Hurwitz Spaces of Genus 2
Covers of Elliptic Curves

Ernst Kani

Abstract:
Let $E$ be an elliptic curve over a field $K$ of characteristic $\neq 2$ and let $N > 1$ be an integer prime to $\text{char}(K)$. The purpose of this paper is to construct the moduli space $H_{E/K,N}$ which “classifies” the set $\text{Cov}_{E/K,N}(K)$ of genus 2 covers of $E$ of degree $N$ which are normalized in a certain sense.

More precisely, it is shown here that the assignment $L \mapsto \text{Cov}_{E/K,N}(L)$ (where $L/K$ is any field extension) extends in a natural way to a Hurwitz functor

$$\mathcal{H}_{E/K,N} : \text{Sch}_K \to \text{Sets}$$

which is analogous to the Hurwitz functors considered by Fulton[Fu] (where the base is $\mathbb{P}^1$), and that we have

**Theorem.** If $N \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,N}/K$ which is an open subset of a certain twist $X_{E/K,N,1}$ of the modular curve $X(N)$ of level $N$; in particular, $H_{E,N} \otimes \overline{K}$ is isomorphic to an open subset of $X(N)/\overline{K}$, where $\overline{K}$ denotes the algebraic closure of $K$.

Since the above “twisted modular curve” $X_{E/K,N,1}/K$ is the moduli space which classifies pairs $(E',\psi)$ where $E'/K$ is an elliptic curve and $\psi : E[N] \cong E'[N]$ is a $K$-rational anti-isometry of the $N$-torsion subgroups of $E$ and $E'$, the above theorem may be viewed as a refinement and extension of the “basic construction” of genus 2 curves (with elliptic differentials) presented in Frey/Kani[FK] in which it was shown how to construct a genus 2 cover of $E/K$ from the data $(E',\psi)$. 