Mazur’s question on mod 11 representations of elliptic curves *

E.J. Kani and O.G. Rizzo

1 Introduction

In 1978, Barry Mazur [14] asked the following question:

**Question 1.** Do there exist two elliptic curves $E_1/\mathbb{Q}, E_2/\mathbb{Q}$ which are not isogenous over $\mathbb{Q}$ such that their associated Galois representations

$$\rho_{E_i, N} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(E_i[N])$$

are (symplectically) isomorphic for some $N \geq 7$?

In 1992 Kraus and Oesterlé [12] found the first such examples for $N = 7$, and recently Halberstadt and Kraus [5] exhibited explicit infinite families with this property (for $N = 7$).

For larger $N$, Mazur found examples for $N = 11$ and $N = 13$. In addition, Frey and his group have found many examples by computer [4].

The purpose of this paper is to prove the following result:

**Theorem 2.** There exist infinitely many one-parameter families of isomorphism classes of pairs of non-isogenous elliptic curves defined over $\mathbb{Q}$ with symplectically isomorphic 11-structure.

The main idea of the proof of this Theorem is to study the geometry (and arithmetic) of the modular diagonal quotient surfaces $Z_{N,1}$ (as introduced in [9]) in the special case $N = 11$. Now the algebraic surface $Z = Z_{N,1}$ has a natural model as a variety over $\mathbb{Q}$ (cf. §3), and an open subvariety of this turns out to be the coarse moduli space of the moduli functor $Z_{N,1}$ which classifies isomorphism classes of triplets $(E_1, E_2, \psi)$, where $\psi : E_1[N] \to E_2[N]$ is a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-isomorphism which preserves the Weil pairings. Thus, via this modular interpretation (cf. §4), the above Theorem is essentially a consequence of the following result (cf. Theorem 19):

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Theorem 3. Let $\bar{Z}_Q$ denote the minimal model of $Z_{11,1}/\mathbb{Q}$. Then the canonical map defines an elliptic fibration $f_{\text{can}} : \bar{Z}_Q \rightarrow \mathbb{P}^1_\mathbb{Q}$ which has an infinite number of sections.

Indeed, by using the result of Mazur [14] (as supplemented by Kenku [11]) on rational isogenies, one easily concludes that all except finitely many of these sections $S_i/\mathbb{Q} \simeq \mathbb{P}^1_\mathbb{Q}$ give rise to infinitely many pairs of non-isogenous elliptic curves.

It is interesting to observe that, although the above proof is constructive “in principle,” it does not allow us to write down even a single pair explicitly. The reason for this is that, while the sections are constructed as the multiples of an explicit point $Q$ of infinite order on the associated elliptic curve $E/\mathbb{Q}(t)$, the point $Q$ itself does not have a modular interpretation (since it lies in the cuspidal part), and its multiples $nQ$ cannot be interpreted explicitly until the elliptic curve $E$ can be determined.

2 Geometric results

2.1 The geometry of $\bar{Z}$

We recall some of the terminology and results of [9]. Let $N$ be a positive integer and let $X(N) = \Gamma(N)\backslash \mathbb{H}^*$ be the modular curve of level $N$, on which $G_N = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1$ acts naturally. We denote the quotient map by $\pi : X(N) \rightarrow X(1) = G_N/X(N)$. Furthermore, let $Z = Z_{N,1} = \Delta \backslash (X(N) \times X(N))$ denote the (singular) modular diagonal quotient surface; here $\Delta = \{(g,g) : g \in G_N\}$ denotes the diagonal subgroup. Then (as in [9]) the inclusion of the subgroups $\{1\} \leq \Delta \leq G_N \times G_N$ induces natural maps

$$X(N) \times X(N) \xrightarrow{\Phi} Z \xrightarrow{\Psi} X(1) \times X(1)$$

such that $\Psi \circ \Phi = \pi \times \pi : X(N) \times X(N) \rightarrow X(1) \times X(1)$ is the natural projection.

Consider the points $i$, $\exp(2\pi i/3)$ and $\infty$ of $\mathbb{H}^*$, and let $\bar{P}_k$ (for $k = 0, 1, \infty$) be their respective images in $X(1)$.

For the remainder of the section, let $N = 11$. By Theorem 2.1 of [9], we have:

Proposition 4. If $Z = Z_{11,1}$, then
1. The set of singularities of $Z$ decomposes into $\Sigma_0 \cup \Sigma_1 \cup \Sigma_\infty$, where $\Sigma_k \subset \Psi^{-1}(\bar{P}_k, \bar{P}_k)$ for $k = 0, 1, \infty$. All singularities are cyclic quotient singularities.

2. $\Sigma_0$ consists of 6 singularities of type $(2,1)$.

3. $\Sigma_1$ consists of 2 singularities of type $(3,1)$ and of 2 singularities of type $(3,2)$.

4. $\Sigma_\infty = \{z_1, \ldots, z_5\}$, where $z_i$ is a singularity of type $(11, q_i)$ with $q_i \equiv \left(\frac{i}{11}\right) \mod 11$ and $1 \leq q_i \leq 11$.

Let $\sigma : \tilde{Z} \to Z$ be the minimal desingularization of $Z$. If $C$ is a curve on $Z$, we will denote its proper transform on $\tilde{Z}$ by $\tilde{C}$.

Recall that the minimal desingularization of a cyclic quotient singularity of type $(n,q)$ is a $(-n_1, \ldots, -n_r)$-chain, where $r$ and $n_1, \ldots, n_r$ are determined by the continued fraction expansion of $n/q$ (cf. [1, § III.2]). Here, by a $(-n_1, \ldots, -n_r)$-chain we mean an open chain $C = C_1 + \cdots + C_r$ of smooth rational curves $C_i$ such that $C_i^2 = -n_i$, for $i = 1, \ldots, r$.

Let $\Psi_i = pr_i \circ \Psi : Z \to X(1) \simeq \mathbb{P}^1$, where $i = 1, 2$ and $pr_i$ is the projection on the $i$-th factor of $X(1) \times X(1)$. Let $\tilde{\Psi}_i = \Psi_i \circ \sigma$.

Proposition 2.5 of [9] implies that:

**Proposition 5.** For $k = 0, 1, \infty$, let $E_k$ denote the reduced divisor on $\tilde{Z}$ whose support is $\sigma^{-1}(\Sigma_k)$, and let $E_{\infty,i}$ be the reduced divisor on $\tilde{Z}$ whose support is $\sigma^{-1}(z_i)$. Then

1. $E_0$ consists of six (disjoint) $(-2)$-curves;

2. $E_1$ consists of two $(-2)$-curves and of two $(-2, -2)$-chains;

3. $E_\infty = E_{\infty,1} + \cdots + E_{\infty,5}$, where $E_{\infty,1}$ is a $(-11)$-curve, $E_{\infty,2}$ is a $(-4, -3)$-chain, $E_{\infty,3}$ is a $(-3, -2, -2, -2, -2)$-chain, $E_{\infty,4}$ is a $(-2, -2, -2, -2, -3)$-chain, $E_{\infty,5}$ is a $(-3, -4)$-chain.

Furthermore, if $\tilde{C}_{k,i}$ denotes the proper transform of $\Psi_i^*(\bar{P}_k)$ in $\tilde{Z}$, then each chain in $E_k$ joins $\tilde{C}_{k,1}$ to $\tilde{C}_{k,2}$, where $k = 0, 1, \infty$. 

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2.2 The elliptic fibration

**Notation.** Let \( n \not\equiv 0 \mod 11 \) be a positive integer which is a square modulo 11. Fix a \( k \in \mathbb{F}_{11}^\times \) such that \( k^2 n \equiv 1 \mod 11 \) and let \( \tau_k = \left( \begin{smallmatrix} 1/k & 0 \\ 0 & k \end{smallmatrix} \right) \in G_{11} \). Let \( T_n = \Phi \left( (\tau_k^{-1} \times 1)T'(1, n) \right) \), where \( T'(1, n) \) is the Hecke correspondence of \( X(11) \times X(11) \) as defined in §3.3 of [16]. We call such a curve on \( Z \) a Hecke curve.

**Remark 6.** \( T_n \) is the curve \( T_{n,k} \) of [9] or one of the curves \( F^{(i)}_n \) of [6].

**Proposition 7.** a) The (lifted) Hecke curves \( \tilde{T}_1, \tilde{T}_3 \) and \( \tilde{T}_4 \) are exceptional \((-1)\)-curves on \( \tilde{Z} \). Moreover, \( \tilde{T}_1 \) meets precisely three components of \( E_0 + E_1 + E_{\infty} \): a \((-2)\)-curve \( \Gamma_0 \leq E_0 \), a \((-3)\)-curve \( \Gamma_1 \leq E_1 \) and \( E_{\infty,1} \). In addition, \( \tilde{T}_1 \) meets each of these components transversely.

b) Let \( \tilde{Z} \) be the surface obtained by blowing down \( \tilde{T}_1, \Gamma_0, \Gamma_1; \tilde{T}_3; \tilde{T}_4 \). Then \( \tilde{Z} \) is minimal.

**Proof.** The assertions of part (a) follow from Claims 3 and 4 of [9] and Remark 4.9 of [8]—but see also [6]. Part (b) is Claim 8 of [9].

**Notation.** If \( C \) (or \( \tilde{C} \)) is a curve on \( \tilde{Z} \), denote its image in \( \bar{Z} \) by \( \bar{C} \).

**Proposition 8.** Let \( S_0 \) (resp. \( S_1 \)) be the \((-4)\)-component of \( E_{\infty,2} \) (resp. \( E_{\infty,5} \)). Then their images \( \tilde{S}_0 \) and \( \tilde{S}_1 \) on \( \tilde{Z} \) satisfy \( S_0^2 = S_1^2 = -3 \) and \( S_0 \cdot S_1 = 1 \).

**Proof.** See the proof of Claim 4 of [9].

**Theorem 9.** Let \( f_{\text{can}} \) be the canonical map of \( \tilde{Z} \). Then

1. \( f_{\text{can}} : \tilde{Z} \to \mathbb{P}^1 \) is an elliptic fibration with no multiple fibres.

2. \( \tilde{S}_0 \) and \( \tilde{S}_1 \) are sections of \( f_{\text{can}} \).

3. Let \( E \) be the elliptic curve over \( \mathbb{C}(t) \) corresponding to \( (\bar{Z}, \bar{S}_0) \) and let \( Q \) be the point of \( E \) corresponding to \( \tilde{S}_1 \). Then \( Q \) has infinite order on \( E \).

**Proof.** We know from (the proof of) [9], Proposition 2.14, that \( \tilde{Z} \) is a minimal smooth surface with geometric genus \( p_g(\tilde{Z}) = 2 \), Euler–Poincaré characteristic \( \chi(\tilde{Z}) = 3 \) and Kodaira dimension \( \kappa(\tilde{Z}) = 1 \). By Proposition 8, \( \tilde{S}_0 \) and \( \tilde{S}_1 \) are two rational smooth curves on \( \tilde{Z} \) of self-intersection \(-3\) meeting transversally. By the following two lemmata, this suffices to prove the theorem. 

\[ \square \]
Lemma 10. Let $E$ be a smooth minimal compact surface with $p_g(E) = 2$, $\chi(E) = 3$ and $\kappa(E) = 1$. Suppose that there is a smooth irreducible rational curve $C$ of self-intersection $-3$ lying on $E$. Then:

1. The canonical map gives an elliptic fibration $f_{\text{can}} : E \to \mathbb{P}^1$;
2. $C$ is a section of $f_{\text{can}}$;
3. $f_{\text{can}}$ has no multiple fibre.

Proof. It follows from the Enriques–Kodaira classification of minimal surfaces that $E$ admits an elliptic fibration $f : E \to B$, where $B$ is some smooth curve of genus $g$ (cf. [1], p. 194). By Kodaira’s formula for the canonical divisor of an elliptic surface (see Corollary V.12.3 of [1]),

$$K \equiv (\chi(E) - 2\chi(B))F + \sum (m_i - 1)F_i$$

$$= (1 + 2g)F + \sum (m_i - 1)F_i,$$

where $F$ is any elliptic fibre, the sum is over all singular fibres $F_i$ of multiplicity $m_i$ and $\equiv$ denotes algebraic equivalence.

By the adjunction formula, $C \cdot K = 1$; thus, if $d = C \cdot F$, equation (1) yields

$$1 = C \cdot K = C \cdot \left( (1 + 2g)F + \sum (m_i - 1)F_i \right)$$

$$= d \left( 1 + 2g + \sum (m_i - 1) \right).$$

Since $g$, $d$ and $m_i$ are all non-negative integers, (2) holds if only if $d = 1$ and $2g + \sum (m_i - 1) = 0$. This proves that $C$ is a section for $f$, that $g = 0$ and that $f$ has no multiple fibres.

Since the geometric genus of $E$ is 2, the canonical map is a rational map $f_{\text{can}} : E \to \mathbb{P}^1$. Since $B = \mathbb{P}^1$, (1) becomes $K \sim F$. Thus, $f_{\text{can}} = f$ up to an automorphism of $\mathbb{P}^1$.

Lemma 11. Let $B$ be a curve defined over a field $K$ of characteristic 0 and let $E \to B$ be a relatively minimal elliptic fibration defined over $K$ which has two distinct sections $C_0$ and $C_1$ that meet. Let $E$ be the elliptic curve over the function field $\kappa(B)$ of $B$ which corresponds to (the generic fibre of) $(E, C_0)$. Then the point $Q$ of $E$ corresponding to the section $C_1$ has infinite order.
Proof. Let \( t_0 \in B \) be a point over which \( C_0 \) and \( C_1 \) meet, let \( R \) be the local ring of \( t_0 \) on \( B \) and \( k \) its residue field. By Theorem IV.6.1 of [18], the special fibre of the Néron model \( \mathcal{E}'/R \) of \( E/K(B) \) is the curve obtained from the fibre \( \mathcal{E}_{t_0} \) by removing all singular points. In particular, \( C_1 \) belongs to the kernel \( E_1 \) of the reduction map (i.e., the specialization map at \( t_0 \)): \( \mathcal{E}'/R \to \tilde{\mathcal{E}}'/k \). Now, by Corollary IV.9.2 of [18], \( E_1 \) is isomorphic to the kernel \( E_1 \) of the reduction map of a minimal Weierstrass model of \( E \) (see Proposition VII.2.2 of [17]), which, since the ground field has characteristic 0, has no non-trivial torsion by Proposition IV.3.2b of [17]. Therefore, \( Q \) has infinite order. \( \square \)

3 Rational models

3.1 The Galois action on \( X(N) \)

The purpose of this section is to show that the surfaces \( Z, \tilde{Z} \) and \( \bar{Z} \) have natural models defined over \( \mathbb{Q} \) and that the fibration and its sections which were defined in the previous section are actually rational over \( \mathbb{Q} \).

As a first step, let us recall some facts about the Galois action on \( X(N) \), following §6.2 of Shimura [16], particularly Theorem 6.6 and Proposition 6.9.

For each positive integer \( N \), let \( X_N \) denote the smooth projective curve over \( \mathbb{Q} \) whose function field is the field \( \mathcal{F}_N = \mathbb{Q}(j, f_{r,s}) \) of modular functions of level \( N \). Here \( j \) is the modular \( j \)-invariant which induces the isomorphism \( j : X(1) \sim \mathbb{P}^1_{\mathbb{C}} \), and the \( f_{r,s} = f_{(r,s)} \) are the Fricke functions for \((r,s) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}\); cf. Lang [13], p. 65ff or Shimura [16], p. 137. If \( \tilde{G}_N = \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\pm1 \), then we have a natural \( \tilde{G}_N \)-action on \( \mathcal{F}_N \) and hence also on \( X_N \) via the rule

\[
g^* f_{r,s} = f_{(r,s)g}.
\]

The induced action on the subfield \( \mathbb{Q}^N := \mathbb{Q}(\zeta_N) \subset \mathcal{F}_N \) is via the determinant, i.e. we have \( g^*|_{\mathbb{Q}^N} = \sigma_{\det(g)}^* \), for all \( g \in \tilde{G}_N \). Here, for any \( a \in (\mathbb{Z}/N\mathbb{Z})^\times \), \( \sigma_a^* \in \text{Aut}(\mathbb{Q}^N) \) is the unique automorphism such that \( \sigma_a^*(\zeta) = \zeta^a \), for any \( N \)-th root of unity \( \zeta \in \mathbb{Q}^N \).

Thus, if \( p_N : X_N \to \text{Spec}(\mathbb{Q}^N) \) denotes the structure map induced by the inclusion \( \mathbb{Q}^N \subset \mathcal{F}_N \), then we have

\[
p_N \circ g = \sigma_{\det(g)} \circ p_N, \quad \text{for all } g \in \tilde{G}_N. \tag{3}
\]
Note that since the fixed field of $\tilde{G}_N$ is $\mathcal{F}_N^{\tilde{G}_N} = \mathbb{Q}(j)$, the quotient of $X_N$ with respect to $\tilde{G}_N$ is

$$X_Q(1) := \tilde{G}_N \backslash X_N \simeq \mathbb{P}^1_{\mathbb{Q}}.$$ 

Next, consider the subgroup

$$H_N = \{ \pm \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (\mathbb{Z}/N\mathbb{Z})^\times \}/\{\pm 1\} \leq \tilde{G}_N,$$
and let $F_N = \mathcal{F}_N^{H_N}$ be its fixed field. Since $H_N \cdot G_N = \tilde{G}_N$ and $H_N \cap G_N = \{1\}$ (and since $\mathbb{Q}_N$ is algebraically closed in $\mathcal{F}_N$), we have the following diagram of linearly disjoint field extensions:

Thus, if we let

$$\pi_Q = \pi_{Q,N} : X_Q(N) := H_N \backslash X_N \to X_Q(1) := \tilde{G}_N \backslash X_N$$

be the map induced on the quotient curves via the inclusion $H_N \subset \tilde{G}_N$, then the curve $X_Q(N)$ (called Shimura’s canonical model of $X(N)$) is absolutely irreducible over $\mathbb{Q}$, for we have that $X_Q(N) \otimes \mathbb{C} = X(N)$ and $\pi_Q \otimes \mathbb{C} = \pi$. More precisely, the above field diagram translates into the following cartesian diagram of curves:

$$\begin{array}{c}
\Spec(\mathbb{Q}) \quad \xleftarrow{q} \quad \Spec(\mathbb{Q}_N) \quad \xleftarrow{q} \quad \Spec(\mathbb{C}) \\
\downarrow \quad \downarrow \quad \downarrow \\
X_Q(1) \quad \xleftarrow{q_{X(1)}} \quad X_Q(N) \quad \xleftarrow{q_{X(N)}} \quad X(N) \\
\downarrow \quad \downarrow \quad \downarrow \\
\Spec(\mathbb{Q}) \quad \xleftarrow{p} \quad \Spec(\mathbb{Q}_N) \quad \xleftarrow{p} \quad \Spec(\mathbb{C}) \\
\downarrow \quad \downarrow \quad \downarrow \\
X(1) \quad \xleftarrow{p_{X(1)}} \quad X(N) \quad \xleftarrow{p_{X(N)}} \quad X(N) \\
\end{array}$$
Proposition 13. The surface $X_Z$ we first show: precisely, we shall construct $X$. We now use Shimura’s canonical model $X$ on $\tau \in H$. $3.2$ A $\mathbb{Q}$-rational model for $Z_{N,1}$

We now use Shimura’s canonical model $X_\mathbb{Q}(N)/\mathbb{Q}$ (and the modular curve $X_\mathbb{Q}(N)/\mathbb{Q}_N$) to construct a $\mathbb{Q}$-rational model $Z_\mathbb{Q}$ of the surface $Z = Z_{N,1}$. More precisely, we shall construct $Z_\mathbb{Q}$ as a suitable quotient of the surface $Y_{N,1} := X_N \times_{\mathbb{Q}_N} X_N$ and show that $Y_\mathbb{Q}(N) := X_\mathbb{Q}(N) \times X_\mathbb{Q}(N)$ maps to $Z_\mathbb{Q}$. To this end we first show:

Proposition 13. The surface $Y_{N,1} := X_N \times_{\mathbb{Q}_N} X_N$ is naturally an irreducible component of $Y_N := X_N \times_{\mathbb{Q}_N} X_N$, and its stabilizer via the $\tilde{G}_N \times \tilde{G}_N$-action on $Y_N$ is

$$\text{Stab}_{\tilde{G}_N \times \tilde{G}_N}(Y_{N,1}) = G := \{(g_1, g_2) \in \tilde{G}_N \times \tilde{G}_N : \det g_1 = \det g_2\}.$$ 

Moreover, we have

$$(p_N \times p_N) \circ g = \sigma_{\det g_1} \circ (p_N \times p_N), \quad \text{if } g = (g_1, g_2) \in G. \quad (4)$$

Furthermore, the base change maps $q_{Y_{N,1}} : Y_{N,1} \to Y_\mathbb{Q}(N) := X_\mathbb{Q}(N) \times X_\mathbb{Q}(N)$ and $q_{Y(1)} : Y_{1,1} \to Y_{1,1}(1) := X_\mathbb{Q}(1) \times X_\mathbb{Q}(1)$ are Galois covers with covering groups $H := \{(h, h) : h \in H_N\}$ to construct a $\mathbb{Q}$-rational model for $Z$. On the other hand, the diagonal (Cartan) subgroup of $\tilde{G}_N$ does not act on $X_\mathbb{Q}(N)$.

Remark 12. Note that $q, q_{X(1)}$ and $q_{X(N)}$ are Galois covers with Galois group $\sim (\mathbb{Z}/N\mathbb{Z})^\times$ and that $\pi_{\mathbb{Q}_N}$ is a (ramified) cover with covering group $G_N$. The action of $G_N$ does not descend to $X_\mathbb{Q}(N)$, since $G_N$ does not normalize $H_N$.

Moreover, we have

$$q_{X(N)} \circ p_N \circ g_1 = p_N \circ g_2 \circ \sigma_{\det(g_1)} \circ p_N, \quad \text{and so we see that } g \text{ factors over } f \text{ if and only if } \sigma_{\det(g_1)} \circ p_N = \sigma_{\det(g_2)} \circ p_N.$$ 

Proof. Put $Y = Y_{N,1}$ and let $pr_{Y,i}$ (resp. $pr_{Y_{N,i}}$) denote the projection onto the $i$-th factor of $Y$ (resp. $Y_N$). Then there is a unique morphism $f = f_{N,1} : Y \to Y_N$ such that $pr_{Y_{N,i}} \circ f = pr_{Y,i}$. Clearly, $f$ is a closed immersion. Now $Y$ is irreducible because it is (smooth and) geometrically irreducible over $\text{Spec}(\mathbb{Q}_N)$ (because $X_N/\mathbb{Q}_N$ is), and hence is a component of $Y_N$ since both have dimension 2.

Now let $g = (g_1, g_2) \in \tilde{G}_N \times \tilde{G}_N$. Then $g \in \text{Stab}(Y)$ if and only if $g$ factors over $f$, i.e. if and only $p_N \circ g_1 = p_N \circ g_2$. Now by equation (3) we have $p_N \circ g_1 = \sigma_{\det(g_1)} \circ p_N$, and so we see that $g$ factors over $f$ if and only $\sigma_{\det(g_1)} \circ p_N = \sigma_{\det(g_2)} \circ p_N$, and this is equivalent to $\det(g_1) = \det(g_2)$, i.e.
to $g \in G$. This proves the formula for the stabilizer and equation (4) follows immediately.

Next we observe that since $X_N = X_Q(N) \otimes Q_N$, we have the cartesian diagram

$$
\begin{array}{ccc}
Y_Q(N) & \xrightarrow{q_{Y(N)}} & Y \\
p \times p & \downarrow & \downarrow p_N \times p_N \\
\text{Spec}(Q) & \leftarrow & \text{Spec}(\mathbb{Q}_N)
\end{array}
$$

in which $p = p_Q \circ \pi_Q : X_Q(N) \rightarrow \text{Spec}(\mathbb{Q})$ is the structure map and $q_{Y(N)}$ is the unique morphism such that $pr_{YQ,i} \circ q_{Y(N)} = q_{X(N)} \circ pr_{Qi,i}$, for $i = 1, 2$. (Here, $pr_{YQ,i} : Y_Q(N) \rightarrow X_Q(N)$ denotes the projection onto the $i$-th factor of $Y_Q(N)$.)

Now since $q$ is a Galois cover with group $\{\sigma_a\} \simeq (\mathbb{Z}/\mathbb{N})^\times$, the same is true for $q_{Y(N)}$; more precisely, $q_{Y(N)}$ is Galois with covering group $\{\tilde{\sigma}_a\}$, where $\tilde{\sigma}_a \in \text{Aut}(Y)$ is the unique automorphism such that $q_{Y(N)} \circ \tilde{\sigma}_a = q_{Y(N)}$ and $(p_N \times p_N) \circ \tilde{\sigma}_a = \sigma_a \circ (p_N \times p_N)$. We observe:

$$
\tilde{\sigma}_a = (h_a, h_a), \quad \text{where } h_a = \left(\begin{array}{cc}
a & 0 \\
b & a
\end{array}\right) \in H_N. \quad (5)
$$

Indeed, by (4) we have $(p_N \times p_N) \circ (h_a, h_a) = \sigma_a \circ (p_N \times p_N)$. Moreover, since $q_{X(N)} \circ h = q_{X(N)}$ for all $h \in H$, we see that $pr_{YQ,i} \circ q_{Y(N)} \circ (h_a, h_a) = q_{X(N)} \circ pr_{Y,i} \circ (h_a, h_a) = q_{X(N)} \circ h_a \circ pr_{Y,i} = q_{X(N)} \circ pr_{Y,i} \circ h_a$, and so $q_{Y(N)} \circ (h_a, h_a) = q_{Y(N)}$ by the defining property of $q_{Y(N)}$. Thus, $(h_a, h_a)$ satisfies the defining property of $\tilde{\sigma}_a$, and so (5) holds. We thus see that $Y_Q(N)$ is the quotient of $Y$ with respect to the subgroup $H$.

Clearly, $Y_Q(N)(1) := X_Q(N)(1) \times_{Q_N} X_Q(N)(1)$ is the quotient of $Y$ with respect to the subgroup $G_N \times G_N \leq G$ with quotient map $\pi_{Q_N} \times \pi_{Q_N}$. Now since $G_N \triangleleft \hat{G}_N$, we have that $G_N \times G_N \triangleleft G$, and hence $\hat{H} = G/\langle G_N \times G_N \rangle$ acts as a group of automorphisms on $Y_Q(N)(1)$. Now the quotient map induces an isomorphism $\hat{H} \simeq \hat{H}$ because $H \cap (G_N \times G_N) = \{1\}$ and $\hat{H} : (G_N \times G_N) = G$, and so by a similar argument as above we see that $q_{Y(1)} : Y_Q(N)(1) \rightarrow Y_Q(1)$ is the quotient map with covering group $\hat{H}$.

**Theorem 14.** Let $\hat{\Delta} = \Delta \cdot H = \{(g, g) : g \in \hat{G}_N\} \leq G$, where as before $\Delta = \{(g, g) : g \in G_N\}$. Then the quotient variety $Z_Q := \hat{\Delta} \backslash Y_{N,1}$ is a $Q$-rational model of $Z$, and we have morphisms

$$
X_Q(N) \times X_Q(N) \xrightarrow{\phi_Q} Z_Q \xrightarrow{\Psi_Q} X_Q(1) \times X_Q(1) \quad (6)
$$
such that the base change of $\Phi_Q$ and $\Psi_Q$ with $C$ is $\Phi$ and $\Psi$. In particular, $\Psi_Q \circ \Phi_Q = \pi_Q \times \pi_Q$.

Proof. As before, write $Y = Y_{N,1}$. Furthermore, let

$$\pi_\Delta : Y \to Z_{Q,N} := \Delta \backslash Y$$

and

$$\pi_\tilde{\Delta} : Y \to Z_Q := \tilde{\Delta} \backslash Y$$

denote the associated quotient maps and spaces; these exist (as varieties) since we are dealing with quotients of (quasi-) projective varieties by finite groups.

Since $\Delta \triangleleft \tilde{\Delta}$, and $H \triangleleft \Delta / \Delta$, we see that $\pi_\Delta = q_{Z(N)} \circ \pi_{\Delta}$, where $q_Z : Z_{Q,N} \to Z_Q$ is the quotient map with covering group $\tilde{\Delta} / \Delta \simeq H$. Furthermore, the inclusions of subgroups $H \leq \Delta \leq G$ induce quotient maps $\Phi_Q : Y_Q(N) = H \backslash Y \to Z_Q = \tilde{\Delta} \backslash Y$ and $\Psi_Q : Z_Q = \tilde{\Delta} \backslash Y \to Y_Q(1) = G \backslash Y$ which fit into the following cartesian diagram:

Thus, the base change of $\Phi_Q$ and $\Psi_Q$ is $\Phi$ and $\Psi$, respectively. Furthermore, we have $\Psi_Q \circ \Phi_Q = \pi_Q \times \pi_Q$ because we have equality for these morphisms after (faithfully flat) base change with $q$, and so they must be equal.

Remark 15. The above proof shows that $Z = Z_{N,1}$ has a canonical $Q$-rational model $Z_Q = Z_{N,1}/Q$; this is the only case required here. In a similar way, however, one can show that every $Z_{N,\varepsilon}$ has a canonical $Q$-rational model $Z_{N,\varepsilon}/Q$. Indeed, we can use essentially the same construction if we replace $Y = Y_{N,1}$ by $Y_{N,\varepsilon} := X_N \times_{Q,N,\varepsilon} X_N$, the fibre product of $X_N$ with itself via the morphisms $p_N : X_N \to \text{Spec}(Q_N)$ and $\sigma_\varepsilon \circ p_N$. In addition, the group $\Delta$ has
to be replaced by the twisted diagonal subgroup \( \Delta_\varepsilon = (1 \times h^{-1}_\varepsilon) \Delta(1 \times h_\varepsilon) = \{(g, a_\varepsilon(g)) : g \in G_N\} \), and similarly, \( \tilde{\Delta}_\varepsilon = \Delta_\varepsilon H \). (Note that \( \Delta_\varepsilon \leq G \) and that the group \( G \) still acts on \( Y_{N,\varepsilon} \).) Thus, the canonical \( \mathbb{Q} \)-rational model of \( Z_{N,\varepsilon} \) is

\[
Z_{N,\varepsilon}/\mathbb{Q} = \tilde{\Delta}_\varepsilon \backslash Y_{N,\varepsilon}.
\]

We further observe that each \( Y_{N,\varepsilon} \) has a canonical embedding \( \tilde{f}_\varepsilon : Y_{N,\varepsilon} \to Y_N := X_N \times_{\mathbb{Q}} X_N \), which leads to the decomposition of \( Y_N \) into its irreducible components:

\[
Y_N := X_N \times_{\mathbb{Q}} X_N = \coprod_\varepsilon Y_{N,\varepsilon}; \quad (7)
\]

note that \( (1 \times h^{-1}_\varepsilon) \circ f_1 = f_\varepsilon \), so \( Y_{N,\varepsilon} \cong Y_{N,1} \) and \( \tilde{G} := \tilde{G}_N \times \tilde{G}_N \) acts transitively on the components. In fact, since \( G < \tilde{G} \), we see that \( \text{Stab}_{\tilde{G}}(Y_{N,\varepsilon}) = G \) for all \( \varepsilon \) and hence \( g(Y_{N,\varepsilon}) = Y_{N,ae} \), where \( a = \det(g_1)/\det(g_2) \) if \( g = (g_1, g_2) \in \tilde{G}_N \times \tilde{G}_N \).

**Proposition 16.** Every Hecke curve \( T_n \) on \( Z_\mathbb{Q} \) is defined over \( \mathbb{Q} \).

**Proof.** Recall that \( T_n = \Phi\left( (\tau_k^{-1} \times 1)T'(1,n) \right) \), where we have \( k^2 n \equiv 1 \mod N \). By Proposition 7.7 of [16], \( T'(1,n) \) is defined over \( \mathbb{Q} \), i.e. we can view \( T'(1,n) \subset X_\mathbb{Q}(N) \times X_\mathbb{Q}(N) \). By Remark 12, the action of \( \tau_k \) commutes with the action of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \), i.e. \( \tau_k \) is defined over \( \mathbb{Q} \), and hence \( \Phi_\mathbb{Q}\left( (\tau_k^{-1} \times 1)T'(1,n) \right) \) is a model for \( T_n \) over \( \mathbb{Q} \).

### 3.3 A model for the elliptic fibration

We now return to the situation of §2, i.e. we assume that \( N = 11 \). Here we have:

**Proposition 17.** The surface \( \tilde{Z} \) admits a rational model \( \tilde{Z}_\mathbb{Q} \). Moreover, each component of \( E_\infty \) is defined over \( \mathbb{Q} \).

**Proof.** By the uniqueness property of the minimal resolution of singularities, the morphism \( \sigma : \hat{Z} \to Z \) descends to a morphism \( \sigma_\mathbb{Q} : \hat{Z}_\mathbb{Q} \to Z_\mathbb{Q} \). This proves the first part.

Since \( \Psi \) is defined over \( \mathbb{Q} \), the set \( \Sigma_\infty \) is Galois invariant. By Proposition 4.4, we see that each singularity of \( \Sigma_\infty \) has a unique type. Since the singularity type is a Galois invariant, it follows that every singularity \( z_i \) of
\( \Sigma_\infty \) is \( \mathbb{Q} \)-rational. Since \( \sigma \) is defined over \( \mathbb{Q} \), the chain \( E_{\infty,i} \) is \( \mathbb{Q} \)-rational. It is clear that \( \tilde{C}_{\infty,1} \) is \( \mathbb{Q} \)-rational as well. Therefore, the component of \( E_{\infty,i} \) which meets \( \tilde{C}_{\infty,1} \) (which is unique by Proposition 5) is \( \mathbb{Q} \)-rational, and hence so are the other components. This proves the second part. \( \square \)

**Proposition 18.** The surface \( \tilde{Z} \) admits a \( \mathbb{Q} \)-rational model \( \tilde{Z}_\mathbb{Q} \). If \( f_{\text{can}} \) is the canonical map of \( \tilde{Z}_\mathbb{Q} \), then \( f_{\text{can}} : Z_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Q} \) is an elliptic fibration which admits the sections \( \tilde{S}_0 \) and \( \tilde{S}_1 \).

**Proof.** Although this statement could be proved by applying the Minimal Model Theorem of Shafarevich–Lichtenbaum to the canonical map of \( \tilde{Z}_\mathbb{Q} \), we prefer to give the following more explicit proof.

Recall from Proposition 7(b), that \( \bar{Z} \) is obtained from \( \tilde{Z} \) by blowing down \( \tilde{T}_1, \Gamma_0, \Gamma_2; \tilde{T}_3; \tilde{T}_4 \). We claim that all of these curves are \( \mathbb{Q} \)-rational.

It follows from Propositions 16 and 17 that every \( \tilde{T}_n \) is \( \mathbb{Q} \)-rational. Arguing as in the proof of Proposition 17, we conclude that \( E_0 \) and \( E_1 \) are \( \mathbb{Q} \)-rational, since they are the unique components of \( E_0 \) and \( E_1 \) meeting \( \tilde{T}_1 \)—cf. Proposition 7(a). This proves that \( \bar{Z} \) is defined over \( \mathbb{Q} \). Therefore, the canonical class of \( \bar{Z} \) is \( \mathbb{Q} \)-rational, and thus \( f_{\text{can}} \) is defined over \( \mathbb{Q} \).

Since \( S_0 \) and \( S_1 \) are the unique components of \( E_\infty \) of self-intersection \(-4\), and they meet \( C_{\infty,1} \) and \( C_{\infty,2} \), respectively, it follows that they are fixed by the Galois action. Thus, their images \( \tilde{S}_0 \) and \( \tilde{S}_1 \) in \( \bar{Z}_\mathbb{Q} \) are \( \mathbb{Q} \)-rational. \( \square \)

**Notation.** The set \( M = M(Z_{N,1}) \) of Mazur’s trivial points of \( Z = Z_{N,1} \) is defined by

\[
M(Z_{N,1}) = \bigcup_n \Psi^{-1}_Q \left( \xi_n \left( X_0'(n)(\mathbb{Q}) \right) \right),
\]

where \( \xi_n : X_0(n) \to T'(1,n) \subset X(1) \times X(1) \) is the normalization map and \( X_0'(n) := X_0(n) \setminus \{\text{cusps}\} \) is the non-cuspidal part of \( X_0(n) \).

In addition, we put \( M = \sigma^{-1}(M) \cap Z_\mathbb{Q}(\mathbb{Q}) \) and let \( M \) denote the image of \( \tilde{M} \) in \( \bar{Z}_\mathbb{Q} \).

**Theorem 19.** The canonical map \( f_{\text{can}} : \tilde{Z}_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Q} \) admits infinitely many sections \( \{S_i\} \) such that \( S_i(\mathbb{Q}) \setminus M \) is infinite for every \( i \).

**Proof.** By Proposition 18, the elliptic curve \( E/\mathbb{C}(t) \) of Theorem 9 is actually defined over \( \mathbb{Q} \), as is the point of infinite order \( Q \). Thus, the multiples of \( Q \)
correspond to infinitely many rational sections \{S_i\} of \(f_{\text{can}}\). Since the genus of \(X_0(n)\) is \(\geq 1\) for \(n \geq 16\), only finitely many sections can be components of some \(\Psi^*\xi_n(X_0(n))\). By restricting the index \(i\), we can assume that no \(S_i\) is contained in \(\Psi^*\xi_n(X_0(n))\). By a theorem of Mazur and Kenku [11], \(X'_0(n)(\mathbb{Q}) = \emptyset\) for \(n > 163\). Thus, \(\tilde{M}\) is contained in finitely many curves of \(\tilde{Z}\), and each section \(S_i\) has a finite intersection with it. Since \(S_i \simeq \mathbb{P}^1_{\mathbb{Q}}\), the theorem follows.

\[\square\]

4 Modular Interpretation

4.1 Product moduli

Fix a positive integer \(N\), and let \(X_N\) be the contravariant functor

\[X_N : \text{Sch}_{/\mathbb{Q}} \rightarrow \text{Sets}\]

from the category of schemes over \(\mathbb{Q}\) to the category of sets which is defined as follows. If \(S\) is a \(\mathbb{Q}\)-scheme, then let \(X_N(S)\) be

\[\left\{ \langle E/S, \alpha \rangle : E/S \text{ elliptic curve}, \alpha : E[N] \sim (\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z})_S \right\},\]

where \(\langle \cdot \rangle\) denotes the class of \(S\)-isomorphisms, and if \(f : T \rightarrow S\) is a morphism of \(\mathbb{Q}\)-schemes, then \(X_N(f) : X_N(S) \rightarrow X_N(T)\) is the map induced by base change, i.e. \(X_N(f)(\langle E/S, \alpha \rangle) = \langle E(T), \alpha(T) \rangle\), where \(E(T) = E \times_S T\) and \(\alpha(T) = \alpha \times_S \text{id}_T : E(T)[N] = E[N] \times_S T \rightarrow (\mathbb{Z}/N\mathbb{Z})^2_T = (\mathbb{Z}/N\mathbb{Z})^2 \times_S T\).

We have a natural forget map \(\pi_N : X_N \rightarrow X_1\) defined by

\[\pi_{N,S}(\langle E/S, \alpha \rangle) = \langle E/S \rangle;\]

here we use the obvious fact that the functor \(X_1\) can be identified with the functor that classifies isomorphism classes of elliptic curves. Clearly, \(\pi_N\) is \(\tilde{G}_N\)-invariant, where the group \(\tilde{G}_N = \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1\) acts on the left on \(X_N\) by \(g \cdot \langle E/S, \alpha \rangle = \langle E/S, g \circ \alpha \rangle\). (Note that \(\langle E/S, -\alpha \rangle = \langle E/S, \alpha \rangle\), so the action factors over the quotient group.)

It is well known (cf. [2], [3]) that the affine curve \(X'_N = X_N \setminus \{\text{cusps}\}\), the non-cuspidal part of the curve \(X_N/\mathbb{Q}\) introduced in subsection 3.1, represents the functor \(X_N\) when \(N \geq 3\), i.e. for every \(\mathbb{Q}\)-scheme \(S\) we have a bijection

\[\mathcal{P}_{X_N,S} : X_N(S) \sim X'_N(S),\]
which is compatible with base change. Moreover, this bijection is \( \tilde{G}_N \)-equivariant, i.e. we have \( \mathcal{P}_{X_N,S}(g(E/S, \alpha)) = g \mathcal{P}_{X_N,S}([E/S, \alpha]) \).

For \( N \leq 2 \) the curve \( X'_N \) only coarsely represents \( X_N \): we then still have the maps \( \mathcal{P}_{X_N,S} \), but they are only bijections when \( S = \text{Spec}(K) \), \( K \) an algebraically closed field. In the case \( N = 1 \) and \( S = \text{Spec}(K) \), any field, \( \mathcal{P}_{X_1,K} \) can be given explicitly; it is the unique map such that the function \( j \in \kappa(X_1) \cong \mathbb{Q}(j) \) evaluates to the \( j \)-invariant \( j(E/K) \) of the elliptic curve:

\[
j(\mathcal{P}_{X_1,K}([E/K])) = j(E/K).
\]

Note that the above \( \tilde{G}_N \)-equivariance of \( \mathcal{P}_{X_N} \) implies the compatibility relation

\[
\mathcal{P}_{X_1,K} \circ \pi_{N,K} = \pi_N \circ \mathcal{P}_{X_N,K},
\]

where \( \pi_N : X_N \to X_1 \cong \tilde{G}_N \backslash X_N \) denotes the quotient morphism.

Recall from subsection 3.1 that \( X_N \) and hence also \( X'_N \) come equipped with the structure maps \( p_N : X_N \to \text{Spec}(\mathbb{Q}_N), \quad p'_N : X'_N \to \text{Spec}(\mathbb{Q}_N) \).

Viewing \( \text{Spec}(\mathbb{Q}_N) \) as the scheme which represents the functor \( \mu^\text{prim}_N \) that associates to each scheme \( S/\mathbb{Q} \) its set of primitive \( N \)-th roots of unity (i.e. the set of morphisms \( \text{Spec}(\mathbb{Q}_N) \to S \)), we see that the morphism \( p'_N \) represents the morphism of functors \( p'_N : X_N \to \mu^\text{prim}_N \) defined by

\[
p'_N,S([E/S, \alpha]) = e_N(\alpha^{-1}(1,0), \alpha^{-1}(0,1)) \in \mu^\text{prim}_N(S), \tag{8}
\]

in which \( e_N = e_N^X : E[N] \times E[N] \to \mathbb{G}_m/S \) denotes the \( e_N \)-pairing (cf. [15] or [10]).

Next, consider the product functor \( \mathcal{Y}_N = \mathcal{X}_N \times \mathcal{X}_N \), as well as the subfunctor \( \mathcal{Y}_{N,1} \) of \( \mathcal{Y}_N \) defined for \( \varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times \) by

\[
\mathcal{Y}_{N,\varepsilon}(S) = \{ ([E_1/S, \alpha_1]; [E_2/S, \alpha_2]) \in \mathcal{Y}_N(S) : e_\varepsilon \circ ((\alpha_2^{-1} \circ \alpha_1) \times (\alpha_2^{-1} \circ \alpha_1)) = e_N \};
\]

like \( \mathcal{X}_N \), these are functors from \( \text{Sch}/\mathbb{Q} \) to \( \text{Sets} \). We have:

**Proposition 20.** If \( N \geq 3 \), the \( \mathbb{Q} \)-schemes \( Y'_N = X'_N \times_\mathbb{Q} X'_N \) and \( Y'_{N,1} = X'_N \times_{\mathbb{Q}_N} X'_{N,1} \) represent the functors \( \mathcal{Y}_N \) and \( \mathcal{Y}_{N,1} \), respectively; i.e. we have natural isomorphisms of functors

\[
\mathcal{P}_{Y_N} : \mathcal{Y}_N \cong h_{Y'_N} \quad \text{and} \quad \mathcal{P}_{Y_{N,1}} = \mathcal{P}_{Y_N} \mathcal{Y}_{N,1} : \mathcal{Y}_{N,1} \cong h_{Y'_{N,1}}.
\]
Proof. Since \( \mathcal{Y}_N \) is just the product functor, the first assertion is obvious. For the second notice that the condition that \( e_N^E_1 = e_N^E_2 \circ (\alpha_2 \times \alpha_2)^{-1} \circ (\alpha_1 \times \alpha_1) \) is by equation (8) equivalent to the condition that \( p'_N(\langle E_1, \alpha_1 \rangle) = p'_N(\langle E_2, \alpha_2 \rangle) \), which means that \( \mathcal{Y}_{N,1} = \mathcal{X}_N \times_{Q_N} \mathcal{X}_N \) is the fibre product of \( \mathcal{X}_N \) with itself over \( \text{Spec}(Q_N) \) via \( p'_N \), and hence is represented by the fibre product \( Y'_{N,1} = X'_{N} \times_{Q_N} X'_{N} \) over \( \text{Spec}(Q_N) \) (viewed as a \( Q \)-scheme).

Remark 21. The same argument shows that \( \mathcal{Y}_{N,\varepsilon} = \mathcal{X}_N \times_{Q_N,\sigma} \mathcal{X}_N \) is the fibre product of \( \mathcal{X}_N \) with itself via the maps \( p'_N \) and \( \sigma \circ p'_N \), and hence is represented by the scheme \( Y'_{N,\varepsilon} = X'_{N} \times_{Q_N,\sigma} X'_{N} \) (cf. Remark 15). Thus, \( \mathcal{P}_{\mathcal{Y}_N} \) restricts to an isomorphism

\[
\mathcal{P}_{\mathcal{Y}_{N,\varepsilon}} = \mathcal{P}_{\mathcal{Y}_N \mid \mathcal{Y}_{N,\varepsilon}} : \mathcal{Y}_{N,\varepsilon} \xrightarrow{\sim} Y'_{N,\varepsilon}.
\]

Note that although \( Y'_{N,\varepsilon} \) is a scheme over \( Q_N \), we view it here as a scheme over \( Q \); in particular, \( Y'_{N,\varepsilon}/Q \) is not geometrically connected.

4.2 Quotient moduli

Let \( \mathcal{Z}_N \) be the contravariant functor of \( Q \)-schemes defined by

\[
\mathcal{Z}_N(S) = \{ \langle E_1/S, E_2/S, \psi \rangle : E_1/S, E_2/S \text{ elliptic curves}, \psi : E_1[N] \xrightarrow{\sim} E_2[N] \}.
\]

We have natural maps of functors \( \Phi_N : \mathcal{Y}_N \to \mathcal{Z}_N \) and \( \Psi : \mathcal{Z}_N \to \mathcal{Y}_1 \) defined for each \( Q \)-scheme \( S \) by the rules

\[
\Phi_N(S)(\langle E_1, \alpha_1; E_2, \alpha_2 \rangle) = \langle E_1, E_2, \alpha_2^{-1} \circ \alpha_1 \rangle,
\]

\[
\Psi(S)(\langle E_1, E_2, \psi \rangle) = \langle E_1, E_2 \rangle.
\]

Clearly, \( \Psi \circ \Phi_N = \pi_N \times \pi_N \) while \( \Phi_N \) is \( \check{G}_N \)-invariant with respect to the diagonal action on \( \mathcal{Y}_N \), and so \( \Phi_N \) factors over the quotient functor \( \Pi_N : \mathcal{Y}_N \to \overline{\mathcal{Y}}_N := \check{G}_N \backslash \mathcal{Y}_N \), i.e. we have an induced map

\[
\overline{\Phi}_N : \overline{\mathcal{Y}}_N := \check{G}_N \backslash \mathcal{Y}_N \to \mathcal{Z}_N
\]

such that \( \Phi_N = \overline{\Phi}_N \circ \Pi_N \). It is easy to see that \( \overline{\Phi}_N \) is an injection. Unfortunately, \( \overline{\Phi}_N \) is not bijective, so \( \mathcal{Z}_N \) is not quite the quotient functor. (However, both functors have the same sheafifications in the étale topology). Nevertheless, \( \mathcal{Z}_N \) still is coarsely represented by the quotient scheme \( Z'_{N} := \check{G}_N \backslash Y'_{N} \),
the non-cuspidal part of $Z_N := \tilde{G}_N \setminus Y_N$. This means that we have a natural map (of functors) $P_{Z_N} : Z_N \to h_{Z'_N}$, where $h_{Z'_N}$ denotes the functor associated to the $\mathbb{Q}$-scheme $Z'_N$ (i.e. $h_{Z'_N}(S) = Z'_N(S) = \text{Hom}_{\text{Sch}}(S, Z'_N)$), which is bijective on geometric points and which is “best possible” in a certain sense. (In particular, it then follows that the above functor $\Phi_N : Y_N \to Z_N$ is compatible with the quotient morphism $\Phi_N : Y'_N \to \tilde{G}_N \setminus Y'_N = Z'_N$.)

Since the proof of this general fact is somewhat technical (cf. [7]), we content ourselves with the following result which suffices for our purposes.

**Proposition 22.** For every field $K \supset \mathbb{Q}$ there is a unique map

$$P_{Z_N,K} : Z_N(K) \to Z'_N(K)$$

which is compatible with base change (of fields) such that the diagram

$\begin{array}{ccc}
Y_N(K) & \xrightarrow{P_{Y_N,K}} & Y'_N(K) \\
\Phi_{N,K} \downarrow & & \downarrow \Phi_N \\
Z_N(K) & \xrightarrow{P_{Z_N,K}} & Z'_N(K) \\
\Psi_K \downarrow & & \downarrow \Psi \\
Y_1(K) & \xrightarrow{P_{Y_1,K}} & Y'_1(K)
\end{array}$

(9)

commutes. Furthermore, $P_{Z_N,K}$ is a bijection if $K$ is algebraically closed.

**Proof.** Suppose first that $K = \overline{K}$ is algebraically closed. Then the map $\Phi_{N,K} : Y_N(K) := \tilde{G}_N \setminus Y_N(K) \to Z_N(K)$ is bijective.

Indeed, since we already know that $\Phi_{N,K}$ is injective, it is enough to show that $\Phi_{N,K}$ and hence $\Phi_{N,K}$ is surjective. For this, let $\langle E_1, E_2, \psi \rangle \in Y_N(K)$. Since $K$ is algebraically closed (and $\text{char}(K) = 0$), there is a $K$-rational level $N$ structure $\alpha_2 : E_2[N] \sim \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ on $E_2$. Put $\alpha_1 := \alpha_2 \circ \psi$. Then $\langle E_1, \alpha_1; E_2, \alpha_2 \rangle \in Y_N(K)$ and $\Phi_{N,K}(\langle E_1, \alpha_1; E_2, \alpha_2 \rangle) = \langle E_1, E_2, \psi \rangle$. This means that $\Phi_{N,K}$ is surjective and so $Z_N(K)$ is the $\tilde{G}_N$-quotient of $Y_N(K)$ by $\Phi_{N,K}$.

On the other hand, since $Z'_N = \tilde{G}_N \setminus Y'_N$ is the geometric quotient of $Y'_N$, we know (since $K$ is algebraically closed) that $Z'_N(K)$ is the (set-theoretic) $\tilde{G}_N$-quotient of $Y'_N(K)$. Thus, since $P_{Y_N,K} : Y_N(K) \to Y'_N(K)$ is bijective, it follows that $(Z_N(K), \Phi_{N,K})$ and $(Z'_N(K), \Phi_N \circ P_{Y_N,K})$ are both set-theoretic quotients of $Y_N(K)$, so there is a unique bijection $P_{Z_N,K} : Z_N(K) \sim Z'_N(K)$.
such that $\mathcal{P}_{Z,N,K} \circ \Phi_{N,K} = \Phi_N \circ \mathcal{P}_{Y,N,K}$, i.e. such that the top part of diagram (9) commutes.

We observe that this bijection is $\text{Aut}(K)$-equivariant, i.e. we have

$$\mathcal{P}_{Z,N,K}((E_1, E_2, \psi)^\sigma) = \mathcal{P}_{Z,N,K}((E_1, E_2, \psi))^\sigma, \text{ for } \sigma \in \text{Aut}(K).$$

Here $\sigma$ acts (on the right) on $Z_N(K)$ by functoriality (i.e. by $Z_N(\sigma)$) and on $Z'(K)$ by the usual action on $K$-rational points.

To prove that the bottom square of diagram (9) commutes, observe that the outer rectangle of (9) commutes because $\Psi \circ \Phi_N = \pi_N \times \pi_N$, $\Psi \circ \Phi_N = \pi_N \times \pi_N$, and $\mathcal{P}_{X_N} \circ \pi_{N,K} = \pi_N \circ \mathcal{P}_{X_N,K}$, and so the assertion follows by a diagram chase since $\Phi_{N,K}$ is surjective (when $K$ is algebraically closed).

Now consider the case of an arbitrary field $K$ with algebraic closure $\overline{K}$ and let $G_K = \text{Gal}(\overline{K}/K)$ denotes its absolute Galois group. Then we can identify $Z'_N(K)$ as the subset of $G_K$-invariant elements of $Z'_N(\overline{K})$, i.e. $Z'_N(K) = Z'_N(\overline{K})^{G_K}$. The inclusion $i^* : K \rightarrow \overline{K}$ induces a morphism $i : \text{Spec}(\overline{K}) \hookrightarrow \text{Spec}(K)$ and hence a map $Z_N(i) : Z_N(K) \rightarrow Z_N(\overline{K})$. Now the image of $Z_N(i)$ is actually $G_K$-invariant because $i \circ \sigma = i$, for all $\sigma \in G_K$, and so we can define $\mathcal{P}_{Z,N,K}$ as the composition

$$\mathcal{P}_{Z,N,K} = \mathcal{P}_{Z_N,\overline{K}}^{G_K} \circ Z_N(i) : Z_N(K) \rightarrow Z_N(\overline{K})^{G_K} \rightarrow Z_N(\overline{K})^{G_K} = Z_N(K).$$

It is then clear that the diagram (9) commutes and that the $\mathcal{P}_{Z_N,K}$ commute with base change. Furthermore, $\mathcal{P}_{Z_N,K}$ is unique, for compatibility with base change (which includes Galois invariance) forces the above definition. \hfill $\Box$

Remark 23. The above proof shows that we have the following formula for $\mathcal{P}_{Z_N,K}$:

$$\mathcal{P}_{Z_N,K}((E_1, E_2, \psi)) = \Phi_N(\mathcal{P}_{Y_N,\overline{K}}((E_1 \otimes \overline{K}, \alpha_2 \circ \psi; E_2 \otimes \overline{K}, \alpha_2))),$$

in which $\alpha_2 : E_2[N]/\overline{\pi} \rightarrow (\mathbb{Z}/N\mathbb{Z})^2$ is any level $N$-structure of $E_2 \otimes \overline{K}$.

Note that for a non-algebraically closed field $K$ the map $\mathcal{P}_{Z_N,K}$ is not injective. Indeed, for any $(E_1, E_2, \psi) \in Z_N(K)$ and any quadratic character $\chi : G_K \rightarrow \{\pm 1\}$ we have

$$\mathcal{P}_{Z_N,K}((E_1, E_2, \psi)) = \mathcal{P}_{Z_N,K}((E_{1,\chi}, E_{2,\chi}, \psi_\chi)).$$
where \( E_{i,\chi} = (E_i)_\chi \) denotes the quadratic twist of \( E_i/K \), and
\[
\psi_\chi : E_{1,\chi}[N] \to E_{2,\chi}[N]
\]
is the \( \chi \)-twist of \( \psi \), i.e. the unique isomorphism such that \( \psi_\chi \otimes K = f_2^{-1} \circ (\psi \otimes K) \circ f_1|_{E_i[N]} \), where \( f_i : E_{i,\chi} \otimes K \cong E_i \otimes K \) is the twist isomorphism associated to \( \chi \). (Thus, \( f_i \) is a \( \overline{K} \)-isomorphism such that \( f_i(id_{E_i \times \sigma}) = \chi(\sigma)(id_{E_i} \times \sigma) \), for all \( \sigma \in G_K \).) Note that it follows from the definition that \( \psi_\chi \circ \overline{K} \circ (id_{E_{1,\chi}} \times \sigma) = (id_{E_{2,\chi}} \times \sigma) \circ \psi_\chi \), so \( \psi_\chi \) exists by descent theory.

However, we do have that \( \mathcal{P}_{Z_N,K} \) is always surjective. For simplicity, we prove the following slightly weaker assertion in which we replace \( Z_N \) by the open subscheme \( Z''_N := \Phi_N(X''_N \times X''_N) \), where \( X''_N = X_N \setminus \pi_N^{-1}\{\bar{P}_0, \bar{P}_1, \bar{P}_\infty\} \).

**Proposition 24.** For every field \( K \) we have \( Z''_N(K) \subset \text{Im}(\mathcal{P}_{Z_N,K}) \).

Before proving this, we first establish the following two facts.

**Lemma 25.** Let \( z \in Z''_N(K) \). Then there exist elliptic curves \( E_1/K \) and \( E_2/K \) and level \( N \) structures \( \alpha_i : E_i[N]/\overline{K} \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \) for \( i = 1,2 \) such that
\[
\Phi_N(\mathcal{P}_{Y_N,K}((E_1 \otimes \overline{K}, \alpha_1; E_2 \otimes \overline{K}, \alpha_2))) = z.
\]

**Proof.** Since \( \Phi_N : Y'_N \to Z'_N \) is a geometric quotient, there exists \( y \in Y'_N(K) \) such that \( \Phi_N(y) = z \). Then by Proposition 20 there are elliptic curves \( E_i/\overline{K} \) and level \( N \) structures \( \alpha_i : E_i[N] \to (\mathbb{Z}/N\mathbb{Z})^2 \) such that \( \mathcal{P}_{Y_N}((E_1/\overline{K}, \alpha_1; E_2/\overline{K}, \alpha_2)) = y \).

Since \( z \in Z''_N(K) \), we have that \( \Psi(z) \in X'_1(K) \times X'_1(K) \cong \mathbb{A}^1(K) \times \mathbb{A}^1(K) \).

But \( \Psi(z) = \Psi(\Phi_N(y)) = (j(E_1/\overline{K}), j(E_2/\overline{K})) \), so we see that \( j(E_i/\overline{K}) \in K \).

This means that there exist elliptic curves \( E'_i/K \) such that \( E'_i \otimes \overline{K} \cong E_i \).

Taking these as representatives of the isomorphism class (and adapting the \( \alpha_i \) accordingly) yields the assertion. \( \square \)

**Lemma 26.** Suppose that \( E_1 \) and \( E_2 \) are elliptic curves over a field \( K \supset \mathbb{Q} \) and \( \psi : E_1[N]/\overline{K} \cong E_2[N]/\overline{K} \) is an isomorphism such that
\[
\mathcal{P}_{Z_N}((E_1 \otimes \overline{K}, E_2 \otimes \overline{K}, \psi)) \in Z''_N(K).
\]

Then there is a character \( \chi : G_K \to \{\pm 1\} \) such that
\[
\psi\chi = \sigma \psi, \quad \text{for all } \sigma \in G_K.
\] (10)
Proof. Let \( \alpha_2 : E_2[N]/\overline{K} \to \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \) be a level \( N \)-structure and put \( \alpha_1 = \alpha_2 \circ \psi \). Then, writing \( E_i/\overline{K} \) in place of \( E_i \otimes \overline{K}/\overline{K} \), we have
\[
\Phi_N((E_1/\overline{K}, \alpha_1; E_2/\overline{K}, \alpha_2)) = (E_1/\overline{K}, E_2/\overline{K}, \psi)
\]
and thus, if we let
\[
z = \mathcal{P}_{Z_N}((E_1/\overline{K}, E_2/\overline{K}, \psi)) \quad \text{and} \quad y = \mathcal{P}_{Y_N}((E_1/\overline{K}, \alpha_1; E_2/\overline{K}, \alpha_2)),
\]
then \( \Phi_N(y) = z \). Since \( z \) is \( G_K \)-invariant, we have for any \( \sigma \in G_K \) that
\[
y^\sigma = \Phi_N^{-1}(z), \quad \text{so} \quad y^\sigma = g_{\sigma} \cdot y,
\]
for some \( g_{\sigma} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). Note that \( g_{\sigma} \) is unique up to \( \pm 1 \) because \( \overline{G}_N = \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1 \) acts freely on \( \Phi_N^{-1}(z) \); here we use the fact that \( z \in Z'_N(K) \).

On the moduli problem, this means that
\[
(E_1/\overline{K}, \alpha_1; E_2/\overline{K}, \alpha_2)^{\sigma} = g_{\sigma} \cdot (E_1/\overline{K}, \alpha_1; E_2/\overline{K}, \alpha_2).
\]
Hence, since \( E_1/\overline{K} \) and \( E_2/\overline{K} \) are \( G_K \)-invariant,
\[
(E_1/\overline{K}, \alpha_1 \sigma; E_2/\overline{K}, \alpha_2 \sigma) = (E_1/\overline{K}, g_{\sigma} \alpha_1; E_2/\overline{K}, g_{\sigma} \alpha_2).
\]
In other words, there are automorphisms \( \chi_{i,\sigma} \) of \( E_i/\overline{K} \) (which depend on the choice of \( g_{\sigma} \)), such that \( g_{\sigma} \alpha_1 \chi_{i,\sigma} = \alpha_2 \sigma \), for \( i = 1, 2 \). Now since \( z \in Z'_N(K) \), we have \( j(E_i/\overline{K}) \neq 0, 1728 \), so \( \text{Aut}(E_i/\overline{K}) = \{\text{id}_{E_i}\} \simeq \{\pm 1\} \), and hence \( \chi_{i,\sigma} \in \{\pm 1\} \). By replacing \( g_{\sigma} \) with \( -g_{\sigma} \) if necessary (and hence replacing \( \chi_{i,\sigma} \) by \( -\chi_{i,\sigma} \)), we may suppose that \( \chi_{2,\sigma} = 1 \), for all \( \sigma \in G_K \). Write \( \chi(\sigma) = \chi_{1,\sigma} \).

We now have, for all \( \sigma \in G_K \),
\[
\psi^\sigma = \alpha_2^{-1} \alpha_1 \sigma = \alpha_2^{-1} g_{\sigma} \alpha_1 \chi(\sigma) = \sigma \alpha_2^{-1} \alpha_1 \chi(\sigma) = \sigma \psi \chi(\sigma),
\]
which is (10). Note that it follows from this equation that \( \chi \) is a character. \( \square \)

of Proposition 24. Let \( z \in Z'_N(K) \). Then by Lemma 25 there are elliptic curves \( E_i/K \) and level \( N \)-structures \( \alpha_i \) on \( E_i \otimes K \) for \( i = 1, 2 \) such that
\[
\mathcal{P}_{Z_N}(E_1/\overline{K}, E_2/\overline{K}, \psi) = z,
\]
where \( \psi = \alpha_2^{-1} \circ \alpha_1 \). Let \( \chi \) be the corresponding quadratic character given by Lemma 26. If \( E_{2,\chi}/K \) is the twist of \( E_2/K \) by \( \chi \) and \( \tau : E_2 \otimes K \sim E_{2,\chi} \otimes K \) the corresponding isomorphism such that \( \tau \chi(\sigma) = \sigma \tau \), then we have \( \langle E_1/K, E_{2,\chi}/K, \tau \psi \rangle = (E_1/K, E_2/K, \psi) \). By equation (10) we have \( \tau \psi^\sigma = \tau \psi(\sigma) \sigma \psi = \sigma \tau \psi \), for every \( \sigma \in G_K \). Thus, \( \tau \psi \) descends to \( K \) and so \( \langle E_1/K, E_{2,\chi}/K, \tau \psi \rangle \in Z_N(K) \). We thus have (cf. Remark 23)
\[
\mathcal{P}_{Z_N,K}(E_1/K, E_{2,\chi}/K, \tau \psi) = \mathcal{P}_{Z_N,K}(E_1/K, E_2/K, \psi) = \mathcal{P}_{Z_N,K}(E_1/K, E_2/K, \psi) = z,
\]
which shows that \( z \in \text{Im}(\mathcal{P}_{Z_N,K}) \), as claimed. \( \square \)
For each \( \varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times \), we also have the following subfunctor \( Z_{N,\varepsilon} \) of \( Z_N \) which is defined by
\[
Z_{N,\varepsilon}(S) = \{ (E_1/S, E_2/S, \psi) \in Z_N(S) : e_N \circ \psi = e_N^\varepsilon \}.
\]
It is then clear that \( Z(S) = \bigsqcup Z_{N,\varepsilon}(S) \).

Restricting the above (functor) morphisms \( \Phi_N \) and \( \mathcal{P}_{Y_N} \) (cf. Remark 21 and Prop. 22) to the component \( Y_{N,\varepsilon} \) of \( Y_N \) yields morphisms
\[
\Phi_{N,\varepsilon} = \Phi_{N|Y_{N,\varepsilon}} : Y_{N,\varepsilon} \to Z_{N,\varepsilon} \quad \text{and} \quad \mathcal{P}_{Y_{N,\varepsilon}} = \mathcal{P}_{Y_N|Y_{N,\varepsilon}} : Y_{N,\varepsilon} \to h_{Y_{N,\varepsilon}}.
\]
On the other hand, we also have the quotient morphism
\[
\Phi_{N,\varepsilon} : Y'_{N,\varepsilon} \to Z'_{N,\varepsilon}/\mathbb{Q} = \hat{\Delta}_{N,\varepsilon} \backslash Y'_{N,\varepsilon}
\]
where \( Z'_{N,\varepsilon}/\mathbb{Q} \) denotes the non-cuspidal part of the surface \( Z_{N,\varepsilon}/\mathbb{Q} \) defined in section 3.2 (cf. Theorem 14 and Remark 15). These morphisms are connected as follows:

**Theorem 27.** For every field \( K \supset \mathbb{Q} \) there is a unique map
\[
\mathcal{P}_{Z_{N,\varepsilon},K} : Z_{N,\varepsilon}(K) \to Z'_{N,\varepsilon}/\mathbb{Q}(K)
\]
which is compatible with base change (of fields) such that
\[
\Phi_{N,\varepsilon} \circ \mathcal{P}_{Y_{N,\varepsilon},K} = \mathcal{P}_{Z_{N,\varepsilon},K} \circ \Phi_{N,\varepsilon}.
\]
(11)

In addition, \( \mathcal{P}_{Z_{N,\varepsilon},K} \) is a bijection if \( K \) is algebraically closed, whereas for an arbitrary field \( K \) we have
\[
Z''_{N,\varepsilon}(K) \subset \text{Im}(\mathcal{P}_{Z_{N,\varepsilon},K}).
\]

**Proof.** Define \( \mathcal{P}_{Z_{N,\varepsilon},K} = \mathcal{P}_{Z_{N,K}|Z_{N,\varepsilon}(K)} : Z_{N,\varepsilon}(K) \to Z'_N(K) \). Then it is clear from Proposition 22 that \( \mathcal{P}_{Z_{N,\varepsilon},K} \) is compatible with base change and that equation (11) holds. In addition, it follows that if \( K \) is algebraically closed, then \( \mathcal{P}_{Z_{N,\varepsilon},K} \) is a bijection onto its image. Now by (11), this image has to be \( Z_{N,\varepsilon}/\mathbb{Q}(K) \) because it is the image of \( \Phi_{N,\varepsilon} \) (and because \( \Phi_{N,\varepsilon} \) is surjective). From this we see in particular that for any \( K \), the image of \( \mathcal{P}_{Z_{N,\varepsilon},K} \) is contained in \( Z_N(K) \cap Z_{N,\varepsilon}/\mathbb{Q}(K) = Z_{N,\varepsilon}/\mathbb{Q}(K) \), and so \( \mathcal{P}_{Z_{N,\varepsilon},K} \) satisfies the requirements of the theorem. \( \square \)
4.3 Proof of Theorem 2

Lemma 28. Let \( \langle E_1/\mathbb{Q}, E_2/\mathbb{Q}, \psi \rangle \in \mathbb{Z}_{N,1}(\mathbb{Q}) \). If \( E_1 \) and \( E_2 \) are \( \mathbb{Q} \)-isogeneous, then \( \mathcal{P}_{\mathbb{Z}_{N,1}}(\langle E_1, E_2, \psi \rangle_{\mathbb{Q}}) \in M(\mathbb{Z}_{N,1}) \), i.e., it is a Mazur trivial point.

Proof. Since \( E_1 \) and \( E_2 \) are \( \mathbb{Q} \)-isogeneous, there is a cyclic \( \mathbb{Q} \)-isogeny between them. If \( n \) is its degree, then by the modular description of \( X_0'(n) \) we have \( \mathcal{P}_{\mathbb{Y}_1}(\langle E_1, E_2 \rangle_{\mathbb{Q}}) \in \xi_n(X_0'(n)(\mathbb{Q})) \), where \( \xi_n \) is the normalization map as in section 3.3. By Proposition 22, this implies that \( \Psi \circ \mathcal{P}_{\mathbb{Z}_{N,1}}(\langle E_1, E_2, \psi \rangle_{\mathbb{Q}}) \in \xi_n(X_0'(n)(\mathbb{Q})) \) and so the assertion follows from the definition of \( M(\mathbb{Z}_{N,1}) \) (cf. section 3.3).

Proof of Theorem 2. Consider the infinite number of sections of \( \tilde{S}_i : \mathbb{P}^1_{\mathbb{Q}} \to \tilde{Z} \) given by Theorem 19. By Propositions 4 and 7, the proper transform \( S_i \) in \( \mathbb{Z}/\mathbb{Q} \) of each \( \tilde{S}_i \) is a curve \( S_i \simeq \mathbb{P}^1_{\mathbb{Q}} \) on \( Z \); by construction, \( |S_i(\mathbb{Q}) \setminus M| = \infty \), where \( M = M(\mathbb{Z}_{11,1}) \). By Theorem 27 and Lemma 28, the points of \( S_i(\mathbb{Q}) \cap Z'(\mathbb{Q}) \setminus M \) correspond to a one-parameter family of isomorphism classes of pairs of elliptic curves over \( \mathbb{Q} \) with symplectically isomorphic 11-structure which are not \( \mathbb{Q} \)-isogeneous, and so we have proved the theorem.

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References


