_* \circ \alpha^{-1} \circ j_1 = (f_2)_* \circ j_2 \circ \varphi = \lambda_E \circ f_2 \circ \varphi$. Thus $\lambda_E \circ f_1 = (f_1)_* \circ j_1 = (f_2)_* \circ \alpha^{-1} \circ j_1 = \lambda_E \circ f_2 \circ \varphi$, which means that $f_1 = f_2 \circ \varphi$, with $\varphi \in \mathrm{Aut}_{K'}(C')$.

To conclude that $f_1 \simeq f_2$, we still have to show that φ is defined over K. This is automatic if the Weierstrass points are defined over K, for then we can choose $K' = K$. Thus, assume that this is not the case but that $N > 2$. Let $g \in \mathrm{Gal}(K'/K)$, and consider the Galois twist $\varphi^g \in \mathrm{Aut}_{K'}(C')$ of φ. Since the f_i's are defined over K we have $f_1 = f_2 \circ \varphi^g$, and so, if $\varphi' = \varphi^g \circ \varphi^{-1}$, then we have $f_2 \circ \varphi' = f_1 \circ \varphi^{-1} = f_2$. Now since $N \geq 3$, we have by Prop. 2.1 that $\varphi' = id_{C'}$, and so $\varphi^g = \varphi$, for all $g \in \mathrm{Gal}(K'/K)$, which means that $\varphi \in \mathrm{Aut}_K(C)$. Thus $f_1 \simeq f_2$, as desired, and so ρ is injective.

Remark. It is easy to see that the sets of Theorem 2.6 are finite; in fact, the cardinality of $\mathrm{Cov}_K(C, E, N)$ can bounded by the number of (primitive) representations of N^2 by a suitable positive definite quadratic form associated to C (multiplied by $\frac{1}{2}\#\mathrm{Aut}(E)$); cf. [Ka1], Theorem 4.5.

As was mentioned in [FK] or [Ku], each minimal genus 2 cover $f : C \to E$ induces a splitting of the Jacobian J_C up to isogeny, i.e. $J_C \sim E \times E'$, and the *complementary elliptic curve* $E' = E'_f$ can be chosen in a canonical way by using the (canonical) principal polarization $\lambda_C : J_C \xrightarrow{\sim} \hat{J}_C$ of C. This curve E'_f plays an important role in the "basic construction" of [FK], which will be reviewed (and extended) in section 5.

Proposition 2.7 *If $f : C \to E$ is a minimal genus 2 cover of degree N, then $E' := \mathrm{Ker}(f_*)$ is an elliptic curve such that*

$$E' \cap f^* J_E = f^* J_E[N] = E'[N]. \tag{9}$$

Thus, if $\pi : J_E \times E' \to J_C$ denotes the unique map such that $\pi \circ i_{J_E} = f^*$ and $\pi \circ i_{E'} = -h'$, where $i_{J_E} : J_E \hookrightarrow J_E \times E'$, $i_{E'} : E' \hookrightarrow J_E \times E'$, and $h' : E' =$

$\mathrm{Ker}(f_*) \hookrightarrow J_C$ *denote the canonical inclusions, then* $\mathrm{Ker}(\pi) = f^*J_E \cap E'$ *and hence* π *is an isogeny of degree* N^2*; in particular,* $J_C \sim J_E \times E' \simeq E \times E'$.

Proof. Since $f_* = \lambda_E^{-1} \circ (f^*)\hat{} \circ \lambda_C$ is the "dual" of f^* and since f^* is a closed immersion, it follows that f_* is surjective and has (geometrically) connected fibres of dimension $(\dim J_C - \dim J_E) = 1$ (cf. section 7 below). Thus, E' is an elliptic curve.

Moreover, since f^* is an injection and since $E' = \mathrm{Ker}(f_*)$, we have $E' \cap f^*J_E = f^*\mathrm{Ker}(f_* \circ f^*) = f^*([N]_{J_E}) = f^*(J_E[N])$. Thus, $E' \cap f^*J_E$ is a finite group (scheme) of order N^2 and of exponent N, and so $E' \cap f^*J_E \le E'[N]$. But $E'[N]$ also has order N^2, and so $E' \cap f^*J_E = E'[N]$, which proves the first statement. The second statement follows immediately from the first.

Corollary 2.8 *The above embedding* $h' : E' \hookrightarrow J_C$*, has degree* N*, and hence there exists a "complementary" minimal* K*-cover* $f' : C \to E'$ *of degree* N *such that* $(f')^* = h' \circ \lambda_{E'}^{-1}$.

Proof. By Theorem 2.6 there is a normalized K-cover $f' : C \to E'$ of degree $N' = \deg_{\lambda_C}(h')$ such that $(f')^* = h' \circ \lambda_{E'}^{-1}$, and then $(f')_* = \lambda_{E'} \circ (h')^*$, where $(h')^* = \lambda_{E'}^{-1} \circ (h')\hat{} \circ \lambda_C : J_C \to E'$.

Recall that the sequence

$$0 \to E' \xrightarrow{h'} J_C \xrightarrow{f_*} J_E \to 0$$

is exact by the definition of h'. Thus, dualizing this sequence and applying λ_C, $\lambda_{E'}$ and $\lambda_{J_E}(= \lambda_E^{-1})$ yields the sequence

$$0 \to J_E \xrightarrow{f^*} J_C \xrightarrow{(h')^*} E' \to 0$$

which is again exact (cf. [La], p. 216); here we have used the identification $f^* = \lambda_C^{-1} \circ (f_*)\hat{} \circ \lambda_{J_E}$. This means in particular that $\mathrm{Ker}(f'_*) = \mathrm{Ker}((h')^*) = f^*J_E$, and thus, if we now apply Prop. 2.7 to f', then we obtain

$$E' \cap f^*J_E = (f')^*J_{E'} \cap \mathrm{Ker}(f'_*) = E'[\deg(f')].$$

On the other hand, since $f^*J_E \cap E' = E'[N]$ by (9), we conclude that $N = \deg(f') = \deg_{\lambda_C}(h')$, as desired.

3 Families of genus 2 covers and the Hurwitz functor $\mathcal{H}_{E/K,N}$

We now want to study families of normalized genus 2 covers over an arbitrary base scheme S, i.e. (normalized) covers $f : C \to E$ where E/S is a (relative) elliptic curve and C/S is a relative curve of genus 2 in the sense of the appendix (cf. section 7). For this, we first observe that every genus 2 curve C/S is *hyperelliptic* in the sense of Lønsted/Kleiman[LK], p. 101:

Lemma 3.1 *If $p : C \to S$ is a relative curve of genus 2, then there exists a unique S-automorphism $\sigma_{C/S} \in \mathrm{Aut}_S(C)$ which induces the hyperelliptic involution on each fibre C_s of p.*

Proof. If $\omega = \omega_{C/S}$ denotes the relative canonical sheaf of C/S, then $p_*(\omega_{C/S})$ is locally free of rank $g = 2$, and the canonical S-morphism $\varphi_\omega : C \to \mathbb{P}(\omega_{C/S})$ is surjective (because this is true fibre-by-fibre), and so condition (i) of Th. 5.5 of [LK] holds, which means that C/S is hyperelliptic.

As in section 2, let E/K be an elliptic curve over a field K with $\mathrm{char}(K) \neq 2$ (or, more generally, over any ring (or any scheme) K in which 2 is invertible). For any K-scheme S let $E_S := E \times_K S$ be the elliptic curve over S obtained from E/K by base-change.

Definition. A genus 2 cover $f : C \to E_S$ of E_S/S of degree N is called *normalized* if it is minimal (cf. §7) and if the direct image $f_*W_{C/S}$ of the hyperelliptic divisor $W_{C/S}$ (cf. [LK]) has the form

$$f_*(W_{C/S}) = 3\varepsilon[0_{E_S/S}] + (\varepsilon + 1)E_S[2]^\# \qquad (10)$$

where (as before) $\varepsilon = 0$ if N is even and $\varepsilon = 1$ if N is odd, and $[0_{E_S/S}]$ denotes the Cartier divisor associated the zero-section $0_{E_S/S}$ and $E_S[2]^\# := E_S[2] - [0_{E/S}]$ (viewed as effective relative Cartier divisors on E_S).

The basic properties of normalized genus 2 covers of E_S/S are summarized in the following theorem.

Theorem 3.2 *(a) If $f : C \to E_S$ is a normalized genus 2 cover of degree N, then so is any base-change $f_{(T)} : C_T = C \times_S T \to E_T = E_S \times_S T$.*

(b) If C/S is flat and locally of finite presentation, and if S is reduced, then an S-morphism $f : C \to E_S$ is a normalized genus 2 cover of degree

N if and only if $f_s : C_s \to E_s = E \otimes \kappa(s)$ *is a normalized genus* 2 *cover of degree* N*, for all* $s \in S$.

(c) If $f : C \to E_S$ *is a normalized genus* 2 *cover then we have*

$$f \circ \sigma_{C/S} = [-1]_{E_S} \circ f. \tag{11}$$

Conversely, if $f : C \to E_S$ *is a minimal cover satisfying (11) and if for at least one* $s \in S$ *the induced map* $f_s : C_s \to E_s$ *is normalized, then* f *is normalized.*

(d) Let $f : C \to E_S$ *be a normalized genus* 2 *cover of degree* N*. If* N *is odd, or if* N *is even and* C/S *has* 6 *distinct Weierstrass sections, then there is a closed* S*-immersion* $j : C \hookrightarrow J = J_{C/S}$ *of* C *into its Jacobian (cf. §7) such that*

$$j \circ \sigma_C = [-1]_J \circ j, \quad \lambda_E \circ f = f_* \circ j, \quad \text{and} \quad 0_J(S) \cap j(C) = \emptyset. \tag{12}$$

In particular, there is always a suitable etale faithfully flat base change S'/S *such that there exists an immersion* $j : C_{(S')} \hookrightarrow J_{(S')}$ *satisfying* (12).

(e) In the situation of (d), the image $\theta = j(C) \subset J$ *is a symmetric theta-divisor on* J*. If* N *is odd, then* θ *is the unique symmetric theta-divisor satisfying*

$$\theta \cap \mathrm{Ker}(f_*)[2] = \mathrm{Ker}(f_*)[2]^{\#} := \mathrm{Ker}(f_*)[2] \setminus [0]. \tag{13}$$

If N *is even, then* θ *is a symmetric theta divisor satisfying*

$$\theta \cap \mathrm{Ker}(f_*)[2] = \emptyset. \tag{14}$$

Moreover, if θ' *is any symmetric theta-divisor on* J *satisfying (14), then* $\theta' = T_x(\theta)$*, for some* $x \in \mathrm{Ker}(f_*)[2](S')$.

(f) Let C/S *be a curve of genus* 2*. If* $f : C \to E_S$ *is a minimal cover of degree* N*, then* $f^* : J_{E_S/S} \to J_{C/S}$ *is an injective homomorphism of degree* N*, i.e. we have* $\lambda_{E_S/S}^{-1} \circ (f^*)\hat{} \circ \lambda_{C/S} \circ f^* = [N]_{J_{E_S/S}}$*. Conversely, if* $h : J_{E_S} \hookrightarrow J_{C/S}$ *is an injective homomorphism of degree* N*, then there exists a normalized cover* $f : C \to E_S$ *of degree* N *such that* $f^* = h$.

Proof. (a) First note that $f_{(T)}$ is minimal because f is (cf. §7). Moreover, since the formation of $W_{C/S}$ commutes with base-change (cf. [LK], Prop. 6.3), and since the same is true for the direct image of relative Cartier divisors by Lemma 7.2 of the appendix, we see that the analogue of (10) also holds for $f_{(T)}$, and so $f_{(T)}$ is also normalized.

(b) If f is a normalized genus 2 cover, then so is f_s, $\forall s \in S$, by part (a). Conversely, if $f_s : C_s \to E_s$ is a genus 2 cover of degree N for all $s \in S$, then C/S is a smooth curve of genus 2 (use [BLR], 2.4/8), and so f is a genus 2 cover of degree N which is minimal (cf. §2.2). Moreover, since by hypothesis (and base-change) we have $(f_*W_{C/S})_s = (f_s)_*W_{C_s/\kappa(s)} = 3\varepsilon[0_{E_s/\kappa(s)}] + (\varepsilon+1)E_s[2]^\# = (3\varepsilon[0_{E_S/S}] + (\varepsilon+1)E_S[2]^\#)_s$, for all $s \in S$, we can conclude from (46) that (10) holds, and so f is normalized.

(c) Since it is enough to prove (11) after a faithfully flat base-change, we may assume that there exists a Weierstrass section w of C/S. Then $P := f(w) \in E_S[2](S)$ because $[P] = f_*[w] \le f_*W_{C/S} \le 3E_S[2]$.

Write $f_1 = f \circ \sigma_{C/S}$ and $f_2 = [-1]_{E_S} \circ f$. Then $f_i : C \to E_S$ are two S-morphisms which by Prop. 2.2 satisfy $(f_1)_s = (f_2)_s, \forall s \in S$, and so by rigidity we there is a section $\eta \in E_S(S)$ such that $f_1 = (\eta \circ p_C) + f_2$; cf. [Mu1], p. 116. But since $\sigma_{C/S}$ (resp. $[-1]$) fixes w (resp. P), we have $f_1(w) = P = f_2(w)$ and so $\eta = 0_{E_S}$, which means that $f_1 = f_2$, as desired.

We now prove the converse. Again, it is enough to prove this after a faithfully flat base change (because distinct relative Cartier divisors stay distinct after a faithfully flat base change). Thus, we may assume that

(†) C/S has six Weierstrass sections and $J_{C/S}/S$ has sixteen 2-torsion sections.

Then E_S/S has four 2-torsion sections $P_0 = 0$, P_1, P_2, P_3 (because (†) implies that f_* maps $J_{C/S}[2](S)$ surjectively to $J_{E_S/S}[2](S)$ as f_* is surjective with integral fibres). Thus, if $w_1, \ldots, w_6$ denote the Weierstrass sections, then we have by (11) that $f(w_i) \in E_S[2](S)$ and so $f_*W_{C/S} = \sum f_*[w_j] = \sum_{i=0}^3 n_i[P_i]$. Specializing this equation at the given $s \in S$ yields $(f_s)_*W_{C_s} = (f_*W_{C/S})_s = \sum n_i[(P_i)_s]$. But since the 2-torsion sections of E_S/S are mapped injectively to those of $E_s/\kappa(s)$, it follows that $(P_0)_s = 0, (P_1)_s, \ldots, (P_3)_s$ are precisely the 2-torsion points of E_s. Thus, since f_s is normalized we obtain $n_0 = 3\varepsilon$ and $n_i = (\varepsilon+1)$, for $i > 0$, and so f is also normalized.

(d) Suppose first that N is odd, and put $W_0 := f^*([0_{E_S}]) \times_C W_{C/S}$. We claim that W_0 is an effective relative Cartier divisor on C/S of degree 3. For this we shall use Lemma 7.3 of the appendix. Since the formation of W_0 commutes with base-change, we see by Prop. 2.4(a) that $\deg((W_0)_s) = 3$ for all $s \in S$. Choose a finite faithfully flat base-change S'/S such that we have 6 distinct Weierstrass sections $w_i \in W_{C_{S'}/S'}$. By the proof of (c) we know that $P_i = f(w_i) \in E_S[2](S')$ and so by specializing to any fibre we see that (after renumbering) $0_{E_S}, P_1, P_2, P_3$ are distinct and that $P_4 = P_5 = P_6 = 0_{E_S}$.

Thus, w_4, w_5, w_6 are 3 sections of W_0, and so by Lemma 7.3 we conclude that W_0 is a relative Cartier divisor of C/S.

Thus, $W_0' = W_{C/S} - W_0$ is a also relative Cartier divisor of degree 3 and we have $f_*W_0' = E_S[2]^\#$ (since this is obviously true after a suitable faithfully flat base change). Put $\mathcal{L} := \mathcal{O}_C(W_0') \otimes \omega_{C/S}^{-1} \in \mathrm{Pic}(C)$. Then $\mathcal{L}$ has relative degree 1, and therefore gives rise to a closed embedding $j_{\mathcal{L}} : C \to J$ (cf. §7). By Prop. 2.4, properties (12) hold fibre-by-fibre, and so by the same argument as in (c) we conclude that (12) holds over S.

If N is even, and we have 6 Weierstrass sections of C/S, then we choose 3 of these to obtain a relative Cartier divisor W_0' with the property that $(f_*W_0')_s = E_s[2]^\#$, for one fixed $s \in S$. It then follows that $f_*W_0' = E_S[2]^\#$ because both sides are sums of 2-torsion sections, and the 2-torsion sections of E_S/S are mapped bijectively to those of the fibre $E_s/\kappa(s)$. Then the invertible sheaf $\mathcal{L} = \mathcal{O}_C(W_0') \otimes \omega_{C/S}^{-1}$ defines an embedding $j = j_{\mathcal{L}} : C \to J$ which (by the same argument as in the case that N is odd) satisfies (12).

The last assertion clearly follows from the first because we can always choose a suitable (étale) faithfully flat base-change so as to obtain 6 distinct Weierstrass sections (since $W_{C/S}$ is etale).

(e) Since $j = j_{\mathcal{L}}$ is the embedding defined by an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(C)$ of relative degree $1 = g-1$, it is clear that $\theta = j(C)$ is a theta-divisor (cf. §7). Furthermore, $[-1]\theta = j(\sigma_C(C)) = \theta$ by the first equation of (12), and so θ is symmetric.

Next we show that if N is odd, then θ satisfies (13). For this we first note that $\mathrm{Ker}(f_*)[2]$ is a finite etale group scheme of rank 4 (because $\mathrm{Ker}(f_*)/S$ is an elliptic curve by (42); cf. also Prop. 5.2 below). Furthermore, since it is enough to verify (13) after a faithfully flat base-change, we may assume without loss of generality that $W_{C/S}$ has six Weierstrass sections $w_1, \ldots, w_6$. Then by the discussion of (e) we see that $\mathrm{Ker}(f_*)[2](S) = \{0, j(w_4), j(w_5), j(w_6)\}$ and so (13) follows.

To see that θ is uniquely characterized by this property, we argue as follows. If θ' is another symmetric theta-divisor, then by Lemma 7.1 of the appendix we know that $\theta' = T_x(\theta)$, for some $x \in J_{C/S}(S)$. Moreover, $x \in J_{C/S}[2](S)$ since θ and θ' are both symmetric. By specializing to a fibre J_s we conclude by the uniqueness assertion of Cor. 2.5 that $\theta_s' = T_{x(s)}(\theta_s)$ and that hence $x(s) = 0 = 0(s)$. But then $x = 0$ since $J_{C/S}[2]$ is a finite etale group scheme, and so $\theta' = \theta$. This proves the assertion in case that N is odd, and the case that N is even is proved similarly.

(f) Since the first assertion is just a restatement of (44), we only need to prove the second statement.

Suppose first that C/S satisfies (†), and let w be a Weierstrass section. Then the image $\theta = j_w(C)$ of the embedding $j_w : C \hookrightarrow J_{C/S}$ is a symmetric theta-divisor of $\lambda_{C/S}$, and so there is a unique automorphism $\sigma \in \mathrm{Aut}_S(C)$ such that $j_w \circ \sigma = [-1] \circ j_w$. Clearly, σ fixes the 6 Weierstrass sections and hence $\sigma = \sigma_C$ is the hyperelliptic involution. Put $f' = h^* \circ j_w$, where $h^* = \hat{h} \circ \lambda_{C/S} : J_{C/S} \to \hat{J}_{E_S/S} = E_S$. Then $f' \circ \sigma_C = (h^*) \circ [-1]_J \circ j_w = [-1]_{E_S} \circ (h^*) \circ j_w = [-1]_{E_S} \circ f'$, so f' satisfies (11). Furthermore, $(f')^* = j_w^* \circ (h^*) = -\lambda_{C/S}^{-1} \circ (\lambda_{C/S} \circ h) = -h$, so f' is minimal.

We now claim that there exists an $x \in E_S[2](S)$ such that $f = T_x \circ f'$ is normalized. To see this, fix an $s \in S$. Then by Theorem 2.6 there is a normalized cover $f_s : C_s \to E_s$ such that $(f_s)^* = -h_s$, and so by the Albanese property (cf. §7) there exists $x_s \in E_s(k)$ (where $k = \kappa(s)$) such that $f'_s = T_{x_s} \circ f_s$. Since f_s and f'_s both satisfy (11), it follows that $x_s \in E_s[2](k)$. Since (†) implies that $E_S[2](S) \to E_s[2](k)$ is surjective, there exists an $x \in E_S[2](S)$ such that $f := T_x \circ f'$ specializes at s to the given f_s. Then f also satisfies (11), and so f is normalized by part (c) above. In addition, $f^* = (f')^* = -h$. Thus, replacing f by $f \circ \sigma_C$ yields the desired f (since $\sigma^* = [-1]_J$).

We now remove the hypothesis that C/S satisfies (†) and hence consider an arbitrary genus 2 curve C/S. Then there exists a finite, faithfully flat base change $\beta : S' \to S$ such that $C_{(S')}/S'$ satisfies (†) and so, by what was just shown, there exists a normalized morphism $f' : C_{(S')} \to E_{(S')}$ such that $(f')^* = h_{(S')}$.

It is clearly enough to show that $f' = f_{(S')}$ for some morphism $f : C \to E_S$, for then f is automatically normalized (since f' is) and satisfies $f^* = h$ (because we have $f^*_{(S')} = h_{(S')}$). Furthermore, the existence of f will follow by faithfully-flat descent (cf. [BLR], Th. 6.1/6(a)) once we have shown that $p_1^* f' = p_2^* f'$, where $p_i = pr_i : S'' := S' \times_S S' \to S'$, $i = 1, 2$, denote the two projections and $p_i^* f' : C_{(S'')} \to E_{(S'')}$ denotes the base-change of f' via p_i.

Since f' is normalized, we have by part (d) (or by construction) that there exists $\mathcal{L} \in \mathrm{Pic}(C_{(S')})$ such that the embedding $j = j_{\mathcal{L}} : C_{(S')} \to J_{C_{(S')}/S'}$ satisfies $[-1] \circ j = j \circ \sigma_{C_{(S')}}$ and $f' = h^*_{(S')} \circ j$. Here, as above, $h^* = \hat{h} \circ \lambda_{C/S}$ and so $h^*_{(S')} = \lambda^{-1}_{E_{S'}/S'} \circ (f')_*$. Put $\mathcal{L}_i = p_i^* \mathcal{L} \in \mathrm{Pic}(C_{(S'')})$. Since $\lambda_{\mathcal{O}(\mathcal{L}_i)} = (\lambda_{C/S})_{(S'')}$ and $\lambda_{C/S}$ is a principal polarization, it follows that $x := cl(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}) \in \mathrm{Pic}^0_{C/S}(S'') = J_{C/S}(S'')$, and so $p_1^* j = j_{\mathcal{L}_1} = T_x \circ j_{\mathcal{L}_2} = T_x \circ p_2^* j$.

In addition we note that $x \in J_{C/S}[2](S'')$ because $[-1] \circ p_i^* j = p_i^* j \circ \sigma_{C_{(S'')}}$, for $i = 1, 2$. Now since h^* is defined over S, $p_i^* h^* = h^*_{(S'')}$ does not depend on i and so we obtain $p_1^* f' = h^*_{(S'')} \circ p_1^* j = h^*_{(S'')} \circ T_x \circ p_2^* j = T_{x'} \circ h^*_{(S'')} \circ p_2^* j = T_{x'} \circ p_2^* f'$ with $x' = h^*_{(S'')}(x) \in E_S[2](S'')$.

To prove that $p_1^* f' = p_2^* f'$, it is thus enough to show that $x' = 0$ or, equivalently, that $x \in \mathrm{Ker}(h^*)(S'')$. For this, consider $\theta_i := p_i^* j(C_{(S'')})$, for $i = 1, 2$; note that $\theta_1 = T_x(\theta_2)$. Since $p_i^* f'$ is normalized, we have by part (e) that θ_i is a symmetric theta-divisor of $\lambda_{C_{(S'')}/S''}$ which satisfies $\theta_i \cap \mathrm{Ker}(h^*_{(S'')})[2] = \mathrm{Ker}(h^*_{(S'')})[2]^{\#}$ (resp., $\theta_i \cap \mathrm{Ker}(h^*_{(S'')})[2] = \emptyset$) if N is odd (resp. if N is even) because $\mathrm{Ker}(p_i^*(f')_*) = \mathrm{Ker}(h^*)$. Thus, by the uniqueness assertion of part (e) we obtain $\theta_1 = \theta_2$ (resp. $\theta_1 = T_y \theta_2$ with $y \in \mathrm{Ker}(h^*)[2](S'')$) and so $x = 0$ (resp. $x = y \in \mathrm{Ker}(h^*)[2](S'')$). (Recall that if $\mathcal{O}(\theta_1) \simeq T_z^*(\mathcal{O}(\theta_1))$, then $z = 0$ because $\lambda_{C/S}$ is a principal polarization.) Thus, in both cases $x \in \mathrm{Ker}(h^*)(S'')$ and so $p_1^* f' = p_2^*(f')$, as desired.

For later use let us also observe that the analogue of Proposition 2.1 carries over to genus 2 families and thus has the following important consequence.

Proposition 3.3 *(a) If $f : C \to E_S$ is a minimal genus 2 cover of degree $N \geq 3$ and $\alpha \in \mathrm{Aut}_S(C)$ is an automorphism such that $f \circ \alpha = f$, then $\alpha = id_C$.*

(b) Let $f_i : C_i \to E_S$, $i = 1, 2$ be two normalized genus 2 covers of E_S/S of degree $N \geq 3$, and let $p : S' \to S$ be a faithfully flat, quasi-compact cover of S. If there exists an S'-isomorphism $\alpha' : (C_1)_{(S')} \xrightarrow{\sim} (C_2)_{(S')}$ such that $(f_2)_{(S')} \circ \alpha' = (f_1)_{(S')}$, then there is a unique S-isomorphism $\alpha : C_1 \xrightarrow{\sim} C_2$ such that $\alpha_{(S')} = \alpha'$, and we have $f_2 \circ \alpha = f_1$.

Proof. (a) First note that it follows from Prop. 2.1 (and Th. 3.2(a)) that $\alpha_s = id_{C_s}$, for all $s \in S$. Thus, if S is reduced, then so is C and thus it follows that $\alpha = id_C$ because C/S is separated. (Indeed, let $\gamma : \mathrm{Ker}(\alpha, \beta) \to X$ be the subscheme of coincidences of α and $\beta = id_C$. Then $\mathrm{Ker}(\alpha, \beta)$ is a closed subscheme of X by [EGA], (I, 5.2.5). Furthermore, γ is surjective (since this is true fibre-by-fibre), hence schematically dominant since C is reduced (use [EGA], (I, 5.4.3)). Thus, γ is an epimorphism (by [EGA], (I, 5.4.6)), and so, since $\alpha \circ \gamma = \beta \circ \gamma$ by definition, it follows that $\alpha = \beta$.)

Now suppose that S is an arbitrary scheme; without loss of generality, however, we may assume that S is locally noetherian. Then the map $S_{red} \to$

S is defined by a locally nilpotent ideal of $\mathcal{O}_S$ (cf. [EGA], (I, 4.5.8)). Now since the scheme of automorphisms $\mathrm{Aut}_S(C)$ is finite and unramified (this is a special case of [DM], Th. (1.11)), it follows that the induced map $\mathrm{Aut}_S(C) \to \mathrm{Aut}_{S_{red}}(C_{red})$ is injective (cf. [EGA], (IV, 17.1.2)(iv)). Thus, $\alpha = id_C$, as desired.

(b) It is clearly enough to construct α such that $\alpha_{(S')} = \alpha'$, for then the second property follows since p is faithfully flat. Furthermore, the existence of α will follow by faithfully-flat descent (cf. [BLR], Th. 6.1/6(a)) once we have shown that $p_1^*\alpha' = p_2^*\alpha'$, where $p_i = pr_i : S'' := S' \times_S S' \to S'$ denote the two projections.

Now since $p_1^*(f_i)_{(S')} = p_2^*(f_i)_{(S')} = (f_i)_{(S'')}$, for $i = 1, 2$, we have $(f_2)_{(S'')} \circ p_j^*\alpha' = p_j^*((f_2)_{(S')} \circ \alpha') = p_j^*((f_1)_{(S')}) = (f_1)_{(S'')}$, for $j = 1, 2$. Thus, if we put $\beta = (p_1^*\alpha')^{-1} \circ p_2^*\alpha' \in \mathrm{Aut}_{S''}((C_1)_{(S'')})$, then we obtain $(f_1)_{(S'')} \circ \beta = (f_1)_{(S'')}$. Now since $(f_1)_{(S'')} : (C_1)_{(S'')} \to E_{(S'')}$ is a normalized genus 2 cover by Th. 3.2(a), we have by part (a) that $\beta = id_{(C_1)_{S''}}$, and so $p_1^*\alpha' = p_2^*\alpha'$, as desired.

Notation. Let us now put, for any K-scheme S,

$$\mathcal{H}_{E/K,N}(S) = \{C \xrightarrow{f} E_S : f \text{ is a normalized genus 2 cover of degree } N\}/\simeq,$$

where two covers $f_i : C_i \to E_S$ are called *isomorphic* (notation: $f_1 \simeq f_2$) if there is an isomorphism $\varphi : C_1 \to C_2$ such that $f_1 = f_2 \circ \varphi$.

By Theorem 3.2(a) we see that if $\beta : S' \to S$ is a K-morphism, then the rule $f \mapsto f_{(S')}$ induces a map $\mathcal{H}_{E/K,N}(\beta) : \mathcal{H}_{E/K,N}(S) \to \mathcal{H}_{E/K,N}(S')$, and so we obtain a contravariant functor

$$\mathcal{H}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets},$$

called the *Hurwitz functor of genus* 2 *covers of* E/K.

Remark. Part (b) of the above Proposition 3.3 shows that if $N \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is a *separated presheaf* in the fpqc-topology (cf. [Mi1], p. 49 and/or [BLR], p. 199), which is a necessary condition for the functor to be (finely) representable; cf. [BLR], Proposition 8.1/1.

In fact, if N is invertible in K, then we shall see later (cf. Theorem 5.18) that $\mathcal{H}_{E/K,N}$ is representable by a smooth affine curve $H_{E/K,N}$ which is an open subscheme of the (affine) modular curve $X_{E,N,-1}$ which will be defined and studied in the next section.

4 The modular curves $X_{E/K,N}$ and $X_{E/K,N,\varepsilon}$

For a given elliptic curve E/K, let

$$\mathcal{X}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets}$$

denote the functor which "classifies $E[N]$-structures of elliptic curves". Thus, if S is any K-scheme, then $\mathcal{X}_{E/K,N}(S) = \{(E', \psi)\}/\simeq$ is the set of isomorphism classes of pairs (E', ψ) consisting of an elliptic curve E'/S and an S-isomorphism $\psi : E_S[N] \xrightarrow{\sim} E'[N]$ of the S-group schemes of N-torsion points of E_S and E'; here two such pairs are *equivalent*, i.e. $(E'_1, \psi_1) \simeq (E'_2, \psi_2)$, if there is an S-isomorphism $\varphi : E'_1 \xrightarrow{\sim} E'_2$ such that $\psi_2 = \varphi \circ \psi_1$. Furthermore, if $\beta : S' \to S$ is any K-morphism, then the map $\mathcal{X}_{E/K,N}(\beta) : \mathcal{X}_{E/K,N}(S) \to \mathcal{X}_{E/K,N}(S')$ is given by base-change, i.e. by the assignment $(E', \psi) \mapsto (E'_{S'}, \psi_{(S')})$.

Theorem 4.1 *If $N \geq 3$ is invertible in K, then the functor $\mathcal{X}_{E/K,N}$ is representable by a smooth affine curve $X_{E/K,N}/K$ which is a twist of the usual (affine) modular curve X'_N/K which classifies elliptic curves with level-N-structures. More precisely, if K'/K is any extension field such that the N-torsion points of E are K-rational over K', then $X_{E/K,N} \otimes K' \simeq X'_N \otimes K'$.*

Proof. This is analogous to Corollary 4.7.2 of [KM] and is proved by exactly the same techniques. To explain this more precisely, fix a finite étale group scheme G/K of rank N^2 and consider the moduli problem $\mathcal{P}_{G/K}$ "which classifies G-structures of elliptic curves". By this we mean the contravariant functor

$$\mathcal{P}_{G/K} : \underline{Ell}_{/K} \to \underline{Sets}$$

from the category $\underline{Ell}_{/K}$ of elliptic curves E'/S over variable K-schemes S (cf. [KM], (4.1), (4.13)) defined by $\mathcal{P}_{G/K}(E/S) = \mathrm{Isom}_{S-gp}(G_S, E'[N])$, the set of all S-group isomorphisms $\alpha : G_S = G \times_K S \xrightarrow{\sim} E'[N]$. (Each such α will be called a G-structure of E'/S.)

Note that in the case that $G = G_N := \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ is the constant group scheme of rank N^2, a G-structure is the same as a (naive) level-N-structure (or $\Gamma(N)$-structure) in the sense of [KM], (3.1). Furthermore, since the above functor $\mathcal{X}_{E/K,N}$ depends only on the K-group scheme $E[N]$ (which has rank N^2), we see that this functor is closely related to the functor $\mathcal{P}_{E[N]/K}$; in fact, we have

$$\mathcal{X}_{E/K,N} = \tilde{\mathcal{P}}_{E[N]/K} : \underline{Sch}_{/K} \to \underline{Sets},$$

where $\tilde{\mathcal{P}}$ is the functor on $\underline{Sch}_{/K}$ associated to $\mathcal{P}$ in the sense of [KM], p. 108 and p. 125; i.e. $\tilde{\mathcal{P}}$ classifies isomorphism classes of elliptic curves with $\mathcal{P}$-structures.

Now the same argument as that of [KM], Corollary 4.7.2 shows that $\mathcal{P}_G$ is representable by a smooth affine curve. Indeed, $\mathcal{P}_{G/K}$ is relatively representable and finite étale (since for a given E'/S, the functor (of S-schemes) $T \mapsto \mathrm{Isom}_{T-gp}(G_T, E'_T[N])$ is represented by a finite étale S-scheme; cf. [KM], 1.6.7). Moreover, $\mathcal{P}_{G/K}$ is rigid for $N \geq 3$ (cf. [KM], Cor. 2.7.2), and so $\mathcal{P}_{G/K}$ is representable by [KM], (4.7.0); in particular, $\tilde{\mathcal{P}}_{G/K}$ is representable by [KM], A.4.2 (p. 126).

Applying this to the case that $G = E[N]$ shows that $\mathcal{X}_{E/K,N} = \tilde{\mathcal{P}}_{E[N]/K}$ is representable by a K-scheme $X_{E/K,N}$. Furthermore, if K'/K is as given, then $E'_K[N] \simeq (G_N)_{/K'}$, and so $\mathcal{P}_{E_{K'}[N]} \simeq \mathcal{P}_{(G_N)_{/K'}/K'}$, which implies that $X_{E/K,N} \otimes K' = X_{E_{K'}/K',N} = (X_N)_{/K'}$. Since the latter is a smooth affine curve over K' (cf. [KM], Cor. 4.7.2), it follows (by faithfully flat descent) that $X_{E/K,N}$ is a smooth affine curve over K.

It is well-known that the curves X_N/K and hence $X_{E/K,N}$ are not geometrically connected; indeed, each is a sum of $\phi(N)$ irreducible components over $\overline{K}$. Now for the curves $X_{E/K,N}$, this decomposition already takes place over K (even though $(X_N)_{/K}$ may still be irreducible as a K-scheme). The reason for this is that it is possible to define a *determinant* of an isomorphism $\psi : E_S[N] \to E'[N]$, as the following lemma shows.

Lemma 4.2 *Let E/S and E'/S be elliptic curves over a scheme S, and suppose $\psi : E[N] \xrightarrow{\sim} E'[N]$ is an S-isomorphism, where N is invertible in S. Then there is a unique automorphism $\det(\psi) \in \mathrm{Aut}_S(\mu_N) \simeq ((\mathbb{Z}/N\mathbb{Z})^\times)_{/S}$ such that*

(15) $$e'_N \circ (\psi \times \psi) = \det(\psi) \circ e_N,$$

where $e_N : E[N] \times E[N] \to \mu_N$ and $e'_N : E'[N] \times E'[N] \to \mu_N$ denote the e_N-pairings of E and E'.

Proof. Viewing $E[N]$ as a (locally constant) étale sheaf of $\mathbb{Z}/N\mathbb{Z}$-modules on S_{et}, let $\wedge^2_A E[N]$ be its second exterior power with respect to $A = \mathbb{Z}/N\mathbb{Z}$ (which is formed analogous to the tensor product of étale sheaves; cf. [Mi1], p. 79); thus, $\wedge^2_A E[N]$ is a (locally constant) étale sheaf of A-modules. Since e_N is an alternating pairing (cf. [KM], p. 90, 505), we have an induced A-homomorphism of étale sheaves $\overline{e}_N : \wedge^2 E[N] \to \mu_N$ because for any étale sheaves M,N of A-modules we have the canonical isomorphism (analogous to [Mi1], Prop. II.3.19)

$$\mathrm{Hom}_A(\wedge_A^2 M, N) \simeq \mathrm{Alt}_A(M^2, N).$$

Now since e_N is non-degenerate (cf. [KM], p. 90) it follows that $\overline{e}_N$ is an isomorphism, and hence $\det(\psi) := \overline{e}'_N \circ \wedge^2(\psi) \circ \overline{e}_N^{-1}$ is the unique automorphism of μ_N such that (15) holds.

By the above lemma we thus see that for any $\varepsilon \in (\mathbb{Z}/NZ)^\times$, the rule

$$\mathcal{X}_{E/K,N,\varepsilon}(S) = \{(E', \psi) \in \mathcal{X}_{E/K,N}(S) : \det(\psi) = [\varepsilon]_{\mu_N}\}$$

defines an open and closed subfunctor $\mathcal{X}_{E/K,N,\varepsilon}$ of $\mathcal{X}_{E/K,N}$ which is therefore represented by an open and closed K-subscheme $X_{E/K,N,\varepsilon}$ of $X_{E/K,N}$, and we have the decomposition

$$X_{E/K,N} = \coprod_{\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times} X_{E/K,N,\varepsilon}.$$

In fact, each $X_{E/K,N,\varepsilon}$ is geometrically irreducible, as we shall now show:

Corollary 4.3 *If $N \geq 3$ is invertible in K, then for each $\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times$, the functor $\mathcal{X}_{E/K,N,\varepsilon}$ is representable by a smooth, affine, geometrically irreducible curve $X_{E/K,N,\varepsilon}/K$ which is a twist of the usual (affine) modular curve $X'(N)/K(\zeta_N)$ of level N which classifies elliptic curves with level-N-structures (of fixed determinant ζ_N). More precisely, if K'/K is any extension field such that the N-torsion points of E are rational over K', then $X_{E/K,N,\varepsilon} \otimes K' \simeq X'(N)_{/K'}$.*

Proof. By Igusa and/or Deligne/Rapoport[DM], Cor. IV.5.6, the modular curve $X(N)/K(\zeta_N)$ is smooth and geometrically irreducible and hence so is $X'(N) = X(N) \setminus \{cusps\}$. Since the functor $\mathcal{X}_{E_{K'}/K',N,\varepsilon}$ is isomorphic (over K') to the functor $\mathcal{X}(N)$ defining $X'(N)$, it follows that $X_{E/K,N,\varepsilon} \otimes K' \simeq X'(N)_{/K'}$, and so $X_{E/K,N,\varepsilon}$ is geometrically irreducible.

Remarks. (a) The modular curves $X_{E/K,N}$ and $X_{E/K,N,1}$ were already studied by Frey[Fr1] and Kraus-Oesterlé[KO], respectively.

(b) Although we had tacitly always assumed above that K is a field, this hypothesis is never used. Thus, the same conclusions hold if K is an arbitrary commutative ring (or even an arbitrary scheme).

5 The basic construction

5.1 The functor $\mathcal{A}_{E/K,N}$

We now return to the Hurwitz functor $\mathcal{H}_{E/K,N}$ which was defined in section 3 and show that it is (finely) representable. To this end we shall generalize the "basic construction" of genus 2 covers presented in [FK], [Ka2] so as to obtain an (open) embedding of functors

$$\Psi : \mathcal{H}_{E/K,N} \hookrightarrow \mathcal{X}_{E/K,N,-1}.$$

Before defining this functor in full generality, let us briefly recall that this "basic construction" shows that any (minimal) genus 2 K-cover $f : C \to E$ of degree N determines a unique pair $(E'_f, \psi_f) \in \mathcal{X}_{E/K,n,-1}(K)$ where E'_f/K is the complementary elliptic curve (cf. Proposition 2.7) and $\psi_f : E[N] \xrightarrow{\sim} E'[N]$ is an *anti-isometry*, i.e. an isomorphism of determinant -1, and that conversely for each such a pair (satisfying a suitable additional hypothesis) one can reconstruct the cover $f : C \to E$.

We shall now see that the same statement holds for (normalized) genus 2 covers of E_S/S over an arbitrary base S. However, instead of proving this directly, it is more convenient to divide the above construction into two steps, which amounts to a factorization of Ψ as

$$\Psi = \Psi' \circ \tau : \mathcal{H}_{E/K,N} \hookrightarrow \mathcal{A}_{E/K,N} \xrightarrow{\sim} \mathcal{X}_{E/K,N,-1},$$

in which τ is (essentially) the Torelli map which associates to a curve its (polarized) Jacobian and $\mathcal{A}_{E/K,N}$ is the functor that classifies principally polarized abelian surfaces with an embedding "of degree N" of E. In order to explain these terms more precisely, we introduce the following definition and notation.

Definition. Let J/S be an abelian scheme with a principal polarization $\lambda : J \xrightarrow{\sim} \hat{J}$. Then an injective homomorphism $h : E_S \hookrightarrow J$ is said to have degree N if we have

(16) $$\hat{h} \circ \lambda \circ h = \lambda_{E_S/S} \circ [N]_{E_S}.$$

Notation. Let E/K be an elliptic curve E/K and $N \geq 2$ be an integer. For any K-scheme S let

$$\mathcal{A}_{E/K,N}(S) = \{(J, \lambda, h)\}/\simeq$$

denote the set of isomorphism classes of triples (J, λ, h) consisting of an abelian scheme J/S of relative dimension 2, a principal polarization $\lambda : J \xrightarrow{\sim} \hat{J}$ of J, and an injective homomorphism $h : E_S \hookrightarrow J$ of degree N. Here, two such triples (J, λ, h) and (J', λ', h') are called *isomorphic* if there exists an S-isomorphism $\alpha : J \xrightarrow{\sim} J'$ such that $\lambda' = \hat{h} \circ \lambda \circ h$ and $h' = \alpha \circ h$; we then write $(J, \lambda, h) \simeq (J', \lambda', h')$. It is immediate that if $\beta : S' \to S$ is any K-morphism, then the triple $\beta^*(J, \lambda, h) := (J_{(S')}, \lambda_{(S')}, h_{(S')})$ obtained from (J, λ, h) by base-change induces an element in $\mathcal{A}_{E/K,N}(S')$ and so we have a (functorial) map

$$\mathcal{A}_{E/K,N}(\beta) : \mathcal{A}_{E/K,N}(S) \to \mathcal{A}_{E/K,N}(S').$$

We have thus defined a contravariant functor

$$\mathcal{A}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets}.$$

As a first step of the *basic construction* we define the "Torelli map" τ which was mentioned above.

Proposition 5.1 *If $f : C \to E_S$ is a minimal genus 2 cover of degree N, then $h_f := f^* \circ \lambda_{E_S/S} : E_S \to J$ is an injective homomorphism of degree N. Thus, the rule $(f : C \to E_S) \mapsto (J_{C/S}, \lambda_{C/S}, h_f)$ defines a morphism of functors*

$$\tau : \mathcal{H}_{E/K,N} \to \mathcal{A}_{E/K,N}.$$

Proof. Since f is minimal, $f^* : J_{E_S} \to J_{C/S}$ is an injective homomorphism (by definition). Now by (44) we have $[N]_{J_{E_S/S}} = f_* \circ f^* = \lambda_{E_S/S}^{-1} \circ \widehat{(f^*)} \circ \lambda_{C/S} \circ f^*$, which shows that f^* and hence h_f has degree N.

5.2 The isomorphism $\Psi' : \mathcal{A}_{E/K,N} \to \mathcal{X}_{E/K,N,-1}$

Our next aim is to construct the morphism $\Psi' : \mathcal{A}_{E/K,N} \to \mathcal{X}_{E/K,N,-1}$. For this, we first prove:

Proposition 5.2 *Let J/S be an abelian scheme of relative dimension 2 with a principal polarization $\lambda : J \xrightarrow{\sim} \hat{J}$, and let $h : E_S \to J$ be an injective homomorphism of degree N. If $h^* := \hat{h} \circ \lambda : J \to J_{E_S/S}$ denotes the "dual" of h, then $E'_h := \mathrm{Ker}(h^*)$ is an elliptic curve over S, and there is a unique isomorphism*

$$\psi_h : E_S[N] \xrightarrow{\sim} E'_h[N]$$

of the N-torsion subgroup schemes of E_S/S and E'_h/S such that $h' \circ \psi_h = h_{|E_S[N]}$, where $h' : E'_h = \mathrm{Ker}(h^*) \hookrightarrow J$ denotes the associated closed immersion.

Proof. Since $h : E_S \hookrightarrow J$ is a closed immersion, its "dual" $h^* : J \to J_{E_S/S}$ is surjective and has connected fibres by (42). Clearly, these fibres have dimension $2 - 1 = 1$ and so $E'_h := \mathrm{Ker}(h^*)$ is an elliptic curve over S which has a natural closed immersion $h' : E'_h \hookrightarrow J$.

Consider the fibre product $H := E_S \times_J E'_h$ via the closed immersions h and h':

$$\begin{array}{ccc} H & \stackrel{pr_2}{\to} & E'_h \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle h'} \\ E_S & \stackrel{h}{\to} & J \end{array}$$

Since h and h' are closed immersions, so are pr_2 and pr_1. Moreover, since $E'_h = \mathrm{Ker}(h^*)$ and $h^* \circ h = \lambda_{E_S/S} \circ [N]_{E_S}$, the image of H with respect to pr_1 is $\mathrm{Ker}(h^* \circ h) = \mathrm{Ker}(\lambda_{E_S/S} \circ [N]_{E_S}) = E_S[N]$ and so H is finite and flat of rank N^2 and is annihilated by multiplication by N. Thus, the image of H under pr_2 is contained in $E'_h[N]$ and so is equal to $E'[N]$ since both group schemes have rank N^2. Thus, if H' denotes the common image of $h \circ pr_1 = h' \circ pr_2$ in J, then the restrictions of h and h' to the respective N-torsion subgroups induce isomorphisms

$$h_{|E_S[N]} : E_S[N] \xrightarrow{\sim} H' \quad \text{and} \quad h'_{|E'_h[N]} : E'_h[N] \xrightarrow{\sim} H',$$

and so $\psi_h := (h'_{|E'_h[N]})^{-1} \circ h_{|E_S[N]} : E_S[N] \xrightarrow{\sim} E'_h[N]$ is the desired isomorphism. Note that since h' is a (closed) immersion (hence a monomorphism), ψ_h is uniquely determined by the indicated property.

Corollary 5.3 *If $(h')^* = (h')\hat{} \circ \lambda : J \to J_{E'_h}$, then we have*

$$h(\mathrm{Ker}(h^* \circ h)) = h'(\mathrm{Ker}((h')^* \circ h')) = E_S \times_J E'_h \simeq E_S[N] \qquad (17)$$

and hence the injection $h' : E'_h \hookrightarrow J$ also has degree N.

Proof. The above proof shows that $h(\mathrm{Ker}(h^* \circ h)) = H' = E_S \times_J E'_h \simeq E_S[N]$. To prove that also $h'(\mathrm{Ker}((h')^* \circ h')) = E_S \times_J E'_h$, we first note that by the definition of h' we have the exact sequence (of abelian schemes)

$$0 \to E'_h \stackrel{h'}{\to} J \stackrel{h^*}{\to} J_{E_S} \to 0. \qquad (18)$$

Dualizing this sequence and composing with λ and $\kappa_{E_S} : E_S \xrightarrow{\sim} \hat{J}_{E_S}$ yields the sequence

$$(19) \qquad 0 \to E_S \xrightarrow{h} J \xrightarrow{(h')^*} J_{E_S} \to 0$$

which is again exact by (43); here we have also used the fact that $\lambda^{-1} \circ (h^*)\hat{} \circ \kappa_{E_S} = \lambda^{-1} \circ \hat{\lambda} \circ (\hat{h})\hat{} \circ \kappa_{E_S} = h$. Thus we have $h(E_S) = \mathrm{Ker}((h')^*)$ and hence, since h' is injective, we obtain $h'(\mathrm{Ker}((h')^* \circ h')) = h(E_S) \cap h'(E'_h) = E_S \times_J E'_h$, which proves (17).

To prove that h' has degree N, i.e. that the equation $(h')^* \circ h' = \lambda_{E'_h} \circ [N]$ holds, it is enough (by rigidity) to prove this fibre-by-fibre, so assume $S = \mathrm{Spec}(F)$, where F is a field. Now since $(h')^* \circ h' = \hat{h'} \circ \lambda \circ h$ is a polarization on E'_h (cf. (37)), it follows that $(h')^* \circ h' = \lambda_{E'_h} \circ [n]$ for some $n > 0$ since all polarizations on E'_h are of this form. But since $\mathrm{Ker}((h')^* \circ h') \simeq E_F[N]$ has rank N^2, it follows that $n = N$, and so h' has degree N, as claimed.

By the above proposition, each triple (J, λ, h) determines a pair (E'_h, ψ_h) and hence an element $\mathcal{X}_{E/K,N}(S)$. It is easy to see that this construction is compatible with the equivalence relation of covers; more precisely, we have:

Proposition 5.4 *Suppose J_i/S are two abelian schemes of relative dimension 2 with principal polarizations $\lambda_i : J_i \xrightarrow{\sim} \hat{J}_i$ and that $h_i : E_S \to J_i$ are injective homomorphisms of degree N, for $i = 1, 2$. If $\alpha : J_1 \xrightarrow{\sim} J_2$ is an isomorphism such that $\lambda_2 = \hat{\alpha} \circ \lambda_1 \circ \alpha$ and $h_2 = \alpha \circ h_1$, then there is an induced isomorphism $\alpha' : E'_{h_1} \xrightarrow{\sim} E'_{h_2}$ such that $\alpha' \circ \psi_{h_1} = \psi_{h_2}$, and α' is uniquely characterized by this property if $N \geq 3$. In particular, the assignment $(J, \lambda, h) \mapsto (E'_h, \psi_h)$ defines a morphism of functors*

$$\Psi' : \mathcal{A}_{E/K,N} \to \mathcal{X}_{E/K,N}.$$

Proof. First note that α' is uniquely determined by this property if $N \geq 3$, for if $\alpha'' : E'_{h_1} \xrightarrow{\sim} E'_{h_2}$ is another such isomorphism, then $\beta := (\alpha')^{-1} \circ \alpha'' \in \mathrm{Aut}_S(E'_{h_1})$ satisfies $\beta_{|E'_{h_1}[N]} = id_{E'_{h_1}[N]}$, and so $\beta = id_{E'_{h_1}}$ (cf. [KM], p. 85), which means that $\alpha'' = \alpha'$.

We now prove the existence of α'. Dualizing the relation $h_2 = \alpha \circ h_1$ (and composing with λ_i) yields $h_2^* = h_1^* \circ \alpha^*$, where $\alpha^* = \lambda_2^{-1} \circ \hat{\alpha} \circ \lambda_1 = \alpha^{-1}$, where the latter equality follows from the hypothesis $\lambda_2 = \hat{\alpha} \circ \lambda_1 \circ \alpha$. Thus $h_1^* = h_2^* \circ \alpha$, and so by the universal property of kernels, α induces a (unique) isomorphism $\alpha' : E'_{h_1} = \mathrm{Ker}(h_1^*) \xrightarrow{\sim} E'_{h_2} = \mathrm{Ker}(h_2^*)$ such that $h'_2 \circ \alpha' = \alpha \circ h'_1$, where $h'_i : E'_{h_i} = \mathrm{Ker}(h_i^*) \hookrightarrow J_i$ are the canonical inclusions.

It is now easy to see that $\alpha' \circ \psi_{f_1} = \psi_{f_2}$. Indeed, using the defining equation of ψ_{h_1} (cf. Prop. 5.2), we have $h_2' \circ \alpha' \circ \psi_{h_1} = \alpha \circ h_1' \circ \psi_{h_1} = \alpha \circ (h_1)_{|E_S[N]} = (h_2)_{|E_S[N]}$, which means that $\alpha' \circ \psi_{h_1}$ satisfies the defining equation of ψ_{h_2}, and so $\psi_{h_2} = \alpha' \circ \psi_{h_1}$, as claimed.

Thus the map $(J, \lambda, h) \mapsto (E_h', \psi_h)$ is compatible with the equivalence relations and hence determines a map $\Psi_S' : \mathcal{A}_{E/K,N}(S) \to \mathcal{X}_{E/K,N}(S)$. Since this map is clearly compatible with base-change, $\Psi' = \{\Psi_S'\}_S$ defines a morphism of functors $\Psi' : \mathcal{A}_{E/K,N} \to \mathcal{X}_{E/K,N}$.

We next show that Ψ' maps $\mathcal{A}_{E/K,N}$ into the component $\mathcal{X}_{E/K,N,-1}$ of $\mathcal{X}_{E/K,N}$ and that $\Psi' : \mathcal{A}_{E/K,N} \to \mathcal{X}_{E/K,N,-1}$ is an isomorphism of functors. For this we shall first prove the following result.

Proposition 5.5 *In the situation of Proposition 5.2, let $A := E_S \times_S E_h'$ denote the product surface over S and let $\pi : A \to J$ be defined by*

$$\pi = h \circ pr_{E_S} + h' \circ pr_{E_h'} \tag{20}$$

where $pr_{E_S} : A \to E_S$ and $pr_{E_h'} : A \to E_h'$ are the projections. Then π is an isogeny of degree N^2 whose kernel is $G_{\psi_h} = \mathrm{Graph}(-\psi_h) \subset E_S[N] \times_S E_f'[N] \subset A$, the graph of the isomorphism $-\psi_h$. Furthermore, if $\pi' : J \to A$ is defined by

$$\pi' := i_{E_S/S} \circ \lambda_{E_S/S}^{-1} \circ h^* + i_{E_h'} \circ \lambda_{E_h'/S}^{-1} \circ (h')^* \tag{21}$$

where $i_{E_S} : E_S \hookrightarrow A$ and $i_{E_h'} : E_h' \hookrightarrow A$ are the canonical embeddings and $(h')^* := (h')\hat{} \circ \lambda$, then we have $\pi' \circ \pi = [N]_A$ and $\pi \circ \pi' = [N]_J$. Furthermore, if $\lambda_A = \lambda_{E_S/S} \otimes \lambda_{E_h'/S} : A \xrightarrow{\sim} \hat{A}$ denotes the product polarization, then we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda_A} & \hat{A} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\hat{\pi}'} \\ J & \xrightarrow{\lambda} & \hat{J} \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\hat{\pi}} \\ A & \xrightarrow{\lambda_A} & \hat{A}\,. \end{array} \tag{22}$$

Proof. Let $k : K_\psi := \mathrm{Ker}(\pi) \hookrightarrow A$ denote the closed immersion defined by the kernel of π and let $p = (pr_{E_S})_{|K_\psi} = pr_{E_S} \circ k : K_\psi \to E_S$, and $p' = (pr_{E'})_{|K_\psi} = pr_{E'} \circ k : K_\psi \to E' = E_h'$ denote the restriction of the projection maps of A to K_ψ. Then by the universal property of kernels it follows from (20) that $(K_\psi, p, -p')$ is the fibre product $E_S \times_J E'$ with respect to the maps h and h'. Thus, by the proof of Prop. 5.2, it follows that p and p' factor over the immersions $j_N : E_S[N] \hookrightarrow E_S$ and $j'_N : E'[N] \hookrightarrow E'$ as $p = j_N \circ p_N$ and $p' = j'_N \circ p'_N$, and that $p_N : K_\psi \xrightarrow{\sim} E_S[N]$ and $p'_N : K_\psi \xrightarrow{\sim} E'[N]$ are isomorphisms. Thus, $\psi_h = -p'_N \circ p_N^{-1}$, and so, if $\gamma = \gamma_{-\psi_h} : E_S[N] \to A[N]$ denotes the graph morphism, then we have

$$(j_N \times j'_N) \circ \gamma \circ p_N \;=\; k : K_\psi \hookrightarrow A$$

because $pr_{E_S} \circ (j_N \times j'_N) \circ \gamma \circ p_N = j_N \circ pr_{E_S[N]} \circ \gamma \circ p_N = j_N \circ id_{E_S[N]} \circ p_N = p = pr_{E_S} \circ k$ and $pr_{E'} \circ (j_N \times j'_N) \circ \gamma \circ p_N = j'_N \circ pr_{E'[N]} \circ \gamma \circ p_N = j'_N \circ (-\psi_h) \circ p_N = j'_N \circ p'_N = p' = pr_{E'} \circ k$. Since p_N is an isomorphism, this equation means that $K_\psi = \mathrm{Graph}(-\psi_h)$ (as closed subschemes of A), as desired. Moreover, since $\mathrm{Graph}(-\psi_h) \simeq E_S[N]$ has rank N^2, we see that π is an isogeny (because A/S and J/S both have relative dimension 2).

In order to verify that $\pi' \circ \pi = [N]_A$, we first observe that

$$(23)\quad \lambda_{E_S/S}^{-1} \circ h^* \circ h = [N]_{E_S},\; h^* \circ h' = 0,\; (h')^* \circ h = 0,\; \lambda_{E'/S}^{-1} \circ (h')^* \circ h' = [N]_{E'}.$$

Indeed, the first and last equation of (23) are a restatement of (16) (since both h and h' have degree N by Cor. 5.3), and the second and third equations follow from the exact sequences (18) and (19).

Now by (20), (21) and (23) we obtain $\pi' \circ \pi = (i_{E_S} \circ \lambda_{E_S/S}^{-1} \circ h^* + i_{E'_h} \circ \lambda_{E'_h/S}^{-1} \circ (h')^*) \circ (h \circ pr_{E_S} + h' \circ pr_{E'_h}) = i_{E_S} \circ [N]_{E_S} \circ pr_{E_S} + i_{E'_h} \circ [N]_{E'_h} \circ pr_{E'_h} = (i_{E_S} \circ pr_{E_S} + i_{E'_h} \circ pr_{E'_h}) \circ [N]_A = [N]_A$, as claimed. Furthermore, from this we get $\pi \circ \pi' \circ \pi = \pi \circ [N]_A = [N]_J \circ \pi$, and so $\pi \circ \pi' = [N]_J$ since π is an isogeny and hence an epimorphism in $\underline{AbSch}_{/S}$.

It remains to show that the diagram (22) commutes. For this we first observe that if $\varphi = \varphi_{E_S,E'} : \hat{A} \xrightarrow{\sim} J_{E_S/S} \times_S J_{E'/S}$ is the canonical isomorphism of (39), then

$$(24)\qquad \varphi \circ \hat{\pi} \circ \lambda = i_{J_{E_S/S}} \circ h^* + i_{J_{E'_h/S}} \circ (h')^*$$

because $\varphi \circ \hat{\pi} \circ \lambda = \varphi \circ (h \circ pr_{E_S})\hat{} \circ \lambda + \varphi \circ (h' \circ pr_{E'_h})\hat{} \circ \lambda = \varphi \circ \hat{pr}_{E_S} \circ \hat{h} \circ \lambda + \varphi \circ \hat{pr}_{E'_h} \circ \hat{h}' \circ \lambda = i_{J_{E_S/S}} \circ h^* + i_{J_{E'_h/S}} \circ (h')^*$.

From (24), (20) and (23) we obtain $\varphi \circ \hat{\pi} \circ \lambda \circ \pi = (i_{J_{E_S/S}} \circ h^* + i_{J_{E'_h/S}} \circ (h')^*) \circ (h \circ pr_{E_S} + h' \circ pr_{E'_h}) = i_{J_{E_S/S}} \circ \lambda_{E_S/S} \circ [N]_{E_S} \circ pr_{E_S} + i_{J_{E'_h/S}} \circ \lambda_{E'_h/S} \circ [N]_{E'_h} \circ pr_{E'_h} = (\lambda_{E_S/S} \times \lambda_{E'_h/S}) \circ [N]_A = \varphi \circ \lambda_A \circ \pi' \circ \pi$. Thus, $\varphi \circ \hat{\pi} \circ \lambda \circ \pi = \varphi \circ \lambda_A \circ \pi' \circ \pi$, and hence $\hat{\pi} \circ \lambda = \lambda_A \circ \pi'$ because φ is an isomorphism and π is an isogeny. Thus, the bottom square of (22) commutes, and hence so does the top square since it is the dual of the bottom square.

Remarks. (a) The commutative diagram (22) is essentially the diagram on p. 157 of [FK], where it is proved by a slightly different method (in the case that $S = \mathrm{Spec}(K)$ and $J = J_{C/K}$).

(b) The maps π and π' can also be characterized by the properties

$$\pi \circ i_{E_S} = h \quad \text{and} \quad \pi \circ i_{E'_h} = h', \tag{25}$$

$$pr_{E_S} \circ \pi' = \lambda_{E_S/S}^{-1} \circ h^* \quad \text{and} \quad pr_{E'_h} \circ \pi' = \lambda_{E'_h/S}^{-1} \circ (h')^*. \tag{26}$$

Corollary 5.6 *We have $N\lambda_A = \hat{\pi} \circ \lambda \circ \pi$, and so* $\mathrm{Graph}(-\psi_h) \le A[N]$ *is an isotropic subgroup of $A[N]$ with respect to $e^{N\lambda_A}$. Thus, $\psi_h : E_S[N] \xrightarrow{\sim} E'[N]$ is an anti-isometry, i.e. $e'_N \circ (\psi_h \times_S \psi_h) = [-1]_{\mu_N} \circ e_N$.*

Proof. By commutativity of the bottom square of (22) we obtain $(\hat{\pi} \circ \lambda) \circ \pi = (\lambda_A \circ \pi') \circ \pi = \lambda_A \circ [N]_A$, which proves the first assertion.

Thus, by the functorial property (38), we have for every S-scheme T and for all T-valued points $x, y \in \mathrm{Graph}(-\psi_f) = \mathrm{Ker}(\pi)$ that $e^{N\lambda_A}(x,y) = e^{\hat{\pi}\circ\lambda\circ\pi}(x,y) \overset{(38)}{=} e^{\lambda}(\pi(x), \pi(y)) = e^{\lambda}(0,0) = 1$, which means that $\mathrm{Graph}(-\psi_h)$ is an isotropic subgroup of $A[N]$ (with respect to $e^{N\lambda_A}$).

Since $\lambda_A = \lambda_1 \otimes \lambda_2$ is the product polarization (with $\lambda_1 = \lambda_{E_S/S}$ and $\lambda_2 = \lambda_{E'_h/S}$), we have $e^{N\lambda_A} = (e^{N\lambda_1} \circ (pr_1 \times pr_1)) \cdot (e^{N\lambda_2} \circ (pr_2 \times pr_2))$. Thus, applying this to $x = (x', -\psi_h(x'))$ and $y = (y', -\psi_h(y'))$, where $x', y' \in E_S[N](T)$, yields $e^{N\lambda_1}(x', y') e^{N\lambda_2}(-\psi_h(x'), -\psi_h(y')) = e^{N\lambda_A}(x,y) = 1$, which proves that $-\psi_h$ (and hence also ψ_h) is an anti-isometry.

This therefore shows that Ψ' maps $\mathcal{A}_{E/K,N}$ into the (-1)-component of $\mathcal{X}_{E/K,N}$, i.e. into $\mathcal{X}_{E/K,N,-1}$. In order to show that Ψ' is an isomorphism, we shall construct an inverse morphism $\Psi'' : \mathcal{X}_{E/K,N,-1} \to \mathcal{A}_{E/K,N}$. To this end, we shall prove:

Proposition 5.7 *Suppose E'/S is an elliptic curve with an anti-isometry $\psi : E_S[N] \xrightarrow{\sim} E'[N]$. Let $A = E_S \times_S E'$ and let $G_\psi :=$* $\mathrm{Graph}(-\psi)$ *$\le E_S[N] \times_S E'[N] = A[N]$ denote the graph of $-\psi$. Then the quotient $J = J_\psi := A/G_\psi$ is a projective abelian scheme and the quotient map $\pi = \pi_\psi : A \to J = A/G_\psi$ is an isogeny of degree N^2. Furthermore, there exists a* *unique isogeny* $\pi' : J \to A$ *such that* $\pi' \circ \pi = [N]_A$ *and* $\pi \circ \pi' = [N]_J$. *In addition,* J *has a unique principal polarization* $\lambda_J : J \xrightarrow{\sim} \hat{J}$ *such that* $\lambda_J \circ \pi = \hat{\pi'} \circ \lambda_A$, *where* $\lambda_A = \lambda_{E_S/S} \otimes \lambda_{E'/S}$ *denotes the product polarization of* $A = E_S \times_S E'$, *and then we also have* $\hat{\pi} \circ \lambda_J = \lambda_A \circ \pi'$.

Proof. Since $G_\psi \simeq E_S[N]$ is a finite, flat closed subscheme of the (strongly) projective group scheme A/S, the quotient group scheme $J := A/G_\psi$ exists and is quasi-projective; cf. Raynaud[Ra], Th. 1(iv) and/or [BLR], Th. 8.2/12. Furthermore, it is immediate that the quotient map $\pi : A \to J = A/G_\psi$ is finite, flat, and surjective and so J/S is projective; cf. [EGA], (II, 6.6.4). Since it is immediate that the fibres of $J_s = A_s/(G_\psi)_s$ are geometrically irreducible, we see that J_s is an abelian scheme.

It is thus clear that $\pi : A \to J$ is an isogeny of order N^2 since $G_\psi \simeq E_S[N]$ has rank N^2. Moreover, since $G_\psi \leq A[N] = \mathrm{Ker}([N]_A)$, it follows by the universal property of quotients (cf. [Mu1], p. 3) that $[N]_A = \pi' \circ \pi$, for some homomorphism $\pi' : J \to A$. Clearly, π' is an isogeny (since π and $[N]_A$ are). By the same argument as in the proof of Prop. 5.5 one concludes that $\pi \circ \pi' = [N]_J$ holds as well.

We now construct λ_J. For this, we first note that there can be at most one isomorphism $\lambda : J \xrightarrow{\sim} \hat{J}$ such that $\lambda \circ \pi = \hat{\pi'} \circ \lambda_A$, for π is an epimorphism. Thus, it is enough to prove that λ exists after a faithfully flat base extension S'/S (and that it is a polarization).

Since ψ is an anti-isometry, G_ψ is an isotropic subgroup with respect to $E^{N\lambda_A}$ (see the proof of Cor. 5.6). Now $\lambda_A = \lambda_\Theta$, where $\Theta = pr_1^*\mathcal{O}([0_{E_S/S}]) \otimes pr_2^*\mathcal{O}([0_{E'/S}])$. Thus, by Lemma 5.8 below we know that after a suitable finite, faithfully flat base extension (which, for ease of notation, will be suppressed in the sequel), there is an invertible sheaf $\mathcal{M}$ on J such that $\Theta^N \simeq \pi^*\mathcal{M}$, and so $\lambda_A \circ [N]_A = \lambda_{\Theta^N} = \lambda_{\pi^*\mathcal{M}} = \hat{\pi} \circ \lambda_\mathcal{M} \circ \pi$, the latter by (37). Thus $\hat{\pi'} \circ \lambda_A \circ [N]_A = \hat{\pi'} \circ \hat{\pi} \circ \lambda_\mathcal{M} \circ \pi = \lambda_\mathcal{M} \circ \pi \circ [N]_A$, and so $\hat{\pi'} \circ \lambda_A = \lambda_\mathcal{M} \circ \pi$ since $[N]_A$ is an epimorphism. Thus $\lambda_J = \lambda_\mathcal{M}$ is the desired polarization. Note that since $\deg(\pi) = \deg(\pi') = N^2$, it follows that $\deg(\lambda_J) = \deg(\lambda_A) = 1$, i.e. that λ_J is a principal polarization.

Finally, to prove that the last equation also holds, we multiply the previous equation by $\hat{\pi}$ to obtain $\hat{\pi} \circ \lambda_J \circ \pi = \hat{\pi} \circ \hat{\pi'} \circ \lambda_A = [N]_{\hat{A}} \circ \lambda_A = \lambda_A \circ [N]_A = \lambda_A \circ \pi' \circ \pi$, which implies that $\hat{\pi} \circ \lambda_J = \lambda_A \circ \pi'$ because π is an epimorphism.

In the above proof we had used the following basic fact about the descent of polarizations.

Lemma 5.8 *Let $\pi : A \to B$ be an isogeny of abelian S-schemes and let $\mathcal{L} \in \mathrm{Pic}(A)$ be an ample invertible sheaf on A. If $\mathrm{Ker}(\pi)$ is an isotropic subgroup of $K(\mathcal{L}) := \mathrm{Ker}(\lambda_{\mathcal{L}})$ with respect to the pairing $e^{\mathcal{L}}$, then there exists a finite, faithfully flat extension S'/S and an invertible sheaf $\mathcal{M}'$ on $B_{(S')}$ such that $\mathcal{L}_{(S')} \simeq \pi^*_{(S')}\mathcal{M}'$, where $\mathcal{L}_{(S')}$ denotes the pullback of $\mathcal{L}$ to $A_{(S')}$.*

Proof. This can be proved by a modification of the proof of the Corollary on p. 231 of Mumford[Mu2] (see also [MB2], Cor. VI.1.3). As in [Mu2], §23 and/or [MB1]), let $\mathcal{G}(\mathcal{L})$ denote the theta-group associated to $\mathcal{L}$; recall that $\mathcal{G}(\mathcal{L})$ fits into an exact sequence

$$0 \to \mathbb{G}_m \to \mathcal{G}(\mathcal{L}) \xrightarrow{p} K(\mathcal{L}) \to 0.$$

Let $H = \mathrm{Ker}(\pi)$ and $G = p^{-1}(H)$. Since $e^{\mathcal{L}}$ is given by the commutator of $\mathcal{G}(\mathcal{L})$, the isotropy of H means that G is commutative. Now by [Mi1], Lemma III.4.17 (and its proof) there is finite faithfully flat extension S'/S such that $G_{(S')} \simeq \mathbb{G}_m \times H_{(S')}$ (as group schemes). Thus, there is a homomorphism $\alpha : H_{(S')} \to G_{(S')}$ such that $p \circ \alpha = id$, which means that $\mathcal{L}_{(S')} = \pi^*(\mathcal{M}')$ by [MB1], Th. 4.1.

Corollary 5.9 *In the situation of Proposition 5.7, the maps $h_\psi := \pi_\psi \circ i_{E_S} : E_S \to J_\psi$ and $h'_\psi := \pi_\psi \circ i_{E'} : E' \to J_\psi$ are injective homomorphisms of degree N whose "duals" $h^*_\psi := \hat{h}_\psi \circ \lambda_J$ and $(h'_\psi)^* := \hat{h}'_\psi \circ \lambda_J$ satisfy the relations*

$$h^*_\psi = \lambda_{E_S/S} \circ pr_{E_S} \circ \pi'_\psi \quad \text{and} \quad (h'_\psi)^* = \lambda_{E'/S} \circ pr_{E'} \circ \pi'_\psi, \tag{27}$$

and hence fit into the exact sequences

$$0 \to E' \xrightarrow{h'_\psi} J_\psi \xrightarrow{h^*_\psi} J_{E_S/S} \to 0 \quad \text{and} \quad 0 \to E_S \xrightarrow{h_\psi} J_\psi \xrightarrow{(h'_\psi)^*} J_{E'/S} \to 0. \tag{28}$$

In particular, the assignment $(E', \psi) \mapsto (J_\psi, \lambda_{J_\psi}, h_\psi)$ defines a morphism of functors

$$\Psi'' : \mathcal{X}_{E/K,N,-1} \to \mathcal{A}_{E/K,N}.$$

Proof. Since ψ is an isomorphism, we have $\mathrm{Ker}(h_\psi) = i^{-1}_{E_S}(G_\psi \cap i_{E_S}(E_S)) = \mathrm{Ker}(\psi) = \{0\}$, and so h_ψ is an injective homomorphism. Similarly, h'_ψ is also an injective homomorphism.

We next verify (27). By definition, $h^*_\psi = \hat{h}_\psi \circ \lambda_J = \hat{i}_{E_S} \circ \hat{\pi} \circ \lambda_J = \hat{i}_{E_S} \circ \lambda_A \circ \pi' = \lambda_{E_S/S} \circ pr_{E_S} \circ \pi'$, where the last equality used (40). This proves the first equation of (27), and the second is proved similarly.

The fact that h_ψ and h'_ψ have degree N follows immediately from (27) because $\hat{h}_\psi \circ \lambda_J \circ h_\psi = h^*_\psi \circ h_\psi \overset{(27)}{=} \lambda_{E_S/S} \circ pr_{E_S} \circ \pi' \circ \pi \circ i_{E_S} = \lambda_{E_S/S} \circ pr_{E_S} \circ [N]_A \circ i_{E_S} = \lambda_{E_S/S} \circ [N]_{E_S}$. Thus, h_ψ has degree N, and a similar computation shows that h'_ψ also has degree N.

Next we show that the sequences (28) are exact. By (27) we have $h^*_\psi \circ h'_\psi = \lambda_{E_S/S} \circ pr_{E_S} \circ \pi' \circ \pi \circ i_{E'} = \lambda_{E_S/S} \circ pr_{E_S} \circ i_{E'} \circ [N]_{E'} = 0$, and similarly, $(h'_\psi)^* \circ h_\psi = 0$. Moreover, since h_ψ and h'_ψ are injective, their duals h^*_ψ and $(h'_\psi)^*$ are surjective and have connected fibres (cf. (42)), and so it follows easily that the sequences (28) are exact.

Finally, we verify that the assignment $(E', \psi) \mapsto cl(J_\psi, \lambda_{J_\psi}, h_\psi)$ defines a functor. Indeed, by the above we know that the (isomorphism class of the) triple $(J_\psi, \lambda_{J_\psi}, h_\psi)$ lies in $\mathcal{A}_{E/K,N}(S)$. Furthermore, it is easy to see that this assignment is compatible with the equivalence relation on $\mathcal{X}_{E/K,N,-1}(S)$, for if $(E'', \psi') \simeq (E', \psi)$ via $\alpha : E' \xrightarrow{\sim} E''$ (with $\alpha \circ \psi = \psi'$), then $\tilde{\alpha} := id_{E_S} \times \alpha : A = E_S \times_S E' \xrightarrow{\sim} A' := E_S \times_S E''$ is an isomorphism such that $\tilde{\alpha}(G_\psi) = G_{\psi'}$, and so we have an induced isomorphism $\overline{\alpha} : J_\psi \xrightarrow{\sim} J_{\psi'}$ such that $\pi_{\psi'} \circ \tilde{\alpha} = \overline{\alpha} \circ \pi_\psi$, from which one easily concludes that α defines an isomorphism $(J_\psi, \lambda_{J_\psi}, h_\psi) \simeq (J_{\psi'}, \lambda_{J_{\psi'}}, h_{\psi'})$.

Thus, we have a well-defined map $\Psi''_S : \mathcal{X}_{E/K,N,-1}(S) \to \mathcal{A}_{E/K,N}(S)$. Since this map is clearly compatible with base-change, the collection $\Psi'' = \{\Psi''_S\}_S$ defines the desired morphism of functors $\Psi'' : \mathcal{X}_{E/K,N,-1} \to \mathcal{A}_{E/K,N}$.

Theorem 5.10 *The morphisms*

$$\Psi' : \mathcal{A}_{E/K,N} \to \mathcal{X}_{E/K,N,-1} \quad \textit{and} \quad \Psi'' : \mathcal{X}_{E/K,N,-1} \to \mathcal{A}_{E/K,N}$$

are inverses of each other and hence are isomorphisms.

Proof. Fix a K-scheme S, and let $cl(J, \lambda, h) \in \mathcal{A}_{E/K,N}(S)$. Then by definition $\Psi''_S(\Psi'_S(cl(J, \lambda, h))) = \Psi''_S(cl(E'_h, \psi_h)) = cl(J_{\psi_h}, \lambda_{J_{\psi_h}}, h_{\psi_h})$, where $E'_h = \mathrm{Ker}(\hat{h} \circ \lambda)$ and $J_{\psi_h} = A_h/G_{\psi_h}$ with $A_h = E_S \times_S E'_h$. Now by Prop. 5.5 the map $\pi : A_h \to J$ is the quotient map with respect to G_{ψ_h}, and hence we have a canonical identification $J = J_{\psi_h}$. Furthermore, since $\lambda \circ \pi = \hat{\pi}' \circ \lambda_{A_h}$ (by Prop. 5.5 again), we have (by definition) $\lambda_{J_{\psi_h}} = \lambda$. Finally, $h_{\psi_h} = i_{E_S} \circ \pi = h$ by equation (25), and so $\Psi''_S(\Psi'_S(cl(J, \lambda, h))) = cl(J, \lambda, h)$.

Conversely, let $cl(E', \psi) \in \mathcal{X}_{E/K,N,-1}(S)$. Then $\Psi'_S(\Psi''_S(cl(E', \psi))) = \Psi'_S(cl(J_\psi, \lambda_{J_\psi}, h_\psi)) = cl(E'_{h_\psi}, \psi_{h_\psi})$. Here $J_\psi = A/G_\psi$ and λ_J are as defined in Prop. 5.7 and $h_\psi = \pi_\psi \circ i_{E_S}$ as in Cor. 5.9. Furthermore, $E'_{h_\psi} = \mathrm{Ker}(h^*_\psi)$, and ψ_{h_ψ} is the unique anti-isometry such that $(h_\psi)' \circ \psi_{h_\psi} = (h_\psi)_{|E_S[N]}$, where $(h_\psi)' : E'_{h_\psi} \hookrightarrow J_\psi$ denotes the canonical inclusion.

From the first exact sequence in (28) we see that $E' = \mathrm{Ker}(h_\psi^*)$, and so there exists a unique isomorphism $\alpha : E' \xrightarrow{\sim} E'_{h_\psi}$ such that $(h_\psi)' \circ \alpha = h'_\psi := \pi_\psi \circ i_{E'}$.

We now claim that

$$(29) \qquad \psi_{h_\psi} = \alpha \circ \psi \quad \text{or, equivalently,} \quad (id \times \alpha)(G_\psi) = G_{\psi_{h_\psi}},$$

where, as above, $G_\psi \le E_S \times_S E'$ and $G_{\psi_{h_\psi}} \le E_S \times_S E'_{h_\psi}$ denotes the graph of $-\psi$ and of $-\psi_{h_\psi}$, respectively.

To prove (29), put $\pi = h_\psi \circ pr_{E_S} + (h_\psi)' \circ pr_{E'_{h_\psi}}$. Then we have $\pi \circ (id \times \alpha) = \pi_\psi$ because $(\pi \circ (id \times \alpha)) \circ i_{E_S} = (h_\psi \circ pr_{E_S} \circ i_{E_S}) + ((h_\psi)' \circ \alpha \circ pr_{E'} \circ i_{E_S} = h_\psi = \pi_\psi \circ i_{E_S}$ and $(\pi \circ (id \times \alpha)) \circ i_{E'} = (h_\psi \circ pr_{E_S} \circ i_{E'}) + ((h_\psi)' \circ \alpha \circ pr_{E'} \circ i_{E'} = (h_\psi)' \circ \alpha = h'_\psi = \pi_\psi \circ i_{E'}$.

Now by Prop. 5.5 we have $\mathrm{Ker}(\pi) = G_{\psi_{h_\psi}}$, and so $G_{\psi_{h_\psi}} = \mathrm{Ker}(\pi) = \mathrm{Ker}((id \times \alpha) \circ \pi_\psi) = (id \times \alpha)(\mathrm{Ker}(\pi_\psi)) = (id \times \alpha)(G_\psi)$ since $\mathrm{Ker}(\pi_\psi) = G_\psi$ by definition of π_ψ. This proves (29), and hence α defines an isomorphism $(E', \psi) \simeq (E'_{h_\psi}, \psi_{h_\psi})$. This means that $\Psi'_S(\Psi''_S(cl(E', \psi))) = cl(E'_{h_\psi}, \psi_{h_\psi}) = cl(E', \psi)$, and so Ψ'_S and Ψ''_S are inverse maps of each other.

Remarks. (a) Note that the above theorem does not require any hypotheses on the base field K, and hence is true even if $\mathrm{char}(K)|2N$. In fact, the hypothesis that K is a field was never used, and so the same result holds if E/K is an elliptic curve over an arbitrary ring (or scheme) K.

(b) The above theorem is actually a special case of general result which is valid for arbitrary abelian varieties. More precisely, let E/K be an abelian variety (or abelian scheme) of (relative) dimension d and let $\lambda : E \to \hat{E}$ be a polarization. Fix an integer $g \ge 1$, and consider, for a K-scheme S, the sets

$$\mathcal{A}^{(g)}_{E/K,\lambda}(S) = \{cl(A, \lambda_A, h)\} \quad \text{and} \quad \mathcal{X}^{(g)}_{E/K,\lambda,-1}(S) = \{cl(B, \lambda_B, \psi)\}$$

in which A/S, B/S are abelian schemes of dimension g, $\lambda_A : A \xrightarrow{\sim} \hat{A}$ is a principal polarization, $h : E \hookrightarrow A$ an injective homomorphism of "type λ", i.e. $\hat{h} \circ \lambda \circ h = \lambda$, $\lambda_B : B \to \hat{B}$ is a polarization and $\psi : \mathrm{Ker}(\lambda) \xrightarrow{\sim} \mathrm{Ker}(\lambda_B)$ is an anti-isometry (with respect to the pairings e^λ and e^{λ_B}). These definitions lead to functors $\mathcal{A}^{(g)}_{E/K,\lambda}$ and $\mathcal{X}^{(g)}_{E/K,\lambda,-1}$ which generalize the functors $\mathcal{A}$ and

$\mathcal{X}$ above; clearly $\mathcal{A}_{E/K,N} = \mathcal{A}^{(2)}_{E/K,N\lambda_{E/K}}$ and $\mathcal{X}_{E/K,N,-1} = \mathcal{X}^{(1)}_{E/K,N\lambda_{E/K},-1}$ (if $d = 1$). Now the above proof of Theorem 5.10 can be modified to show that we have in general an isomorphism of functors:

$$\Psi : \mathcal{A}^{(g)}_{E/K,\lambda} \xrightarrow{\sim} \mathcal{X}^{(g-d)}_{E/K,\lambda,-1}. \tag{30}$$

By combining Theorem 5.10 with the fundamental representation result Corollary 4.3 we obtain

Corollary 5.11 *If $N \geq 3$ is invertible in K, then the smooth affine curve $X_{E/K,N,-1}$ represents the functor $\mathcal{A}_{E/K,N}$.*

5.3 The Torelli map $\tau : \mathcal{H}_{E/K,N} \to \mathcal{A}_{E/K,N}$

We now turn to study the Torelli map $\tau : \mathcal{H}_{E/K,N} \to \mathcal{A}_{E/K,N}$ in more detail. The following result may be viewed as a version of *Torelli's theorem* for (special) genus 2 families of curves.

Proposition 5.12 *If $N \geq 3$, then the Torelli map $\tau : \mathcal{H}_{E/K,N} \to \mathcal{A}_{E/K,N}$ is a monomorphism.*

Proof. Fix a K-scheme S and suppose that $f_i : C_i \to E_S$, $i = 1, 2$, are two normalized genus 2 covers of E_S/S and such that $\tau_S(cl(f_1)) = \tau_S(cl(f_2))$. We want to show that $cl(f_1) = cl(f_2)$, i.e. that there exists an S-isomorphism $\varphi : C_1 \to C_2$ such that $f_1 = f_2 \circ \varphi$.

The hypothesis $\tau_S(cl(f_1)) = \tau_S(cl(f_2))$ means that there exists an isomorphism $\alpha : J_{C_1} \xrightarrow{\sim} J_{C_2}$ such that $\hat{\alpha} \circ \lambda_{C_2/S} \circ \alpha = \lambda_{C_1/S}$ and $\alpha \circ h_1 = h_2$, where $h_i := f_i^* \circ \lambda_{E_S/S}$. Since $(f_i)_* = \lambda^{-1}_{J_{E_S/S}} \circ (f_i^*)\hat{} \circ \lambda_{C_i/S} = h_i^*$, the proof of Prop. 5.4 shows that we also have $(f_1)_* = (f_2)_* \circ \alpha$. We now treat the case that N is odd or even separately.

(a) Suppose first that N is odd. Then there exists a sheaf $\mathcal{L}_i \in \mathrm{Pic}(J_{C_i/S})$ of relative degree 1 such that $(f_i)_* \circ j_{\mathcal{L}_i} = \lambda_{E_S/S} \circ f_i$, for $i = 1, 2$; cf. Th. 3.2(d). Put $\theta_i = j_{\mathcal{L}_i}(C_i)$, which is a symmetric theta-divisor of $J_{C_i/S}$, i.e. $\lambda_{\mathcal{O}(\theta_i)} = \lambda_{C_i/S}$ and $[-1]\theta_i = \theta_i$. Since $\lambda_{\mathcal{O}(\alpha^*\theta_2)} = \hat{\alpha} \circ \lambda_{\mathcal{O}(\theta_2)} \circ \alpha = \lambda_{\mathcal{O}(\theta_1)}$, we see that $\theta_1' := \alpha^{-1}\theta_2$ is a theta divisor of $J_{C_1/S}$. Clearly, θ_1' is symmetric because $[-1]\theta_1' = [-1]\alpha^{-1}\theta_2 = \alpha^{-1}[-1]\theta_2 = \theta_1'$. Furthermore, by (13) we have $\theta_1' \cap \mathrm{Ker}((f_1)_*)[2] = \alpha^{-1}(\theta_2 \cap \mathrm{Ker}((f_2)_*)[2]) = \alpha^{-1}(\mathrm{Ker}((f_2)_*)[2]^\#) \overset{(13)}{=} \mathrm{Ker}((f_1)_*)[2]^\#$. Thus, θ_1 and θ_1' are two symmetric theta-divisors satisfying

(13), and so by the uniqueness assertion of Th. 3.2(e) it follows that $\theta_1 = \theta_1'$. This means that $\alpha^{-1}_{|\theta_2} : \theta_2 \xrightarrow{\sim} \theta_1$ is an isomorphism, and hence there is a unique S-isomorphism $\varphi : C_1 \xrightarrow{\sim} C_2$ such that $j_{\mathcal{L}_2} \circ \varphi = \alpha \circ j_{\mathcal{L}_1}$. Then $f_2 \circ \varphi = (f_2)_* \circ j_{\mathcal{L}_2} \circ \varphi = (f_2)_* \circ \alpha \circ j_{\mathcal{L}_1} = (f_1)_* \circ j_{\mathcal{L}_1} = f_1$, as desired.

(b) Now suppose that $N \geq 4$ is even. Prop. 3.3(b) it is enough to verify that the assertion is true after a finite faithfully flat base change S'/S which we can choose by Th. 3.2(d) in such a way that there exist sheaves $\mathcal{L}_i \in \mathrm{Pic}((J_{C_i/S})_{(S')})$ of relative degree 1 such that $(f_i)_* \circ j_{\mathcal{L}_i} = \lambda_{E_S/S} \circ f_i$, for $i = 1, 2$. (In addition, we can assume that $C_{(S')}/S'$ has a section.) By a similar argument as for the odd case we conclude that there exists an $x \in \mathrm{Ker}(f_1)_*[2](S')$ such that $T_x\alpha^{-1}\theta_2 = \theta_1$, where $\theta_i = j_{\mathcal{L}_i}((C_i)_{(S')})$, and so there is a unique S'-isomorphism $\varphi : (C_1)_{(S')} \xrightarrow{\sim} (C_2)_{(S')}$ such that $j_{\mathcal{L}_2} \circ \varphi = \alpha \circ T_{-x} \circ j_{\mathcal{L}_1}$. Since $-x \in \mathrm{Ker}(f_1)_*[2]$, a similar computation as in the odd case shows that $f_2 \circ \varphi = f_1$, and so the assertion follows.

Corollary 5.13 *If $f : C \to E_S$ is a normalized genus 2 cover of degree N, then $E_f' = \mathrm{Ker}(f_*)$ is an elliptic curve over S and we have a unique anti-isometry $\psi_f : E_S[N] \xrightarrow{\sim} E_f'[N]$ such that $(f')^* \circ \psi_f = (f_*)_{|E_S[N]}$, where $(f')^* : E_f' \hookrightarrow J_{C/S}$ denotes the canonical embedding. Furthermore, the rule $(f : C \to E_S) \mapsto (E_f', \psi_f)$ defines a functor*

$$\Psi = \Psi' \circ \tau : \mathcal{H}_{E/K,N} \to \mathcal{X}_{E,N,-1}$$

which is a monomorphism if $N \geq 3$.

Proof. Since $\Psi := \Psi' \circ \tau$ is a monomorphism for $N \geq 3$ by Th. 5.10 and Prop. 5.12, all the assertions follow once we have shown that (E_f', ψ_f) is a representative of $\Psi_S'(\tau_S(cl(f)))$. But this is clear, for by definition $\Psi_S'\tau_S(cl(f)) = \Psi_S'(cl(J_{C/S}, \lambda_{C/S}, h_f)) = cl(E_{h_f}', \psi_{h_f})$, where $h_f = f^* \circ \lambda_{E_S/S}$, $E_{h_f}' = \mathrm{Ker}(h_f^*) = \mathrm{Ker}(f_*) = E_f'$, and $\psi_f := \psi_{h_f}$ is uniquely determined by $h_f' \circ \psi_f = (h_f)_{|E_S[N]}$, with $h_f' = (f')^* : E_f' \hookrightarrow J_{C/S}$.

The final step in the *basic construction* is to identify the image of $\mathcal{H}_{E/K,N}$ in $\mathcal{A}_{E/K,N}$ with respect to the Torelli map τ. The criterion that will be given below amounts essentially to the condition that the theta-divisor(s) associated to the polarization λ_J of an abelian surface J/S be *smooth* over S. However, since a given polarization need not have a theta-divisor that is rational over S, this condition has to be suitably modified. To this end we introduce the following concept.

Definition. A principal polarization $\lambda : J \to \hat{J}$ of an abelian scheme J/S is called *theta-smooth* at $s \in S$ if the principal polarization

$$\lambda_{\overline{s}} : J_{\overline{s}} := J \otimes \overline{\kappa(s)} \xrightarrow{\sim} \hat{J}_{\overline{s}}$$

obtained by base change with $\overline{\kappa(s)}$, the algebraic closure of $\kappa(s)$, has a theta-divisor $\theta_{\overline{s}}$ which is smooth over $\overline{\kappa(s)}$.

We now prove the following fundamental fact which is also of independent interest.

Proposition 5.14 *If J/S is an abelian scheme of relative dimension 2 with a principal polarization $\lambda : J \to \hat{J}$, then following conditions are equivalent:*

(i) (J, λ) is a Jacobian, i.e. there is a smooth curve C/S of genus 2 such that $(J_{C/S}, \lambda_{C/S}) \simeq (J, \lambda)$.

(ii) λ is theta-smooth for all $s \in S$.

(iii) There exists a finite, faithfully flat base extension $\beta : S' \to S$ such that $\lambda_{(S')}$ has an associated theta-divisor θ' which is smooth over S'.

Proof. (i) $\Rightarrow$ (ii): Let $s \in S$. Since $C_{\overline{s}}$ has genus $2 = \dim(J_{\overline{s}})$, the image $j_x(C_{\overline{s}})$ of $j_x : C_{\overline{s}} \hookrightarrow J_{\overline{s}}$ (for any $x \in C_{\overline{s}}(\overline{\kappa(s)})$) is a smooth theta-divisor of $J_{\overline{s}}$ associated to $\lambda_{\overline{s}}$, and so λ is theta-smooth at s, for all $s \in S$.

(ii) $\Rightarrow$ (iii): By Lemma 5.15 below we know that there exists a finite, faithfully flat cover $\beta : S' \to S$ such that $\lambda_{(S')}$ has a theta-divisor θ', and then $\theta'_{\overline{s'}}$ is a theta-divisor of $\lambda_{\overline{s'}}$, for all $s' \in S'$. Since $\theta'_{\overline{s'}}$ is unique up to a translation on $J_{\overline{s'}}$, condition (ii) implies that $\theta'_{\overline{s'}}$ is smooth over $\overline{\kappa(s')}$, and hence $\theta'_{s'}$ is smooth over $\kappa(s')$. Thus, $\theta'_{s'}$ is smooth for all $s' \in S'$, and so it follows that θ' is smooth over S'.

(iii) $\Rightarrow$ (i): We first prove that there exists a smooth curve C/S such that $C' := C_{(S')} \simeq \theta'$. For this we shall use the method of descent of Grothendieck (cf. [BLR], Th. 6.1/7) applied to the pair $(\theta', \mathcal{L}')$, where $\mathcal{L}' = \omega_{\theta'/S'}$; note that $\omega_{\theta'/S'}$ is an ample invertible sheaf on θ' because θ' is a smooth curve of genus 2 over S'.

To apply this method, consider $S'' := S' \times_S S'$ with the two projections $p_i : S'' \to S'$ and let $\theta''_i := p_i^* \theta' = \theta'_{(p_i)}$ be the base change of θ' via p_i, for $i = 1, 2$. Now θ''_1 and θ''_2 are both theta-divisors on $J'' := J_{(S'')}$ with respect to the principal polarization $\lambda'' := \lambda_{(S'')}$, and so by Lemma 7.1 there exists a unique section $x \in J''(S'')$ such that $T_x^*(\theta''_2) = \theta''_1$, i.e. $t := (T_x)_{|\theta''_1} : \theta''_1 \xrightarrow{\sim} \theta''_2$ is an isomorphism of S''-curves. Furthermore, since $p_i^*\mathcal{L}' = \omega_{\theta_i''/S''} = \Omega^1_{\theta_i''/S''}$, it follows that we have a canonical sheaf isomorphism $\omega : t^*p_2^*\mathcal{L}' \simeq p_1^*\mathcal{L}'$. Using the fact that x and ω are uniquely defined, it is now easy to check that the pair (t, ω) satisfies the cocyle condition and hence defines a descent datum on $(\theta', \mathcal{L}')$. Thus, by Grothendieck's theorem ([BLR], Th. 6.1/7), there exists a scheme C/S (and a sheaf $\mathcal{L} \in \mathrm{Pic}(C)$) such that $C_{(S')} \simeq \theta'$. Furthermore, since θ'/S' is a smooth curve of genus 2, so is C/S (since S'/S is faithfully flat).

To show that $J_{C/S} \simeq J$, let $j' : C' := C_{(S')} \simeq \theta' \hookrightarrow J'$ denote the embedding constructed above, and consider the morphism $f' : C' \times_{S'} C' \to \hat{J}'$ defined by $f' = \lambda \circ s \circ j' \times j'$, where $s : J' \times_{S'} J' \to J'$ denotes the minus map $(x, y) \mapsto x - y$. We claim that $f' = f_{(S')}$, for some $f : C \times_S C \to \hat{J}$. By descent theory ([BLR], Th. 6.1/6(a)), it is enough to show that $p_1^* f' = p_2^* f'$, where as above $p_i : S'' = S' \times_S S' \to S'$ are the projections.

Suppose first that $x'', y'' \in C(S'')$ are two sections. Then $(p_i^* f')(x'', y'') = \lambda(p_i^* j'(x'') - p_i^* j'(y'')) = cl((T^*_{p_i^* j'(x'')} p_i^* \mathcal{O}(\theta') \otimes (p_i^* \mathcal{O}(\theta'))^{-1}) \otimes (T^*_{p_i^* j'(y'')} p_i^* \mathcal{O}(\theta') \otimes (p_i^* \mathcal{O}(\theta'))^{-1})^{-1}) = cl(T^*_{p_i^* j'(x'')} p_i^* \mathcal{O}(\theta') \otimes (T^*_{p_i^* j'(y'')} p_i^* \mathcal{O}(\theta'))^{-1})$. Now by the above we know that there exists an $x \in J(S'')$ such that $p_2^* j' = T_x \circ p_1^* j'$ and $T_x^* p_2^* \mathcal{O}(\theta') \simeq p_1^* \mathcal{O}(\theta')$, and so $T^*_{p_2^* j'(x'')} p_2^* \mathcal{O}(\theta') \simeq T^*_{p_1^* j'(x'')} T_x^* p_2^* \mathcal{O}(\theta') \simeq T^*_{p_1^* j'(x'')} p_1^* \mathcal{O}(\theta')$ (and similarly for x'' replaced by y''.) This therefore gives $p_1^*(f')(x'', y'') = cl(T^*_{p_1^* j'(x'')} p_1^* \mathcal{O}(\theta') \otimes (T^*_{p_1^* j'(y'')} p_1^* \mathcal{O}(\theta'))^{-1}) = cl(T^*_{p_2^* j'(x'')} p_2^* \mathcal{O}(\theta') \otimes (T^*_{p_2^* j'(y'')} p_2^* \mathcal{O}(\theta'))^{-1}) = p_2^*(f')(x'', y'')$. Repeating the same argument with arbitrary T-valued points (for a scheme T/S'') shows that we have $p_1^*(f') = p_2^*(f')$.

Thus, by descent theory, there exists a unique morphism $f : C \times_S C \to \hat{J}$ such that $f_{(S')} = f'$. It is immediate from the definition that $f \circ \delta = 0$, where δ is the diagonal map, and so by the Albanese property $(***)$ (applied to $\lambda^{-1} \circ f : C \times C \to J$) there exists a unique homomorphism $h : J_{C/S} \to J$ such that $h \circ f_{C/S} = \lambda^{-1} \circ f$. Now h is an isomorphism because after base-change it is clear from the definition that $f' = \lambda \circ f_{C'/S'}$.

Above we had used:

Lemma 5.15 *If $\lambda : A \to \hat{A}$ is a polarization of an abelian scheme A/S, then there exists a finite, faithfully flat base extension $\beta : S' \to S$ such that $\lambda_{(S')}$ is defined by an invertible sheaf $\mathcal{L}' \in \mathrm{Pic}(A_{(S')})$. Furthermore, if λ is a principal polarization, then $\mathcal{L}'$ can be chosen such that $\mathcal{L}' \simeq \mathcal{O}(\theta')$ for a* *theta-divisor* θ' *on* $A_{(S')}$.

Proof. First note that by Mumford[Mu1], p. 121, there exists $\mathcal{L} \in \mathrm{Pic}(A)$ such that $\lambda_{\mathcal{L}} = 2\lambda$. Then $A[2] \leq K(\mathcal{L})$ is an isotropic subgroup, and so by Lemma 5.8 there exists a finite, faithfully flat base extension $\beta_0 : S'_0 \to S$ such that $\mathcal{L}_{(S'_0)} \simeq [2]^*(\mathcal{L}'_0)$, for some $\mathcal{L}'_0 \in \mathrm{Pic}(A_{(S'_0)})$. Then by a similar argument as in [Mu2], p. 231, there exist, after a suitable finite, faithfully flat base-change $\beta_1 : S' \to S'_0$, invertible sheaves $\mathcal{L}' \in \mathrm{Pic}(A')$ and $\mathcal{M}' \in \mathrm{Pic}(S')$ such that $\mathcal{L}_{(S')} \simeq (\mathcal{L}')^2 \otimes p^*_{A'}(\mathcal{M}')$, where $A' = A_{(S')}$. Then $2\lambda_{(S')} = \lambda_{\mathcal{L}_{(S')}} = \lambda_{(\mathcal{L}')^2} = 2\lambda_{\mathcal{L}'}$, and so $\lambda_{(S')} = \lambda_{\mathcal{L}}$. This proves the first assertion.

To prove the second assertion, consider $\mathcal{M} := p_*\mathcal{L}'$, where $p : A_{(S')} \to S'$ is the structure map. Then $\mathcal{M}$ is locally free of rank 1 cf. [Mu1], p. 123; recall that λ is now assumed to be a principal polarization. Thus $\mathcal{M} \in \mathrm{Pic}(S)$, and hence $\mathcal{L}'' := \mathcal{L}' \otimes (p^*\mathcal{M})^{-1}$ satisfies $p_*(\mathcal{L}'') \simeq O_S$. Via this isomorphism, the global section $id_{\mathcal{O}_S} \in \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S) = \Gamma(S, \mathcal{O}_S)$ gives rise to a global section of $\mathcal{L}''$ which defines a relative Cartier divisor θ' representing $\mathcal{L}''$ (see the discussion on p. 212ff of [BLR]), and so the second assertion follows.

Corollary 5.16 *If* λ *is a principal polarization of an abelian scheme* J/S *of relative dimension* 2*, then the subset* $S^{\theta-sm}(\lambda) \subset S$ *of those points of* S *at which* λ *is theta-smooth is an open subset of* S.

Proof. By the lemma there exists a finite faithfully flat base extension $\beta : S' \to S$ such that $\lambda_{(S')}$ has a theta-divisor $\theta \subset J_{(S')}$. Then by definition (cf. [EGA], (IV, 17.3.7)) the set $\theta^{sm} \subset \theta$ of smooth points of θ/S' is an open subset of θ and hence $S'_0 = S' \setminus p(\theta \setminus \theta^{sm})$ is also open (since the structure map $p = p_{J_{(S')}} : J_{(S')} \to S'$ is proper). By definition (and [EGA], (IV.17.8.2)) S'_0 is the set of points $s \in S$ where θ_s is smooth, and so $S'_0 = \beta^{-1}(S^{\theta-sm}(\lambda))$. Since β is faithfully flat (and hence is open and surjective), it follows that $S^{\theta-sm}(\lambda)$ is open in S.

Notation. For each K-scheme S let

$$\mathcal{J}_{E/K,N}(S) = \mathcal{A}^{\theta-sm}_{E/K,N}(S) \subset \mathcal{A}_{E/K,N}(S)$$

denote the subset consisting of those classes $cl(J, \lambda, h)$ such that J is theta-smooth (for all $s \in S$). Since any base-change of a theta-smooth abelian scheme is again theta-smooth, this defines a subfunctor $\mathcal{J}_{E/K,N}$ of $\mathcal{A}_{E/K,N}$.

Proposition 5.17 *The functor $\mathcal{J}_{E/K,N}$ is an open subfunctor of $\mathcal{A}_{A/K,N}$. Furthermore, the Torelli morphism $\tau : \mathcal{H}_{E/K,N} \to \mathcal{A}_{E/K,N}$ factors over $\mathcal{J}_{E/K,N}$ and defines a surjection*

$$\tau : \mathcal{H}_{E/K,N} \to \mathcal{J}_{E/K,N}$$

which is an isomorphism for $N \geq 3$. In particular, τ is an open embedding of functors (if $N \geq 3$).

Proof. Let S be a K-scheme and let $F : \mathbf{h}_S \to \mathcal{J}_{E/K,N}$ be a morphism of functors. If we write $F(id_S) = cl(J, \lambda, h) \in \mathcal{A}_{E/K,N}(S)$, then by Cor. 5.16 the set $U := S^{\theta-sm}(\lambda) \subset S$ at which λ is theta-smooth is an open subset of S, and so $F(id_S)_{|U} := (J_{(U)}, \lambda_{(U)}, h_{(U)} \in \mathcal{J}_{E/K,N}(U)$. Clearly, $f(id_S)_{|U}$ represents the fibre product $\mathcal{J}_{E/K,N} \times_{\mathcal{A}_{E/K,N}} \mathbf{h}_S$, which means that $\mathcal{J}_{E/K,N}$ is an open subfunctor of $\mathcal{A}_{E/K,N}$. This proves the first assertion.

Since each Jacobian is theta-smooth, it is immediate that τ maps into $\mathcal{J}_{E/K,N}$. To prove that τ is surjective, let S be a K-scheme and let $cl(J, \lambda, h) \in \mathcal{J}_{E/K,N}(S)$. Then by Prop. 5.14 there exists a smooth curve C/S of genus 2 such that $(J_{C/S}, \lambda_{C/S}) \simeq (J, \lambda)$. Moreover, by Th. 3.2(f) there exists a normalized cover $f : C \to E_S$ of degree N such that $f^* = h$. Thus $\tau_S(cl(f)) = cl(J, \lambda, h)$, and so τ is surjective. Moreover, if $N \geq 3$ then τ is an isomorphism because τ_S is injective by Cor. 5.13.

This, therefore, completes the "basic construction". We can now summarize our results as follows.

Theorem 5.18 *The functor $\Psi : \mathcal{H}_{E/K,N} \to \mathcal{X}_{E/K,N,-1}$ is an open embedding of functors if $N \geq 3$; in particular, Ψ is relatively representable. Thus, if $N \geq 3$ is invertible in K, then $\mathcal{H}_{E/K,N}$ is represented by a smooth curve $H_{E/K,N}/K$ which is an open subscheme of the (twisted) modular curve $X_{E/K,N,-1}$. In particular, the fibres of $H_{E/K,N}/K$ are geometrically irreducible.*

Proof. Recall that $\Psi = \Psi' \circ \tau$ (cf. Cor. 5.13). Since Ψ' is an isomorphism by Theorem 5.10 and τ is an open embedding by Prop. 5.17 (if $N \geq 3$), the first assertion follows.

Now if $N \geq 3$ is invertible in K, then by Cor. 4.3 the twisted modular curve $X_{E/K,N-1}$ represents the functor $\mathcal{X}_{E/K,N,-1}$, and so by the first assertion $\mathcal{H}_{E/K,K}$ is represented by an open subscheme $H_{E/K,N}$ of $X_{E/K,N,-1}$. Since $X_{E/K,N,-1}$ is a smooth K-curve with geometrically connected fibres (cf.

Cor. 4.3), the same is true for $H_{E/K,N}$ if (and only if) each geometric fibre $(H_{E/K,N})_{\overline{s}}$ is not empty.

To see that this is the case, write $k = \kappa(\overline{s})$ and let E'/k be an elliptic curve which is not isogenous to E_k. Since k is algebraically closed (and $\mathrm{char}(k) \not| N$), there exists an anti-isometry $\psi : E_k[N] \xrightarrow{\sim} E'[N]$, which determines a triple $cl(J, \lambda, h) = \Psi''_k(cl(E_k, E', \psi)) \in \mathcal{A}_{E/K,N}(k)$. But then λ is automatically theta-smooth (cf. [FK], Prop. 1.4 or Theorem 6.1 below), and so $\mathcal{H}_{E/K,N}(k) \neq \emptyset$ by Prop. 5.17.

Remark. Note that the above Theorem 5.18 includes Theorem 1.1 of the introduction as a special case.

The "basic construction" itself is the following statement, which follows from the fact that $\tau : \mathcal{H}_{E/K,N} \to \mathcal{J}_{E/K,N}$ is surjective (Proposition 5.17) and that Ψ' is bijective (Theorem 5.10).

Corollary 5.19 (Basic Construction) *If E'/S is an elliptic curve and $\psi : E_S[N] \to E'[N]$ is an anti-isometry which is "theta-smooth" (in the sense that the induced principal polarization λ_J on $J_\psi = (E_S \times_S E')/\mathrm{Graph}(-\psi)$ is theta-smooth), then there is a normalized genus 2 cover $f : C \to E_S$ of degree N such that the associated anti-isometry ψ_f is equivalent to ψ. Moreover, every normalized genus 2 cover of degree $N \geq 2$ arises in this way.*

Remarks. (a) A small blemish in the above result is that the notion of a theta-smooth anti-isometry is defined in a round-about way. However, by using the results of [Ka2] it is possible to give a definition that is *intrinsic* to the data (E', ψ) as will be explained in the next section; cf. Theorem 6.1.

(b) Although the basic idea and underlying steps of this "basic construction" are as in [FK], there is one notable difference in the method of proof. In [FK], the existence of the curve C associated to a (theta-smooth) anti-isometry ψ was proved by using (for N odd) a characterization of a suitable symmetric theta-divisor in terms of its linear equivalence class; cf. [FK], Proposition 1.1. This was replaced here by a different characterization in terms of a condition involving certain 2-torsion points of J_ψ; cf. Theorem 3.2(e) (together with some descent arguments).

6 The degeneracy locus $D_{E/K,N}$

In the previous section we had constructed the moduli space $H_{E/K,N}$ as an open subset of the twisted modular curve $X_{E/K,N-1}$. In order to complete the description of $H_{E/K,N}$, we need to analyze the "degeneracy locus"

$$D_{E/K,N} := X_{E/K,N,-1} \setminus H_{E/K,N}.$$

As a set[1], $D_{E/K,N}$ is the disjoint union of its fibres $(D_{E/K,N})_s = D_{E_s/\kappa(s),N}$, and so it is enough to describe these. Thus, we may assume in the sequel that $K = \kappa(s)$ is a field.

For an algebraically closed field K, the results of [Ka2] yield an explicit description of the set $D_{E/K,N}$. To be able connect these results with the set $D_{E/K,N}$, we first recall that Proposition 5.17 yields a natural identification

(31) $$D_{E/K,N} = \{cl(E',\psi) \in X_{E/K,N,-1}(K) : \psi \text{ is reducible}\}$$

where, as in [Ka2], p. 98, an anti-isometry $\psi : E[N] \to E'[N]$ is called *reducible* if its associated theta-divisor θ on J_ψ is reducible. Note that since the theta-divisor θ is reducible if and only if it is not smooth, the anti-isometry ψ is reducible if and only if it is not theta-smooth in the sense of Corollary 5.19.

Several characterizations of reducible anti-isometries were given in [Ka2]; one of these is the following.

Theorem 6.1 *An isometry $\psi : E[N] \xrightarrow{\sim} E'[N]$ is reducible if and only if there exist four isogenies $f_i : E \to E_i$ and $f'_i : E_i \to E'$, for $i = 1, 2$, satisfying the following conditions:*

$$\begin{aligned}
&(32) & \deg(f_1) + \deg(f_2) &= N, \\
&(33) & \operatorname{Ker}(f_1) \cap \operatorname{Ker}(f_2) &= \{0\}, \\
&(34) & f'_1 \circ f_1 &= -f'_2 \circ f_2, \\
&(35) & (f'_i)^* \circ \psi &= (f_i)_{|E[N]}, \quad \textit{for } i = 1, 2,
\end{aligned}$$

where $(f'_i)^* = \lambda_{E_i}^{-1} \circ \hat{f}'_i \circ \lambda_{E'} : E' \to E_i$ denotes the "dual" of f'_i.

[1] Although $D_{E/K,N}$ is a priori just a closed subset of $H_{E/K,N}$, we can give it a unique subscheme structure by observing that it is the underlying set of the subscheme which represents a suitable closed subfunctor of $\mathcal{X}_{E/K,N,-1}$. However, since this extra structure is not required in the sequel, we do not explain this in more detail here.

In particular, if N is prime, then ψ is reducible if and only if there exists an integer k with $0 < k < N$ and an isogeny $f : E \to E'$ of degree $k(N-k)$ such that

$$f_{|E[N]} = k\psi.$$

Proof. The first assertion is Corollary 2.4 of [Ka2] (with f_2' replaced by $-f_2'$), and the second is explained in Remark 2.5.

In addition, the main results of [Ka2] and [Ka3] can be re-interpreted to yield a formula for the number of points in the degeneracy locus $D_{E/K,N}$ (which is finite by Theorem 1.1).

Theorem 6.2 *The number of points in $D_{E/K,N}$ is*

$$(36)\ \#D_{E/K,N} = \frac{1}{2}\sum_{k=1}^{N-1} \sigma(E, k(N-k), N) \leq \frac{1}{24N}(5N-6)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}),$$

where $\sigma(E, n, N)$ denotes the number of subgroup schemes $H \leq E$ of order n such that $E[p] \not\leq H$, for all primes $p|N$. Furthermore, equality holds in (36) if and only if $\mathrm{char}(K)\not| N!$.

Proof. As in [Ka2], p. 107, let $r(E, E', N)$ denote the number of reducible anti-isometries $\psi : E[N] \to E'[N]$. Then from formula (31) it follows that

$$\#D_{E/K,N} = \sum_{E'} \frac{r(E, E', N)}{\#\mathrm{Aut}(E')},$$

where the sum extends over a system of representatives of the isomorphism classes of elliptic curves E'/K. But this sum is exactly the left hand side of the "mass formula" (4.1) of Theorem 4.1 of [Ka2], p. 115, and so the (first) equality of (36) is just a restatement of the formula (4.1). Similarly, the inequality of (36) is a restatement Theorem 6 of [Ka3], in which it is also asserted that equality holds if and only if $\mathrm{char}(K)\not| N!$.

Remark. The above result shows that if $\mathrm{char}(K) = 0$, then $D_{E/K,N}$ consists asymptotically of $\frac{5}{4\pi^2}N^3$ points because

$$\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = N^3 \prod_{p|N}\left(1 - \frac{1}{p^2}\right) \sim \frac{1}{\zeta(2)}N^3 = \frac{6}{\pi^2}N^3.$$

7 Appendix: Jacobians of relative curves

The purpose of this appendix is review (and extend) some of the basic facts concerning the Jacobians of relative curves C/S. The main references for these facts are [Mu1], chapter 6, and [BLR], chapter 9. In addition, we review some facts concerning relative Cartier divisors.

Definition. If $g \geq 1$, then a *(relative) curve of genus g* is a proper, smooth morphism $p : C \to S$ (of finite presentation) whose fibres C_s are geometrically connected curves of genus g. If $g = 1$, then we assume in addition that there exists a section $\eta : S \to C$; in this case it follows from [Mu1], p. 124, that C/S is an abelian scheme of relative dimension 1 with zero-section $0_C = \eta$.

Basic Facts: 1) Projectivity: Each relative curve C/S is strongly projective in the sense of Altman-Kleiman; cf. [BLR], p. 211. More precisely, if $g \geq 2$, then the relative dualizing sheaf $\omega_{C/S}$ is S-ample (in fact, $\omega_{C/S}^{\otimes n}$ is relatively very ample for $n \geq 3$ by [DM], p. 78), and if $g = 1$ then the invertible sheaf $\mathcal{O}_C(\eta)$ associated to the section η is S-ample (cf. [BLR], Remark 9.3/2) (in fact, $\mathcal{O}_C(\eta)^{\otimes n}$ is relatively very ample for $n \geq 3$).

2) Jacobians: The (relative) Picard-functor $\mathrm{Pic}_{C/S}$ is representable by a smooth, separated S-scheme $\mathrm{Pic}_{C/S}$ which is locally of finite presentation, and we have a decomposition

$$\mathrm{Pic}_{C/S} = \coprod_{n \in \mathbb{Z}} J_{C/S}^{(n)}$$

where $J_{C/S}^{(n)} = (\mathrm{Pic}_{C/S})^n$ denotes the open and closed subscheme of $\mathrm{Pic}_{C/S}$ consisting of all line bundles of degree n; cf. [BLR], Th. 9.3/1. Thus, for each S-scheme T we have a functorial injection

$$cl : \mathrm{Pic}^{(n)}(X_T)/p_{(T)}^*\mathrm{Pic}(T) \hookrightarrow J_{C/S}^{(n)}(T) = \mathrm{Hom}(T, J_{C/S}^{(n)})$$

which is surjective if and only there exists a universal sheaf on $C \times_S J^{(n)}$; cf. [MB3], p. 34. Furthermore, $J_{C/S}^{(0)} = (\mathrm{Pic}_{C/S})^0$ coincides with the identity component $J_{C/S} := \mathrm{Pic}^0_{C/S}$ of $\mathrm{Pic}_{C/S}$ and is an abelian scheme over S ([BLR], loc. cit. and Prop. 9.4/4). In addition, each $J_{C/S}^{(n)}$ is strongly projective over S and is an S-torsor under $J_{C/S}$; cf. [BLR], Th. 8.2/5, Th. 9.3/1 or [MB3], p. 34. In particular, if there is an invertible sheaf $\mathcal{L} \in \mathrm{Pic}^{(n)}(C)$ of relative degree n, then the map $g \mapsto g \cdot cl(\mathcal{L})$ defines an isomorphism $T_{\mathcal{L}} : J_{C/S} \xrightarrow{\sim} J_{C/S}^{(n)}$ of S-schemes.

3) Duality and Polarizations: The dual abelian scheme $\hat{J}_{C/S} := \mathrm{Pic}^0_{J/S}$ of $J = J_{C/S}$ exists and there is an explicit isomorphism $\lambda_{C/S} : J_{C/S} \xrightarrow{\sim} \hat{J}_{C/S}$; which is described in more detail in 5) below.

More generally, for any (projective) abelian scheme A/S, its dual $\hat{A} = \mathrm{Pic}^0_{A/S} = \mathrm{Pic}^\tau_{A/S}$ exists and is a (projective) abelian scheme (cf. [Mu1], p. 117 and [FC], p. 3); here, $\mathrm{Pic}^\tau_{A/S}(T)$ consists of those $\mathcal{L} \in \mathrm{Pic}(A_T)/\mathrm{Pic}(T)$ such that (a power of) $\mathcal{L}_t$ is algebraically equivalent to 0, for all $t \in T$. By functoriality, any homomorphism $f : A \to B$ of abelian schemes induces a homomorphism $\hat{f} = f^* : \hat{B} \to \hat{A}$, called the dual homomorphism. Furthermore, the canonical map $\kappa_A : A \to \hat{\hat{A}}$ is an isomorphism (cf. [FC], p. 3; this follows from [Mu2], p. 132), and we have $f = \kappa_B^{-1} \circ \hat{\hat{f}} \circ \kappa_A$.

Each invertible sheaf $\mathcal{L} \in \mathrm{Pic}(A)$ defines an S-homomorphism $\lambda_\mathcal{L} : A \to \hat{A}$ by the rule $x \mapsto cl(T_x^*\mathcal{L} \otimes \mathcal{L}^{-1})$ (functorially in S); cf. [Mu1], p. 120. Note that if $h : B \to A$ is any S-homomorphism, then we have

$$\lambda_{h^*\mathcal{L}} = \hat{h} \circ \lambda_\mathcal{L} \circ h, \tag{37}$$

as is immediate from the definition of $\hat{h}$.

If $\mathcal{L}$ is (relatively) ample, then $\lambda_\mathcal{L}$ is an isogeny and hence a *polarization* in the sense of Mumford[Mu1], p. 120. (However, not every polarization is of this form.) Note that if $\lambda : A \to \hat{A}$ is a polarization, then λ is symmetric in the sense that $\hat{\lambda} \circ \kappa_A = \lambda$; cf. [FC], p. 4.

Each polarization $\lambda = \lambda_\mathcal{L} : A \to \hat{A}$ induces a (non-degenerate) pairing

$$e^\lambda : K(\lambda) \times_S K(\lambda) \to (\mathbb{G}_m)_{/S},$$

where $K(\lambda) = \mathrm{Ker}(\lambda)$; cf. [Mu2], p. 228ff and [FC], p. 5. This pairing is related to the usual e_N-pairing $e_N : A[N] \times_S \hat{A}[N] \to \mu_N$ by the formula $e_N \circ (id_{A[N]} \times \lambda) = e^{N\lambda}$, if λ is a principal polarization (and N is invertible in S); cf. formula (4) on p.228 of [Mu2]. We also note that if $\pi : B \to A$ is an isogeny, then we have the formula

$$e^{\hat{\pi}\circ\lambda\circ\pi} = e^\lambda \circ \pi \times \pi \quad \text{on} \quad \pi^{-1}K(\lambda) \times \pi^{-1}K(\lambda); \tag{38}$$

cf. property (1) of [Mu2], p. 228 (combined with the fact that $\hat{\pi}\circ\lambda_\mathcal{L}\circ\pi = \lambda_{\pi^*\mathcal{L}}$, as is evident from the definitions).

4) Products of Abelian Schemes: It is immediate that the category $\underline{AbSch}_{/S}$ of abelian S-schemes (with group homomorphisms) is an additive category (in the sense of [HS], p. 75). Thus, if A_1, A_2 are two abelian S-schemes, then $A = A_1 \times_S A_2$ is a co-product of A_1 and A_2 in $\underline{AbSch}_{/S}$ with respect to the maps $i_{A_1} = id_{A_1} \times 0_{A_2} : A_1 = A_1 \times_S S \to A$ and $i_{A_2} = 0_{A_2} \times id_{A_2} : A_2 = S \times_S A_2 \to A$; cf. [HS], Prop. II.9.1. Furthermore, by duality it then follows that $\hat{A}$ is a product of $\hat{A}_1$ and $\hat{A}_2$ with respect to the morphisms $\hat{i}_{A_1}$ and $\hat{i}_{A_2}$, and so there is a unique isomorphism

(39) $\varphi_{A_1,A_2} : \hat{A} \xrightarrow{\sim} \hat{A}_1 \times \hat{A}_2$ such that $pr_{\hat{A}_k} \circ \varphi_{A_1,A_2} = \hat{i}_{A_k}$, for $k = 1, 2$;

we then also have that $\varphi_{A_1,A_2} \circ \hat{pr}_{A_k} = i_{\hat{A}_k}$, for $k = 1, 2$.

Note that if $\lambda_k : A_k \to \hat{A}_k$, $k = 1, 2$ are two polarizations, then the unique map $\lambda = \lambda_1 \otimes \lambda_2 : A \to \hat{A}$ such that $\varphi_{A_1,A_2} \circ \lambda = \lambda_1 \times_S \lambda_2$, or, equivalently, such that

(40) $$\hat{i}_{A_k} \circ (\lambda_1 \otimes \lambda_2) = \lambda_k \circ pr_{A_k}, \quad \text{for } k = 1, 2,$$

is again a polarization (since this is true fibre-by-fibre), called the *product polarization* of $A = A_1 \times_S A_2$ defined by λ_1 and λ_2.

For later use we also note that if $f : A \to B$ is a homomorphism of abelian schemes then

(41) $$f \text{ is finite} \quad \Leftrightarrow \quad \hat{f} \text{ is surjective.}$$

(Indeed, since f is finite (respectively, surjective) if and only if f_s has this property for all $s \in S$ (observe that f is proper and use [EGA], (IV, 18.12.4), respectively, (I, 3.6.1)), it is enough to verify this for $S = \mathrm{Spec}(K)$, K a field, in which case this is known; cf. [La], p. 125.) Furthermore we have

(42) f is a closed immersion $\Leftrightarrow$ $\mathrm{Ker}(\hat{f})$ is an abelian subscheme of $\hat{B}$ and $\hat{f}$ is surjective.

(Again, this is known in the case that the base is a field (cf. [La], p. 216). The general case be reduced to the field case by noting that f is a closed immersion if and only if $f_s : A_s \to B_s$ has this property for all $s \in S$ (use [EGA], (IV, 18.12.6)) and similarly, $\mathrm{Ker}(\hat{f})$ is an abelian subscheme if and only if $\mathrm{Ker}(f_s) = \mathrm{Ker}(f)_s$ is an abelian subvariety (i.e. is geometrically reduced and irreducible).) Note that it follows easily from (42) that if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence of abelian S-schemes, then the dual sequence is also exact (and conversely):

(43) $$0 \to \hat{C} \xrightarrow{\hat{g}} \hat{B} \xrightarrow{\hat{f}} \hat{A} \to 0.$$

5) Theta-divisors: If $J_{C/S}$ is the Jacobian of a relative curve C/S of genus g, then there is a canonical principal polarization $\lambda_{C/S} : J_{C/S} \xrightarrow{\sim} \hat{J}_{C/S}$; cf. [Mu1], p. 118 or [MB3], p. 40. In general, $\lambda_{C/S}$ *is not* of the form $\lambda_{\mathcal{L}}$, for some $\mathcal{L} \in \mathrm{Pic}(C)$, even if $S = \mathrm{Spec}(K)$, K a field. However, if C/S has a section or, more generally, if C/S has an invertible sheaf of relative degree $g-1$, then there exits a *theta-divisor* θ on $J_{C/S}$, i.e. an effective relative Cartier divisor $\theta \subset J_{C/S}$ such that $\lambda_{C/S} = \lambda_{\mathcal{O}(\theta)}$; cf. [MB3], p. 40. More precisely, θ is constructed as follows. Let $\Theta = Im(C^{g-1} \to J^{(g-1)}_{C/S})$, which is an effective relative Cartier divisor on $J^{(g-1)}$ (for any base S); cf. [MB3], p. 35. If we have an invertible sheaf $\mathcal{L}$ of relative degree $g-1$, then it induces an isomorphism $T_{\mathcal{L}} : J_{C/S} \xrightarrow{\sim} J^{(g-1)}_{C/S}$, and the inverse image $\theta_{\mathcal{L}} = T^*_{\mathcal{L}}\Theta$ of Θ under $T_{\mathcal{L}}$ is a theta-divisor on $J_{C/S}$.

Note that any two theta-divisors are translates of each other:

Lemma 7.1 *Let D be an effective relative Cartier divisor on a strongly projective abelian scheme $p : A \to S$ and let $\mathcal{L} = \mathcal{O}_A(D)$ be its associated invertible sheaf.*

(a) If $H^0(A_s, \mathcal{L}_s) = 1$ and $H^1(A_s, \mathcal{L}_s) = 0$, for all $s \in S$, then the canonical map $p_*\mathcal{O}_A = \mathcal{O}_S \to p_*\mathcal{L}$ is an isomorphism. Thus, if D' is any effective relative Cartier divisor such that $cl(\mathcal{O}_A(D')) = cl(\mathcal{L})$ in $\mathrm{Pic}_{A/S}(S)$, then $D' = D$ (as effective Cartier divisors).

(b) If $\mathcal{L}$ is ample and defines a principal polarization $\lambda_{\mathcal{L}} : A \xrightarrow{\sim} \hat{A}$, and D' is an effective relative Cartier divisor such that $\lambda_{\mathcal{O}_A(D')} = \lambda_{\mathcal{L}}$ (or such that $cl(\mathcal{O}_A(D' - D)) \in \mathrm{Pic}^0_{A/S}(S)$), then $D' = T^*_x(D)$, for some $x \in A(S)$.

Proof. (a) Without loss of generality we may assume that S is locally noetherian. Since D is an effective divisor, it defines an injection $f_D : \mathcal{O}_A \hookrightarrow \mathcal{O}_A(D) = \mathcal{L}$, which induces an injection $p_*(f_D) : p_*(\mathcal{O}_A) = \mathcal{O}_S \hookrightarrow p_*(\mathcal{L})$. Now the hypotheses on $\mathcal{L}$ imply that $R^1p_*\mathcal{L} = 0$ and that $p_*\mathcal{L}$ is locally free and that $p_*\mathcal{L}_s \otimes \kappa(s) \xrightarrow{\sim} H^0(A_s, \mathcal{L}_s)$ is an isomorphism; cf. [Ha], Th. III.12.11. Since $H^0(A_s, \mathcal{O}_{A_s}) \xrightarrow{\sim} H^0(A_s, \mathcal{L}_s)$ is an isomorphism, it follows that $p_*(f_D)_s : p_*(O_A)_s \xrightarrow{\sim} p_*(\mathcal{L})_s$ is surjective, for all $s \in S$, and hence so is $p_*(f_D) : p_*(\mathcal{O}_A) \to p_*(\mathcal{L})$ by Nakayama. Thus, $p_*(f_D)$ is an isomorphism.

Moreover, since the above result is also true after an arbitrary base-change, it follows that $p_*\mathcal{L}$ is cohomologically flat in dimension 0. Thus, by [BLR], Prop. 8.2/7, the set of effective relative Cartier divisors $D' \in \mathrm{Div}^+(A/S)$ of A/S with $cl(\mathcal{O}_A(D')) = cl(\mathcal{L})$ (in $\mathrm{Pic}_{A/S}(S)$) is parametrized by $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is the dual of $p_*(\mathcal{L}) \simeq \mathcal{O}_S$. But then $\mathbb{P}(\mathcal{F}) = S$, and so $\mathrm{Hom}_S(S, \mathbb{P}(\mathcal{F})) = \{id_S\}$, which proves the assertion.

(b) First note that if $\lambda_{\mathcal{L}'} = \lambda_{\mathcal{L}}$ then $\lambda_{\mathcal{L}\otimes\mathcal{L}^{-1}} = 0$ and hence $\mathcal{L}' \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(A/S)$ (by the definition of $\mathrm{Pic}^0(A/S)$); in particular, $cl(\mathcal{L}' \otimes \mathcal{L}^{-1}) \in \mathrm{Pic}^0_{A/S}(S) = \mathrm{Pic}^0(A/S)/p^*\mathrm{Pic}(S)$.

Now since $\lambda_{\mathcal{L}} : A \xrightarrow{\sim} \hat{A} = \mathrm{Pic}^0_{A/S}$ is an isomorphism, there exists an $x \in A(S)$ such that $cl(\mathcal{O}_A(D' - D)) = cl(T_x^*\mathcal{L} \otimes \mathcal{L}^{-1})$, and so $cl(\mathcal{O}_A(D')) = cl(T_x^*\mathcal{L}) = cl(\mathcal{O}(T_x^*D))$ in $\mathrm{Pic}_{A/S}(S)$. By the Riemann-Roch Theorem and the Vanishing Theorem ([Mu2], p. 150) $\mathcal{L}$ (and hence also $T_x^*\mathcal{L}$) satisfies the hypotheses of part (a), and hence it follows that $D' = T_x^*(D)$, as desired.

6) The Autoduality of Jacobians: If $a : S \to C$ is a section of C/S, then there is an S-morphism $j_a : C \to J_{C/S}$ which represents the map $z \mapsto cl(\mathcal{O}_C(z - a))$; cf. [MB3], p. 43. Furthermore, the induced map $j_a^* : \hat{J}_{C/S} = \mathrm{Pic}^0_{J/S} \to J_{C/S} = \mathrm{Pic}^0_{C/S}$ is an isomorphism which satisfies $\lambda_{C/S} \circ j_a^* = -id_J$ ([MB3], p. 43); in particular, j_a^* does not depend on a. Note that j_a is a closed immersion since this is known to be true fibre-by-fibre and so we can apply [EGA] (IV, 18.12.6) again. Furthermore, j_a has the following universal property (usually called the *Albanese property* of $J_{C/S}$):

$(*)$ *If A/S is an abelian scheme and $f : C \to A$ an S-morphism such that $f(a) = 0_A$, the zero-section of A/S, then there is a unique S-homomorphism $f_* : J_C \to A$ such that $f = f_* \circ j_a$; in fact, f_* is given by the formula $f_* = -\kappa_A^{-1} \circ (f^*)^* \circ \lambda_{C/S}$.*

[*Proof.* Suppose first that there exists an f_* with $f = f_* \circ j_a$. Then the map $f^* : \hat{A} = \mathrm{Pic}^0_{A/S} \to J_{C/S} = \mathrm{Pic}^0_{C/S}$ induced by f satisfies $\lambda_{C/S} \circ f^* = \lambda_{C/S} \circ (f_* \circ j_a)^* = \lambda_{C/S} \circ j_a^* \circ (f_*)^* = -(f_*)^*$. Dualizing this equation and composing it with κ_A^{-1} and κ_J yields $f_* = \kappa_A^{-1} \circ (f_*)^{**} \circ \kappa_J = -\kappa_A^{-1} \circ (\lambda_{C/S} \circ f^*)^* \circ \kappa_J = -\kappa_A^{-1} \circ (f^*)^* \circ \lambda_{C/S}^* \circ \kappa_J = -\kappa_A^{-1} \circ (f^*)^* \circ \lambda_{C/S}$, the latter by the symmetry of $\lambda_{C/S}$. This proves the uniqueness of f_* and the asserted formula for f_*.

To prove the existence of f_*, define f_* by this formula. Since this definition commutes with base-change, we have for each $s \in S$ that $(f_*)_s \circ j_{a,s} = f_s$ because the autoduality property holds over a field (cf. [Mi2], p. 185) (and by uniqueness). But then $f, f_* \circ j_a : C \to A$ are two S-morphisms whose fibres agree, and so by rigidity ([Mu1], p. 116) there is a section η of $p_A : A \to S$ such that $f = (\eta \circ p_C) \cdot (f_* \circ j_a)$. But since $f(a) = 0_A = (f_* \circ j_a)(a)$, it follows that $\eta = 0_A$, and so $f = f_* \circ j_a$, as claimed.]

More generally, if $\mathcal{L} \in \mathrm{Pic}(C)$ has relative degree 1, then there is a closed immersion $j_{\mathcal{L}} : C \to J_{C/S}$ which represents the map $z \mapsto cl(\mathcal{O}_C(z) \otimes \mathcal{L}^{-1})$, and we have $\lambda_{C/S} \circ j_{\mathcal{L}}^* = -id_J$. Furthermore, *if C/S has a section*, then it follows from $(*)$ that $j_{\mathcal{L}}$ satisfies the following universal property:

$(**)$ *If $f : C \to A$ any S-morphism to an abelian scheme A/S, then there is a unique S-homomorphism $f_* : J_{C/S} \to A$ and section $\eta \in A(S)$ such that $f = T_\eta \circ f_* \circ j_{\mathcal{L}}$.*

There is another universal property (also called the *Albanese property*) which is sometimes more useful since it does not require the existence of sections and hence applies to an arbitrary base S. For this, let $s_{C/S} : C \times_S C \to J_{C/S}$ be the morphism defined by the rule $s_{C/S}(x, y) = cl(\mathcal{O}(x - y))$, where $x, y \in C(T)$ are T-valued points and T is any S-scheme. Note that if $\delta : C \to C \times_S C$ denotes the diagonal morphism, then $s_{C/S} \circ \delta = 0$ (the constant map). Then by a similar argument as in [Mi2], we have the following analogue of [Mi2], Prop. 6.4:

$(***)$ *If $f : C \times_S C \to A$ is any S-morphism to an abelian scheme A/S such that $f \circ \delta = 0$, then there is a unique S-homomorphism $f_* : J_{C/S} \to A$ such that $f = f_* \circ s_C$.*

7) Covers of Relative Curves: If N and $g' \geq 1$ are positive integers, then a *genus g' cover of C/S of degree N* is an S-morphism $f : C' \to C$ where C'/S is curve of genus g' such that for each $s \in S$ the induced morphism $f_s : C'_s \to C_s$ has degree N. Since each f_s is then automatically finite, surjective and flat (the latter since the local rings of C_s are discrete valuation rings), it follows (by using [EGA], (IV, 11.3.11)) that f a finite, flat and surjective morphism. Each such cover induces homomorphisms

$$f^* : J_{C/S} \to J_{C'/S} \quad \text{and} \quad f_* := \lambda_{C/S}^{-1} \circ (f^*)^* \circ \lambda_{C'/S} : J_{C'/S} \to J_{C/S},$$

where first is given by functoriality and the second by dualizing the first (and composing with $\lambda_{C'/S}$ and $\lambda_{C/S}^{-1}$). Note that if C'/S has a section $a' \in C'(S)$, then the homomorphism $(j_{f(a)} \circ f)_* : J_{C'/S} \to J_{C/S}$ defined by the Albanese property $(*)$ is the same as f_*, i.e. we have $f_* = (j_{f(a)} \circ f)_*$, as can be seen by using the formula for $(j_{f(a)} \circ f)_*$ given in $(*)$ (and a short computation). Thus we have for any section $x' \in C'(S)$:

$$f_*(cl(\mathcal{O}_{C'}(x' - a'))) = cl(\mathcal{O}_C(f(x') - f(a'))).$$

We also observe that we have the formula

$$f_* \circ f^* = [N]_{J_{C/S}}, \tag{44}$$

where $[N]_{J_{C/S}}$ denotes the multiplication by N map on $J_{C/S}$. (Indeed, if $S = \mathrm{Spec}(K)$ is a field, then this is well-known (by using the previous formula). Thus, for an arbitrary base S, we see that the formula holds fibre-by-fibre, and hence is true over S by rigidity ([Mu1], p. 116) since both sides are homomorphisms of abelian schemes.)

Definition. An S-cover $f : C' \to C$ (of degree N) is called *minimal* if any one of the following equivalent conditions holds:

(i) $f^* : J_{C/S} \to J_{C'/S}$ is a closed immersion;

(ii) $(f_s)^* : J_{C_s/\kappa(s)} \to J_{C'_s/\kappa(s)}$ is a closed immersion for all $s \in S$;

(iii) $f_* : J_{C'/S} \to J_{C/S}$ is surjective and $\mathrm{Ker}(f_*)$ is an abelian subscheme of $J_{C'/S}$.

[To see that these conditions are equivalent, note that the following implications hold:

(iii) $\Rightarrow$ (i): Use (42).

(i) $\Rightarrow$ (ii): Clear, since $(f_s)^* = (f^*)_s$ and since the base change of any closed immersion is again a closed immersion by [EGA], (I, 4.3.6).

(ii) $\Rightarrow$ (iii): By (42) we know that $(f_*)_s = (f_s)_*$ is surjective for all $s \in S$, and hence so is f_*. Thus f_* is flat (use [EGA] (IV, 11.3.11)), and hence so is $\mathrm{Ker}(f_*)/S$. Furthermore, f_* has connected fibres (since each $(f_*)_s$ does by (42)), and so $\mathrm{Ker}(f_*)$ is an abelian subscheme of $J_{C/S}$.]

Note that condition (i) shows that if f is minimal, then so is any base change $f_{(T)} : C'_{(T)} \to C_{(T)}$ (because the base change of any closed immersion is again a closed immersion).

For later use we also append here some basic facts about relative Cartier divisors.

8) Relative Cartier Divisors: These are defined for any scheme X/S which is flat and of finite presentation; cf. [EGA], §IV.21.15, [KM], §1.1 or [BLR], p. 212ff. For any base-change S'/S, the pull-back $D_{S'}$ of a relative Cartier divisor $D \in \mathrm{Div}(X/S)$ is defined and is a relative Cartier divisor of $X_{(S')}/S'$; cf. [EGA], (IV, 21.15.8).

If $f : C' \to C$ is an S-cover of curves, then we have induced homomorphisms

$$f^* : \mathrm{Div}(C/S) \to \mathrm{Div}(C'/S) \quad \text{and} \quad f_* : \mathrm{Div}(C'/S) \to \mathrm{Div}(C/S)$$

between the groups $\mathrm{Div}(C/S)$ and $\mathrm{Div}(C'/S)$ of relative Cartier divisors, and these map effective divisors to effective divisors (cf. [EGA], §IV.21.15). In addition, we have $f_*f^*D = ND$, for all $D \in \mathrm{Div}(C/S)$; cf. [EGA], (IV, 21.5.6).

We note that the formation of f^* commutes with base-change by [EGA], (IV, 21.15.9), and the same is true for f_* by Lemma 7.2 below. In addition, for divisors $D \in \mathrm{Div}(C/S)$ and $D' \in \mathrm{Div}(C'/S)$ we have

$$\deg(f^*D) = \deg(f)\deg(D) \quad \text{and} \quad \deg(f_*D') = \deg(D').$$

Indeed, the first formula is Lemma 1.2.8 of [KM], and the second follows (by base-change) from the corresponding formula for curves over a field.

Furthermore, if $a' \in C'(S)$ is a section and $[a'] \in \mathrm{Div}(C'/S)$ denotes the associated relative Cartier divisor (cf. [KM], Lemma 1.2.2), then we have

$$f_*[a'] = [f(a')],$$

as can be seen by tracing through the definitions. (The discussion of [KM], p. 33, could also be used here.)

Lemma 7.2 *Let $p : X \to S$ be flat and locally of finite presentation, and let $f : X' \to X$ be finite and flat. Then for every relative Cartier divisor $D' \in \mathrm{Div}(X'/S)$, the direct image $f_*D' \in \mathrm{Div}(X/S)$ is a relative Cartier divisor of X/S, and its formation commutes with every base-change $g : S' \to S$ in the sense that we have the equality*

$$(f_{(S')})_*(g^*_{(X')}(D') = g^*_{(X)}(f_*D'), \quad \textit{for all } D' \in \mathrm{Div}(X'/S). \tag{45}$$

Proof. The first assertion is proved in [EGA], (IV, 21.15.8). To prove the second, recall first that the pullbacks of relative Cartier divisors is always defined ([EGA], (IV, 21.15.9)), and so both sides of (45) are defined. Thus, the same proof of [EGA], Prop. (IV, 21.5.8) applies (where this formula is proven for arbitrary Cartier divisors but only for flat base-change so as to guarantee that the relevant pull-backs are defined).

Over general base S, equality of two relative Cartier divisors $D, D' \in \mathrm{Div}^+(C/S)$ of C/S cannot be decided in the fibres C_s; indeed, this is already false for sections. However, *if S is reduced*, then it follows from [KM], Cor. 1.3.5 that

$$D = D' \quad \Leftrightarrow \quad D_s = D'_s, \text{ for all } s \in S, \tag{46}$$

where $D_s, D'_s \in \mathrm{Div}^+(C_s)$ denote the effective Cartier divisors induced by D and D' on the fibre C_s.

The following lemma gives some other criteria for deciding equality (in a slightly more general setting):

Lemma 7.3 *Let C/S be a relative curve and let $Z \subset C$ be a closed subscheme which is finite over S and which has the property that $d := \deg(Z_s)$ is independent of $s \in S$.*

(a) If there exists an effective Cartier divisor $D \subset Z$ of degree d, then $Z = D$.

(b) If Z has d sections $z_1, \ldots, z_d \in Z(S)$ such that for every $s \in S$, the points $z_1(s), \ldots, z_d(s) \in Z_s$ are distinct, then Z is an effective relative Cartier divisor of C/S which is etale over S, and we have $Z = [z_1] + \ldots + [z_d]$.

(c) If there exists a finite, faithfully flat base extension S'/S such that $Z \times_S S'$ satisfies the hypothesis of (b), then $Z \in \mathrm{Div}^+(C/S)$ is etale over S.

Proof. (a) Let $f : D \hookrightarrow Z$ denote the closed immersion. For any $s \in S$, the induced map $f_s : D_s \leq Z_s$ is an isomorphism because both are closed subschemes of degree d of smooth curve C_s. Now by hypothesis D is flat, of finite presentation over S, and so f is an isomorphism by [EGA], (IV, 17.9.5).

(b) Let $D = [z_1] + [z_2] + \ldots + [z_d] \in \mathrm{Div}^+(C/S)$; clearly, $z_1, \ldots, z_d$ are sections of D, and D is etale over S; in fact, $D = S \coprod \ldots \coprod S$ (cf. [KM], Lemma 1.8.3). We thus have a natural morphism $f : D \to Z$ which induces isomorphisms $f_s : D_s \to Z_s$, for all $s \in S$. Thus, by the argument in (a) we see that $f : D \simeq Z$ is an isomorphism.

(c) By (b), $Z \times_S S'$ is etale over S', hence Z is etale over S by [EGA], (IV, 17.7.3).

Remark. If, in the situation of Lemma 7.3, the base S is an *integral* noetherian scheme, then Z is automatically an effective relative Cartier divisor. (Indeed, since C/S and hence Z/S is projective, and the (constant) Hilbert polynomial of Z_s is $d = \deg(Z_s)$ which does not depend on $s \in S$ by hypothesis, it follows that Z is flat over S by [Ha], Th. III.9.9.)

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