The Space of Binary Theta Series

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1 Introduction

The purpose of this paper is to study the space Θ_D generated by the binary theta series ϑ_f attached to the primitive, positive-definite binary quadratic forms $f(x, y) = ax^2 + bxy + cy^2$ of discriminant $D = b^2 - 4ac < 0$. It is curious that while much has been written about the space generated by theta series attached to quadratic forms in $2k \geq 4$ variables (cf. [10] and the references therein), the binary case does not seem to have been treated in detail in the literature.

By the work of Weber[25], Hecke[8] and Schoeneberg[18], it is known that Θ_D is a subspace of the space $M_1(|D|, \psi_D)$ of modular forms of weight 1, level $|D|$ and Nebentypus ψ_D, where $\psi_D = (\frac{D}{\cdot})$ is the Kronecker-Legendre character. Contrary to the case of higher weight, the space of binary theta series is often a proper subspace of $M_1(|D|, \psi_D)$ (cf. Remark 16) and so it is of interest to be able to identify it inside the space of modular forms.

As a first step towards this, we explain in this paper how each theta series ϑ_f can be expressed as a linear combination of the *canonical* (extended) Atkin-Lehner basis of $M_1(|D|, \psi_D)$; cf. Remark 30(b). This will be used in the next paper[12] to give an *intrinsic* description of the space of theta series.

To this end we first observe that Θ_D has a natural basis $\{\vartheta_\chi\}$ indexed by the characters $\chi \in \mathrm{Cl}(D)^*$ of the class group $\mathrm{Cl}(D)$ of forms of discriminant D. It turns out that each ϑ_χ is a normalized eigenform with respect to Hecke algebra $\mathbb{T}(D)$; the latter is the algebra generated by all Hecke operators T_n with $(n, D) = 1$.

Theorem 1 *The space Θ_D is a $\mathbb{T}(D)$-submodule of $M_1(|D|, \psi_D)$ of multiplicity one, and has a canonical basis $\{\vartheta_\chi\}$ consisting of normalized $\mathbb{T}(D)$-eigenforms. Furthermore, ϑ_χ is a cusp form if and only if χ is not a quadratic character.*

This theorem implies the following interesting result.

Theorem 2 *We have $\Theta_D = \Theta_D^E \oplus \Theta_D^S$, where $\Theta_D^E = \Theta_D \cap E_1(|D|, \psi_D)$ denotes the Eisenstein space part and $\Theta_D^S = \Theta_D \cap S_1(|D|, \psi)$ denotes the cusp space part of Θ_D. Moreover,*

$$\dim \Theta_D^E = g_D \quad \textit{and} \quad \dim \Theta_D^S = \tfrac{1}{2}(h_D - g_D), \tag{1}$$

where $h_D = |\mathrm{Cl}(D)|$ denotes the number of classes of forms of discriminant D, and $g_D = [\mathrm{Cl}(D) : \mathrm{Cl}(D)^2]$ denotes the number of genera.

Note that it follows from (1) that Θ_D has no non-zero cusp forms if and only if $h_D = g_D$, i.e. if and only if D is an *idoneal discriminant.* As is explained in Remark 17(b) below, this can be viewed as an alternate version of Theorem 3 of Kitaoka[13].

If D is a *fundamental* discriminant, i.e. if $D = d_K$, where d_K is the discriminant of $K := \mathbb{Q}(\sqrt{D})$, then each ϑ_χ is a primitive form (newform) (cf. Remark 16), and hence in this case the ϑ_χ's are part of the canonical Atkin-Lehner basis. However, in the general case this is no longer true for every $\chi \in \mathrm{Cl}(D)^*$ because some of the characters $\chi \in \mathrm{Cl}(D)^*$ are not *primitive*, i.e. they are lifts $\chi = \chi' \circ \pi$ of characters $\chi' \in \mathrm{Cl}(D')^*$ of some "lower level" $D'|D$ via the canonical map $\pi = \pi_{D,D'} : \mathrm{Cl}(D) \to \mathrm{Cl}(D')$. In the general case the situation is as follows.

Theorem 3 *Let χ be a character on the class group $\mathrm{Cl}(D)$, where $D = f_D^2 d_K$.*

(a) *There is a unique divisor $f_\chi | f_D$ and a unique primitive character χ_{pr} on the class group $\mathrm{Cl}(D_\chi)$, where $D_\chi = f_\chi^2 d_K$, such that $\chi = \chi_{pr} \circ \bar{\pi}_{D,D_\chi}$.*

(b) *The form $\vartheta_{\chi_{pr}} \in \Theta_{D_\chi}$ is a primitive form of level $|D_\chi|$ whose L-function is the Hecke L-function associated to a suitable Hecke character $\tilde{\chi}_{pr}$, i.e. $L(s, \vartheta_{\chi_{pr}}) = L(s, \tilde{\chi}_{pr})$. Moreover, there exist constants $c_n(\chi) \in \mathbb{R}$ such that*

$$\vartheta_\chi(z) \;=\; \sum_{n|\bar{f}_\chi^2} c_n(\chi) \vartheta_{\chi_{pr}}(nz), \tag{2}$$

where $\bar{f}_\chi = f_D/f_\chi$. Furthermore, the function $n \mapsto c_n(\chi)$ is multiplicative and has the generating function

$$C(s,\chi) \;:=\; \sum_{n|\bar{f}_\chi^2} c_n(\chi) n^{-s} \;=\; L(s,\vartheta_\chi)/L(s,\vartheta_{\chi_{pr}}) \;=\; L(s,\vartheta_\chi)/L(s,\tilde{\chi}_{pr}). \tag{3}$$

Note that while $L(s, \vartheta_{\chi_{pr}}) = L(s, \tilde{\chi}_{pr})$ is a classical Hecke L-function and hence is well-understood, the L-function $L(s, \vartheta_\chi)$ is more complicated and is, in fact, unknown. Thus, (3) does not help in determining the constants $c_n(\chi)$. Instead, we calculate $C(s,\chi)$ directly in Theorem 28 by using facts about ideals in quadratic orders which are presented in the Appendix (§6). As a consequence, we thus obtain an explicit expression for L-function $L(s,\chi)$; cf. Corollary 31. In particular, we obtain

Corollary 4 *If $p^{\bar{e}_p} || \bar{f}_\chi$ denotes the largest power of p dividing $\bar{f}_\chi$, then the p-Euler factor $L_p(s,\chi)$ of $L(s,\chi)$ at $p|\bar{f}_\chi$ is*

$$L_p(s,\chi) \;=\; \frac{1 - p^{(1-2s)\bar{e}_p}}{1 - p^{1-2s}} + \frac{\left(1 - \frac{1}{p}\psi_{D_\chi}(p)\right) p^{(1-2s)\bar{e}_p}}{1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}}.$$

At the end of §5, we present some special cases and numerical examples of these results; cf. Examples 35 and 37.

After the first version of this paper was completed, Norm Hurt drew my attention to the papers of Sun and Williams [22], [23] which are partially related to some of the topics mentioned here. Indeed, Corollaries 31 and 4 can be viewed (for $D < 0$) as a generalization of many of their results; cf. Remark 32(c) for more details.

2 The basis $\{\vartheta_\chi\}$ of Θ_D

As in the introduction, let $f(x,y) = ax^2 + bxy + cy^2$ be a primitive, positive definite binary quadratic form of discriminant $D = b^2 - 4ac < 0$. Thus $a, b, c \in \mathbb{Z}$ and $(a, b, c) = 1$, and $D = f_D^2 d_K$, where d_K is the (fundamental) discriminant of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. For a fixed D, let Q_D denote the set of all such forms. The *binary theta series* attached to $f \in Q_D$ is the function on the upper half-plane $\mathfrak{H}$ given by

$$\vartheta_f(z) \;=\; \sum_{x,y\in\mathbb{Z}} e^{2\pi i f(x,y)z} \;=\; \sum_{n=0}^{\infty} r_n(f) e^{2\pi i n z},$$

where $r_n(f) = \#\{(x,y) \in \mathbb{Z}^2 : f(x,y) = n\}$ denotes the number of representations of n by f. The following fact is fundamental for most of this paper.

Proposition 5 *If $f \in Q_D$, then ϑ_f is a modular form of level $|D|$ and Nebentypus $\psi_D = \left(\frac{D}{\cdot}\right)$. Thus $\Theta_D := \sum_{f\in Q_D} \mathbb{C}\vartheta_f$ is a subspace of $M_1(|D|, \psi_D)$.*

Proof. See Schoeneberg[19], ch. IX, Theorems 4 and 5, or Miyake[16], Corollary 4.9.5(3).

Remark 6 (a) This result was proved by Weber[25] in 1893 (see his formula §15 (21)), except that he did not verify that ϑ_f is holomorphic at all the cusps. In 1926 Hecke[8] proved a more general result from which above result can be deduced (cf. Remark 11(c)). Siegel[21] proved something weaker but for quadratic forms in an arbitrary number $m \geq 2$ of variables; cf. [21], Hilfssatz 30 and 31 and also Satz 4, where the case $m = 2$ is excluded. In 1939 Schoeneberg[18] proved a very general result about theta series which includes the above result as a special case.

(b) The L-function associated to the modular form ϑ_f is

$$Z_f(s) \;=\; L(s, \vartheta_f) \;=\; \sum_{n\geq 1} \frac{r_n(f)}{n^s} \;=\; \sum_{(x,y)\neq(0,0)} f(x,y)^{-s}, \quad \text{where } \mathrm{Re}(s) > 1.$$

This function is often called the *Epstein zeta-function* of f (cf. e.g. [4]), even though it was introduced by Dirichlet in 1839 (cf. [5], §6.18 (p. 358)) and was studied intensively by him and by Kronecker[14] (and by others) many years before Epstein.

Recall that the group $\mathrm{GL}_2(\mathbb{Z})$ acts on binary quadratic forms by change of coordinates, and that this action preserves the set Q_D. It is immediate that $r_n(fT) = r_n(f)$, for all $n \geq 0$ and $T \in \mathrm{GL}_2(\mathbb{Z})$, so $\vartheta_{fT} = \vartheta_f$. We can thus index the theta series by the quotient set

$$\overline{Q}_D \;=\; Q_D/\mathrm{GL}_2(\mathbb{Z}) \;=\; Q_D/\approx,$$

where $\approx$ denotes the equivalence relation induced by $\mathrm{GL}_2(\mathbb{Z})$-equivalence.

Proposition 7 *The set* $\{\vartheta_f : f \in \overline{Q}_D\}$ *is a basis of the space* Θ_D. *Thus*

$$\dim \Theta_D \;=\; \overline{h}_D := \#\overline{Q}_D.$$

As we shall see, this follows easily from the following basic facts about binary quadratic forms which are due to Dirichlet[5] and Weber[24].

Lemma 8 *Let* $f, f_1, f_2 \in Q_D$ *be primitive quadratic forms. Then:*

(a) *There exist infinitely many prime numbers* p *with* $r_p(f) > 0$.

(b) *If there is a prime number* $p \nmid D$ *with* $r_p(f_i) > 0$, *for* $i = 1, 2$, *then* $f_1 \approx f_2$.

Proof. (a) This was first proved by Dirichlet[5] for prime discriminants and by Weber[24] in the general case. A proof using class field theory is given in Cox[3], Theorem 9.12.

(b) Weber[25] states this (elementary) result on p. 259 and points out that it follows easily from his paper [24] (or from a paper of Schering). It is restated and proved (without references) as Satz 1 in Piehler[17].

Proof of Proposition 7. Let $f_1, \ldots, f_N \in Q_D$ be a system of representatives of $\overline{Q}_D$; thus, $N = \overline{h}_D$. It is clear from what was said above that $\vartheta_{f_1}, \ldots, \vartheta_{f_N}$ generate Θ_D. To prove that they are linearly independent, suppose that $c_1\vartheta_{f_1} + \ldots + c_N\vartheta_{f_N} = 0$, for some $c_1, \ldots, c_N \in \mathbb{C}$. Thus $c_1 r_n(f_1) + \ldots + c_N r_n(f_N) = 0$, for all $n \geq 0$.

Fix i. By Lemma 8(a) we know that there is a prime $p = p_i \nmid D$ with $r_p(f_i) > 0$, and from Lemma 8(b) it follows that $r_p(f_j) = 0$ for all $j \neq i$. We thus have that $c_i r_p(f_i) = 0$, so $c_i = 0$, and hence $\{\vartheta_{f_i}\}$ is a basis of Θ_D, as claimed.

We now introduce another basis of the space Θ_D. For this, recall that by Gauss's theory of composition of forms (cf. Gauss[6]) the set

$$\mathrm{Cl}(D) \;:=\; Q_D/\mathrm{SL}_2(\mathbb{Z}) \;=\; Q_D/\!\sim$$

has the structure of an abelian group. The identity of $\mathrm{Cl}(D)$ is the class of the principal form 1_D which is defined by $1_D(x, y) = x^2 + \varepsilon xy + \frac{\varepsilon - D}{4} y^2$, for $D \equiv \varepsilon \pmod 4$ and $\varepsilon \in \{0, 1\}$. Note also that the $\mathrm{SL}_2(\mathbb{Z})$-equivalence relation $\sim$ is related to the previous $\mathrm{GL}_2(\mathbb{Z})$-equivalence relation $\approx$ by

$$f_1 \approx f_2 \quad \Leftrightarrow \quad f_1 \sim f_2 \text{ or } f_1 \sim f_2^{-1}, \tag{4}$$

where f_2^{-1} denotes a representative of the inverse class of f_2 (in the group $\mathrm{Cl}(D)$). From this it follows easily that

$$\overline{h}_D \;=\; \frac{1}{2}(g_D + h_D), \quad \text{where } h_D = |\mathrm{Cl}(D)| \text{ and } g_D = [\mathrm{Cl}(D) : \mathrm{Cl}(D)^2]. \tag{5}$$

Let $\chi \in \mathrm{Cl}(D)^* := \mathrm{Hom}(\mathrm{Cl}(D), \mathbb{C}^\times)$ be a character on $\mathrm{Cl}(D)$, and put

$$\vartheta_\chi \;:=\; \frac{1}{w_D} \sum_{f \in Cl(D)} \chi(f)\vartheta_f \;\in\; \Theta_D, \tag{6}$$

where $w_D = r_1(1_D)$. (Thus, as is well known, $w_D = 2$ when $D < -4$ and $w_{-4} = 4$ and $w_{-3} = 6$.) Note that the terms on the right hand side are not linearly independent since we sum over $\mathrm{Cl}(D)$ in place of $\overline{Q}_D$. However, by using (4) and noting that $\chi(f^{-1}) = \overline{\chi(f)}$, we can re-write (6) in the form

$$\vartheta_\chi = \frac{1}{w_D} \sum_{\bar{f} \in \overline{Q}_D} w(\bar{f}) \mathrm{Re}(\chi(\bar{f})) \vartheta_{\bar{f}}, \tag{7}$$

where $w(\bar{f}) = \#\{f \in \mathrm{Cl}(D) : f \approx \bar{f}\}$. (Thus, $w(f) = 1$, if $f^2 \sim 1_D$ and $w(f) = 2$ otherwise.) From this expression we see immediately that $\vartheta_{\chi^{-1}} = \vartheta_\chi$, and so it is useful to index the ϑ_χ's by the set $\overline{\mathrm{Cl}(D)}^* = \mathrm{Cl}(D)^*/(\chi \mapsto \chi^{-1})$. We then have:

Proposition 9 *If $f \in Q_D$ is a primitive quadratic form, then*

$$\vartheta_f = \frac{w_D}{h_D} \sum_{\chi \in Cl(D)^*} \overline{\chi}(f) \vartheta_\chi = \frac{w_D}{h_D} \sum_{\chi \in \overline{Cl(D)}^*} w(\chi) \mathrm{Re}(\chi(f)) \vartheta_\chi, \tag{8}$$

where $w(\chi) = \#\{\chi, \chi^{-1}\}$. Thus $\{\vartheta_\chi : \chi \in \overline{\mathrm{Cl}(D)}^*\}$ is also a basis of Θ_D.

Proof. The second identity of (8) follows immediately from the fact that $\vartheta_{\chi^{-1}} = \vartheta_\chi$. To prove the first identity, recall that the orthogonality relations for group characters imply that

$$\sum_{\chi \in Cl(D)^*} \chi(f) = \begin{cases} h_D & \text{if } f \sim 1_D, \\ 0 & \text{otherwise.} \end{cases}$$

From this, together with the definition (6) of ϑ_χ, it follows that

$$\begin{aligned} w_D \sum_{\chi \in Cl(D)^*} \overline{\chi}(f) \vartheta_\chi &= \sum_{\chi \in Cl(D)^*} \overline{\chi}(f) \sum_{f_1 \in Cl(D)} \chi(f_1) \vartheta_f \\ &= \sum_{\chi \in Cl(D)^*} \chi(f^{-1}) \sum_{f_1 \in Cl(D)} \chi(f_1) \vartheta_{f_1} \\ &= \sum_{f_1 \in Cl(D)} \vartheta_{f_1} \sum_{\chi \in Cl(D)^*} \chi(f^{-1} f_1) = h_D \vartheta_f, \end{aligned}$$

which proves the first identity of (8) and hence (8) itself.

From (8) we see that the set $\{\vartheta_\chi : \chi \in \overline{\mathrm{Cl}(D)}^*\}$ generates Θ_D. Since $\mathrm{Cl}(D) \simeq \mathrm{Cl}(D)^*$, it follows that $\#(\overline{\mathrm{Cl}(D)}^*) = \#(\mathrm{Cl}(D)/(f \mapsto f^{-1})) = \#\overline{Q}_D = \dim \Theta_D$, where the last two identities follow from (4) and Proposition 7, respectively. Thus, the set $\{\vartheta_\chi : \chi \in \overline{\mathrm{Cl}(D)}^*\}$ is a basis of Θ_D.

We now examine the Fourier coefficients $a_n(\chi) := a_n(\vartheta_\chi)$ of the modular form ϑ_χ more closely. For this we shall use a basic result due to Dedekind that the class group $\mathrm{Cl}(D)$ can be identified with the group $\mathrm{Cl}(\mathfrak{O}_D) = \mathrm{Pic}(\mathfrak{O}_D) = I(\mathfrak{O}_D)/P(\mathfrak{O}_D)$ of classes of invertible (fractional) ideals of the order $\mathfrak{O}_D$ of discriminant D in $K = \mathbb{Q}(\sqrt{D})$. (Here we use the terminology and notation of Appendix §6.1.) More precisely, if $L(f) = a\mathbb{Z} + \frac{-b+\sqrt{D}}{2}\mathbb{Z}$ denotes the quadratic lattice in K associated to the form $f(x,y) = ax^2 + bxy + cy^2 \in Q_D$, then the rule $f \mapsto L(f)$ defines an isomorphism

$$\lambda_D : \mathrm{Cl}(D) \xrightarrow{\sim} \mathrm{Cl}(\mathfrak{O}_D) := I(\mathfrak{O}_D)/P(\mathfrak{O}_D);$$

cf. Cox[3], Theorem 7.7 (p. 137). We observe:

Proposition 10 *Let $\chi \in \mathrm{Cl}(D)^*$. If $n \geq 1$, then the n-th Fourier coefficient of ϑ_χ is given by the formula*

$$a_n(\chi) = \sum_{\mathfrak{a} \in Id_n(\mathfrak{O}_D)} \chi^*(\mathfrak{a}), \tag{9}$$

where $Id_n(\mathfrak{O}_D)$ denotes the set of invertible ideals of $\mathfrak{O}_D$ of norm n and $\chi^* = \chi \circ \lambda_D^{-1} \in \mathrm{Cl}(\mathfrak{O}_D)^*$. Moreover, $a_0(\chi) = 0$ except when $\chi = 1$ is trivial, and then $a_0(1) = \frac{h_D}{w_D}$.

Proof. Recall that if $f \in Q_D$, then $f(x,y) = ax^2 + bxy + cy^2 = N_K(ax - \beta_f y)/N(L)$, where $\beta_f = \frac{-b+\sqrt{D}}{2}$, N_K denotes the field norm and $N(L) = a$ is the norm of the lattice $L := L(f)$; cf. [3], p. 137. We thus obtain (in the notation of the appendix) that

$$\vartheta_f(z) = \sum_{x,y \in \mathbb{Z}} e^{2\pi i f(x,y) z} = \sum_{\alpha \in L(f)} e^{2\pi i N_K(\alpha) z / N(L(f))} = 1 + w_D \sum_{\substack{\mathfrak{a} \in Id(\mathfrak{O}_D) \\ \mathfrak{a}L(f) \in P(\mathfrak{O}_D)}} e^{2\pi i N(\mathfrak{a}) z}, \tag{10}$$

where the last equality follows from the fact that $\mathfrak{a} := \alpha L(f)^{-1}$ is an integral $\mathfrak{O}_D$-ideal if and only if $\alpha \in L(f)$ (where $\alpha \in K^\times$), together with the fact that $|\mathcal{O}_D^\times| = w_D$. From this it thus follows from the definition that

$$\vartheta_\chi(z) = \frac{1}{w_D} \sum_{f \in Cl(D)} \chi(f) \left(1 + w_D \sum_{\substack{\mathfrak{a} \in Id(\mathfrak{O}_D) \\ \mathfrak{a}L(f) \in P(\mathfrak{O}_D)}} e^{2\pi i N(\mathfrak{a}) z} \right) = c_\chi + \sum_{\mathfrak{a} \in Id(\mathfrak{O}_D)} \chi^*(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z},$$

where $c_\chi = \frac{1}{w_D} \sum_{f \in Cl(D)} \chi(f)$. This proves (9). Note that $c_\chi = 0$ except when $\chi = 1$; in the latter case clearly $c_\chi = \frac{h_D}{w_D}$.

Remark 11 (a) The above result shows that the L-function associated to ϑ_χ is

$$L(s, \vartheta_\chi) = \sum_{n \geq 1} \frac{a_n(\chi)}{n^s} = \sum_{\mathfrak{a} \in Id(\mathfrak{O}_D)} \frac{\chi^*(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where $\chi^* = \chi \circ \lambda_D^{-1}$. Thus, $L(s, \vartheta_\chi)$ coincides with the L-function $L_{\mathfrak{O}_D}(s, \chi^*)$ defined by the first equation on the bottom p. 280 of the first edition of [15].

(b) For later reference we note that for the trivial character $\chi = 1$ the equations (6) and (9) yield that

$$a_n(1) \;=\; \frac{1}{w_D} \sum_{f \in Cl(D)} r_n(f) \;=\; \#Id_n(\mathfrak{O}_D). \tag{11}$$

This number will be computed computed below (cf. Remark 32(b)) and is well-known when $(n, f_D) = 1$; cf. (60).

(c) Note that it follows from the first part of (10) that if $f \in Q_D$, then

$$\vartheta_f(z) \;=\; \sum_{i=1}^{f_D} \vartheta(f_D z;\, \alpha_i \sqrt{d_K},\, \mathfrak{a}_f,\, f_D \sqrt{d_K}),$$

where $\vartheta(z; \rho, \mathfrak{a}, Q\sqrt{d_K}) \;=\; \sum_{\mu \equiv \rho(\mathfrak{a} Q \sqrt{d_K})} e^{2\pi i z N_K(\mu)/(N(\mathfrak{a})Q|d_K|)}$ is as in Hecke[8], $\mathfrak{a}_f = L(f)\mathcal{O}_K$, and $\{\alpha_i\}$ is a system of coset representatives of $L(f)/f_D\mathfrak{a}_f$. Thus, Θ_D is a subspace of the space generated by dilations of Hecke's theta-functions.

By using results about ideals in the ring $\mathfrak{O}_D$ (cf. Appendix, §6.2), we obtain the following important result.

Theorem 12 *If $\chi \in \mathrm{Cl}(D)^*$, then the function $n \mapsto a_n(\chi)$ is multiplicative. Thus ϑ_χ is a normalized eigenfunction with eigenvalue $a_n(\chi)$ with respect to the Hecke operator T_n whenever $(n, f_D) = 1$.*

Proof. Put $\chi^* = \chi \circ \lambda_D^{-1} \in \mathrm{Cl}(\mathfrak{O}_D)^*$, and write $Id_n = Id_n(\mathfrak{O}_D)$. If $(m, n) = 1$, then from (9) together with Proposition 44 of §6.2 we obtain that

$$a_{mn}(\chi) = \sum_{\mathfrak{a} \in Id_{mn}} \chi^*(\mathfrak{a}) = \sum_{\mathfrak{b} \in Id_m} \sum_{\mathfrak{c} \in Id_n} \chi^*(\mathfrak{b}\mathfrak{c}) = \sum_{\mathfrak{b} \in Id_m} \chi^*(\mathfrak{b}) \sum_{\mathfrak{c} \in Id_n} \chi^*(\mathfrak{c}) = a_m(\chi) a_n(\chi),$$

which shows that the function $n \mapsto a_n(\chi)$ is multiplicative. Moreover, since $a_1(\chi) = \chi^*(\mathfrak{O}_D) = 1$, we see that ϑ_χ is normalized.

From this it follows from Hecke[9], Satz 42, that ϑ_χ is a T_n-eigenfunction, at least when $(n, |D|) = 1$. By using the results of [16], this can be refined to yield the above assertion also for $(n, f_D) = 1$, as we shall now show.

Fix $n \geq 1$ with $(n, f_D) = 1$, and consider the function $g := (\vartheta_\chi)|_1 T_n - a_n(\chi)\vartheta_\chi \in M_1(|D|, \psi_D)$. For any $m \geq 1$ with $(m, n) = 1$ we have by [16], Lemma 4.5.14, and the above result that the m-th Fourier coefficient of g is $a_m(g) = a_{nm}(\chi) - a_n(\chi)a_m(\chi) = 0$. Thus, since ψ_D has conductor $|d_K|$ and since $\frac{|D|}{|d_K|} = f_D^2$, we see that g satisfies the hypothesis of Theorem 4.6.8(1) of [16] (with $l = n$), and so $g = 0$. This means that ϑ_χ is a T_n-eigenfunction with eigenvalue $a_n(\chi)$, as asserted.

3 The Eisenstein and cuspidal parts of Θ_D

By the basic theory of modular forms, the space $M_k(N, \psi)$ of modular forms of level N, weight k and Nebentypus ψ has a canonical decomposition

$$M_k(N, \psi) = E_k(N, \psi) \oplus S_k(N, \psi) \tag{12}$$

into its Eisenstein part $E_k(N, \psi)$ and cuspidal part $S_k(N, \psi)$; cf. [16], Theorem 4.7.2 (together with Theorem 2.1.7).

Since $\Theta_D \subset M_1(|D|, \psi_D)$, we can define its Eisenstein and cuspidal part by

$$\Theta_D^E := \Theta_D \cap E_1(|D|, \psi_D) \quad \text{and} \quad \Theta_D^S := \Theta_D \cap S_1(|D|, \psi_D),$$

respectively.

We now want to find canonical bases for these spaces. As we shall see, the ϑ_χ's serve this purpose: it turns out that either $\vartheta_\chi \in \Theta_D^E$ or $\vartheta_\chi \in \Theta_D^S$; cf. Theorem 14 and Remark 17(a) below.

To verify this, we shall compare the coefficients of ϑ_χ to those of a suitable modular form attached to some *Hecke character* on the ray class group mod $f_D\mathfrak{O}_K$ of $K = \mathbb{Q}(\sqrt{D})$. To achieve this, recall first that the map $\mathfrak{a} \mapsto \mathfrak{a} \cap \mathfrak{O}_D$ induces an isomorphism

$$\varphi_D : I_K(f_D)/P_{K,\mathbb{Z}}(f_D) \xrightarrow{\sim} \mathrm{Cl}(\mathfrak{O}_D),$$

where $I_K(f)$ is the group of fractional ideals of $\mathfrak{O}_K$ which are prime to f and $P_{K,\mathbb{Z}}(f)$ is the subgroup of $I_K(f)$ generated by principal ideals of the form $\alpha\mathfrak{O}_K$, where $\alpha \in \mathfrak{O}_K$ satisfies $\alpha \equiv a \pmod{f\mathfrak{O}_K}$, for some $a \in \mathbb{Z}$ with $(a, f) = 1$; cf. Cox[3], Proposition 7.22. We then have:

Proposition 13 *Let $\chi \in \mathrm{Cl}(D)^*$, and let $\tilde{\chi} := \chi \circ \lambda_D^{-1} \circ \varphi_D$ be the associated Hecke character on $I_K(f_D)$. If $n \geq 1$, then*

$$a_n(\chi) = a_n(\tilde{\chi}) := \sum_{\mathfrak{a} \in Id_n(\mathfrak{O}_K)} \tilde{\chi}(\mathfrak{a}), \quad \textit{provided that } (n, f_D) = 1. \tag{13}$$

Proof. By Proposition 45 we know that the map $\mathfrak{a} \mapsto \mathfrak{a} \cap \mathfrak{O}_D$ induces a bijection

$$Id_n(\mathfrak{O}_K) \xrightarrow{\sim} Id_n(\mathfrak{O}_D), \quad \text{whenever } (n, f_D) = 1,$$

and so the assertion (13) follows in view of (9).

In the next sections we shall study the precise relation between $a_n(\chi)$ and $a_n(\tilde{\chi})$ also in the case that $(n, f_D) > 1$. As we shall see, the formula (13) is in general no longer true in this case, and has to be replaced by the more complicated formula (31) below. Nevertheless, the above relation (13) suffices to derive many useful properties about the ϑ_χ's. To formulate these in a convenient manner, it is useful to introduce the following notation.

Notation. If f_1 and f_2 are two modular forms and if N an integer, then we write

$$f_1 \sim_N f_2 \quad \overset{\text{def}}{\Leftrightarrow} \quad a_n(f_1) = a_n(f_2), \quad \text{for all } n \geq 1 \text{ with } (n,N) = 1.$$

With this notation, we can then reformulate (13) in terms of the modular form $f(z;\tilde{\chi})$ (cf. [16], p. 183) as follows:

$$\vartheta_\chi(z) \ \sim_{f_D} \sum_{\mathfrak{a}\in Id(\mathfrak{O}_K)\cap I_K(f_D)} \tilde{\chi}(\mathfrak{a}) e^{2\pi i N(\mathfrak{a})z} \ \sim_{f_D} \ f(z;\tilde{\chi}). \tag{14}$$

Theorem 14 *Let $\chi \in \mathrm{Cl}(D)^*$ be a character. If χ is not quadratic, i.e. if $\chi^2 \neq 1$, then ϑ_χ is a cusp form, and otherwise ϑ_χ is in the Eisenstein space.*

Proof. Let $\tilde{\chi} := \chi \circ \lambda_D^{-1} \circ \varphi_D$ be the Hecke character on $I_K(f_D)$ associated to χ.

Assume first that $\chi^2 \neq 1$. In this case we have that $\tilde{\chi} \neq \psi \circ N_K$, for any Dirichlet character ψ mod f_D. Indeed, if this were the case, then ψ is necessarily a quadratic character because if $(a, f_D) = 1$, then $\psi(a^2) = \psi(N_K(a\mathfrak{O}_K)) = \tilde{\chi}(a\mathfrak{O}_K) = 1$ since $a\mathfrak{O}_K \in P_{K,\mathbb{Z}}(f_D)$. Thus $\tilde{\chi}$ is also a quadratic character, contrary to the hypothesis.

Since $\tilde{\chi} \neq \psi \circ N_K$, we know by [16], Theorem 4.8.2, that $f(z;\tilde{\chi}) \in S_1(N, \psi_D)$ where $N = |d_K|N_K(f_D\mathfrak{O}_K) = |D|$; clearly $f(z;\tilde{\chi})$ is a $\mathbb{T}(N)$-eigenfunction. Since $\vartheta_\chi \sim_{f_D} f(z;\tilde{\chi})$ by (14), it follows from Lemma 15(c) below (together with Theorem 12) that ϑ_χ is also a cusp form.

Now suppose that $\chi \in \mathrm{Cl}(D)^*$ is a quadratic character. Then by Gauss's genus theory there exist fundamental discriminants D_1, D_2 such that $D = c^2 D_1 D_2$ for some integer c and such that $\tilde{\chi} = \psi_{D_1} \circ N_K = \psi_{D_2} \circ N_K$; cf. Weber[26], §104 and §109.

Since the pair (ψ_{D_1}, ψ_{D_2}) satisfies condition (4.7.2)(ii) on p. 176 of [16], there is a form $f_1 := f_1(z;\psi_{D_1},\psi_{D_2}) \in E_1(|D_1D_2|, \psi_{D_1D_2}) \subset E_1(|D|, \psi_D)$ such that its L-function is $L(s, f_1) = L(s, \psi_{D_1})L(s, \psi_{D_2})$; cf. [16], Theorem 4.7.1. We now claim:

$$\vartheta_\chi(z) \ \sim_D \ f_1(z;\psi_{D_1},\psi_{D_2}). \tag{15}$$

Indeed, since $\tilde{\chi} = \psi_{D_2} \circ N_K$, it follows from (13) that for $(n, D) = 1$ we have that

$$a_n(\chi) = a_n(\tilde{\chi}) = \sum_{\mathfrak{a}\in Id_n(\mathfrak{O}_K)} \psi_{D_2}(N(\mathfrak{a})) = \psi_{D_2}(n)\#Id_n(\mathfrak{O}_K) = \psi_{D_2}(n)\sum_{t|n}\psi_{d_K}(t), \tag{16}$$

the latter by (60). Since $\psi_{d_K}(t) = \psi_{D_1}(t)\psi_{D_2}(t)$ and $\psi_{D_2}(nt) = \psi_{D_2}(t^2\frac{n}{t}) = \psi_{D_2}(\frac{n}{t})$, we thus see that $a_n(\chi) = \sum_{t|n}\psi_{D_1}(t)\psi_{D_2}(\frac{n}{t}) = a_n(f_1;\psi_{D_1},\psi_{D_2})$, which proves (15). Using this, it follows from Lemma 15(c) below that $\vartheta_\chi \in E_1(|D|, \psi_D)$.

Above we had used the following general fact about $\mathbb{T}(N)$-eigenfunctions which is proved implicitly on p. 179 of [16]. Here, as usual, $\mathbb{T}(N) \subset \mathrm{End}_{\mathbb{C}}(M_k(N,\psi))$ is the Hecke algebra generated by the Hecke operators T_n with $(n, N) = 1$.

Lemma 15 (a) *If* $f \in M_k(N,\psi)$ *is a* $\mathbb{T}(N)$*-eigenfunction, then either* $f \in S_k(N,\psi)$ *or* $f \in E_k(N,\psi)$.

(b) *If* $f_1, f_2 \in M_k(N,\psi)$ *are two non-zero* $\mathbb{T}(N)$*-eigenfunctions which have the same* $\mathbb{T}(N)$*-eigenvalues, then either both are cusp forms or both are in* $E_k(N,\psi)$.

(c) *Let* $f_1, f_2 \in M_k(N,\psi)$ *be two normalized* $\mathbb{T}(N)$*-eigenfunction with* $f_1 \sim_N f_2$. *Then* f_1 *is a cusp form if and only if* f_2 *is a cusp form, and similarly* $f_1 \in E_k(N,\psi)$ *if and only if* $f_2 \in E_k(N,\psi)$.

Proof. (a) If not, then $f = f_1 + f_2$ with $f_1 \in S_k(N,\psi)$, $f_2 \in E_k(N,\psi)$ and $f_i \neq 0$ for $i = 1,2$. Then as in the proof of Theorem 4.7.2 of [16] on p. 179, f_i is a $\mathbb{T}(N)$-eigenfunction with the same eigenvalues as f and there exist normalized eigenfunctions $\tilde{f}_1 \in S_k(N,\psi)$ and $\tilde{f}_2 = f_k(z;\chi_1,\chi_2) \in E_k(N,\psi)$ with the same $\mathbb{T}(N)$-eigenvalues as f. Thus $L(s,\tilde{f}_1) \approx_N L(s,\tilde{f}_2)$, where $\approx_N$ means equality except for the Euler factors at the primes $p|N$. But this is impossible because $\Gamma(s)L(s,\tilde{f}_1)$ is entire while $\Gamma(s)L(s,\tilde{f}_2)$ has a pole at $s = k$, as is explained in [16], p. 179.

(b) If false, then $f = f_1 + f_2$ is a $\mathbb{T}(N)$-eigenfunction which contradicts (a).

(c) Since f_i is a normalized eigenfunction, the T_n-eigenvalue of f_i is $a_n(f_1) = a_n(f_2)$. Thus f_1 and f_2 satisfy the hypotheses of (b), and so the assertion follows.

Remark 16 In the case that $D = d_K$ is a fundamental discriminant, it follows from the above theorem that all the ϑ_χ's are *primitive forms* in the sense of section 4 below. Indeed, since in this case $\psi_D = \psi_{d_K}$ is a primitive character, it follows from the general theory of newforms (primitive forms in the terminology of [16], p. 164) that all $\mathbb{T}(D)$-eigenfunctions of $S_1(|D|,\psi_D)$ are newforms; cf. [16], Lemma 4.6.9(1) and Theorem 4.6.12. In particular, it follows immediately that the $\vartheta_\chi = f(z,\tilde{\chi})$ are newforms when $\chi^2 \neq 1$.

A similar result also holds for the Eisenstein series ϑ_χ when $\chi^2 = 1$, but this requires a different argument. Here we use instead the structure theorem of the Eisenstein space $E_1(|D|,\psi_D)$; cf. [16], p. 179. Specialized to the present situation, it yields that

$$E_1(|d_K|,\psi_{d_K}) = \bigoplus_{d_K = D_1 D_2} \mathbb{C} f_1(z;\psi_{D_1},\psi_{D_2}), \tag{17}$$

where the sum is over all factorizations $d_K = D_1 D_2$ of d_K into fundamental discriminants D_1, D_2 (and the factorization $d_K = D_2 D_1$ is considered to be the same as the factorization $d_K = D_1 D_2$), and $f_1(z;\psi_{D_1},\psi_{D_2})$ is as in the above proof. To verify this from formula (4.7.17) of [16], note first that if $\psi_{d_K} = \psi_1\psi_2$ is any factorization into characters ψ_i with conductors M_i such that $M_1 M_2 | d_K$, then necessarily $|d_K| = M_1 M_2$ and $(M_1, M_2) = 1$. By adjusting signs, we thus have the factorization $d_K = D_1 D_2$ into fundamental discriminants (with $|D_i| = M_i$). Moreover, since ψ_i is necessarily quadratic (because $(M_1, M_2) = 1$), it follows that $\psi_i = \psi_{D_i} = \left(\frac{D_i}{\cdot}\right)$ is the Kronecker-Legendre character. This verifies (17).

From (17) it follows that we can improve the relation (15) to an equality because all $\mathbb{T}(D)$-eigenspaces of $E_1(D,\psi_D)$ are 1-dimensional by (17), and so all the ϑ_χ with $\chi^2 = 1$ are "primitive" as well.

We observe that it follows from (17) that

$$\Theta^E_{d_K} \;:=\; \Theta_D \;\cap\; E_1(|d_K|,\psi_{d_K}) \;=\; E_1(|d_K|,\psi_{d_K}).$$

However, the analogous statement for $\Theta^S_{d_K} = \Theta_D \;\cap\; S_1(|d_K|,\psi_{d_K})$ is in general not true, as the discussion of Serre[20], §9, and/or [12] shows. In particular, the table on p. 258 of [20] shows that we have $\Theta^S_{-p} \neq S_1(p,\psi_{-p})$ for $p = 139, 163, 211, 227, 283$.

We are now ready to prove Theorems 1 and 2 of the introduction.

Proof of Theorem 1. By Proposition 9 and Theorem 12, the subspace Θ_D of $M_1(|D|,\psi_D)$ has the basis $\{\vartheta_\chi\}$ consisting of (normalized) $\mathbb{T}(D)$-eigenfunctions, and hence is a $\mathbb{T}(D)$-submodule.

To verify that Θ_D has multiplicity one, we have to show that ϑ_{χ_1} and ϑ_{χ_2} belong to different $\mathbb{T}(D)$-eigenspaces whenever $\vartheta_{\chi_1} \neq \vartheta_{\chi_2}$, i.e., whenever $\chi_1 \notin \{\chi_2, \chi_2^{-1}\}$.

For this, we first note that there is an $f \in Q_D$ such that $\chi_1(f) \neq \chi_2(f), \overline{\chi}_2(f)$. Indeed, the hypothesis on χ_1, χ_2 implies that $\exists f_i \in Q_D$ such that $\chi_1(f_1) \neq \chi_2(f_1)$ and $\chi_1(f_2) \neq \chi_2^{-1}(f_2) = \overline{\chi}(f_2)$. Now if $\chi_1(f_1) \neq \overline{\chi}_2(f_1)$, then we can take $f = f_1$ and if $\chi_1(f_2) \neq \chi_2(f_2)$, then we can take $f = f_2$. If neither of these cases holds, then we can take $f \sim f_1 f_2$ (product in $\mathrm{Cl}(D)$) because here we have that $\chi_1(f_1) = \overline{\chi}_2(f_1)$ and $\chi_1(f_2) = \chi_2(f_2)$, and so $\chi_1(f) = \chi_1(f_1)\chi_1(f_2) \neq \chi_2(f_1)\chi_1(f_2) = \chi_2(f_1)\chi_2(f_2) = \chi_2(f)$, and $\chi_1(f) = \chi_1(f_1)\chi_1(f_2) \neq \overline{\chi}_2(f_1)\chi_1(f_2) = \overline{\chi}_2(f_1)\overline{\chi}_2(f_2) = \overline{\chi}_2(f)$.

With f as above, choose a prime $p \nmid D$ such that $r_p(f) > 0$; cf. Lemma 8(a). Then by (7) and Lemma 8(b) we see that

$$a_p(\chi_i) = \frac{w(f)}{w_D}\mathrm{Re}(\chi_i(f))r_p(f), \quad \text{for } i = 1, 2.$$

Since $\chi_1(f) \neq \chi_2(f), \overline{\chi}_2(f)$ and $|\chi_i(f)| = 1$, it follows that $\mathrm{Re}(\chi_1(f)) \neq \mathrm{Re}(\chi_2(f))$, and so $a_p(\chi_1) \neq a_p(\chi_2)$. Since $a_p(\chi_i)$ is the T_p-eigenvalues of ϑ_{χ_i} by Theorem 12, we see that ϑ_{χ_1} and ϑ_{χ_2} lie in distinct $\mathbb{T}(D)$-eigenspaces, so Θ_D has multiplicity one.

The last assertion of Theorem 1 follows immediately from Theorem 14.

Proof of Theorem 2. Let V_E and V_S denote the $\mathbb{C}$-subspace of Θ_D generated by $\{\vartheta_\chi : \chi^2 = 1\}$ and by $\{\vartheta_\chi : \chi^2 \neq 1\}$, respectively. By Theorem 14 we know that $V_E \subset \Theta^E_D$ and $V_S \subset \Theta^S_D$, and by Proposition 9 we know that $V_E + V_S = \Theta_D$. Since $\Theta^E_D \cap \Theta^S_D = \{0\}$ by (12), it follows that $V_E = \Theta^E_D$ and $V_S = \Theta^S_D$ and that hence $\Theta_D = \Theta^E_D \oplus \Theta^S_D$.

Let $\mathrm{Cl}(D)^*[2] = \{\chi \in \mathrm{Cl}(D)^* : \chi^2 = 1\}$ denote the group of quadratic characters. By Proposition 9 we know that the set $\{\vartheta_\chi : \chi \in \mathrm{Cl}(D)^*[2]\}$ is linearly independent and hence is a basis of $\Theta_D^E = V_E$. Thus $\dim(\Theta_D^E) = |\mathrm{Cl}(D)^*[2]| = [\mathrm{Cl}(D) : \mathrm{Cl}(D)^2] = g_D$, and hence $\dim(\Theta_D^S) = \dim(\Theta_D) - \dim(\Theta_D^E) = \bar{h}_D - g_D = \frac{1}{2}(h_D - g_D)$, the latter by Proposition 7 and equation (5), respectively. This proves (1).

Remark 17 (a) For later reference we note that the above proof of Theorem 2 shows more precisely that $\{\vartheta_\chi : \chi \in \mathrm{Cl}(D)^*[2]\}$ is a basis of Θ_D^E and that $\{\vartheta_\chi : \chi \in \overline{\mathrm{Cl}(D)}^*, \chi^2 \neq 1\}$ is a basis of Θ_D^S.

(b) It follows immediately from (1) that

$$\Theta_D \subset E_1(|D|, \psi_D) \quad \Leftrightarrow \quad \Theta_D^S = \{0\} \quad \Leftrightarrow \quad h_D = g_D. \tag{18}$$

The discriminants $D < 0$ which satisfy the last condition (or, equivalently, the condition that $\mathrm{Cl}(D)$ is an elementary abelian 2-group) are called *idoneal discriminants* because a number $n \geq 1$ is *idoneal* (in the sense of Euler) if and only if $-4n$ is an idoneal discriminant; cf. [11] for a recent survey about idoneal numbers.

The above assertion (18) can be viewed as an alternate version of Theorem 3 of Kitaoka[13], which states that

$$\vartheta_{1_D} \in E_1(|D|, \psi_D) \quad \Leftrightarrow \quad h_D = g_D. \tag{19}$$

Indeed, if $h_D = g_D$, then clearly $\vartheta_{1_D} \in \theta_D \subset E_1(|D|, \psi_D)$ by (18). To prove the converse, note first that by (8) we have that

$$\vartheta_{1_D} = \frac{1}{w_D} \sum_{\chi \in \overline{Cl(D)}^*} w(\chi) \vartheta_\chi$$

(because $\chi(1_D) = 1$ for all $\chi \in \mathrm{Cl}(D)^*$). Thus, by part (a) we see that $\vartheta_{1_D} \in E_1(|D|, \psi_D) \Leftrightarrow \chi^2 = 1, \forall \chi \in \mathrm{Cl}(D)^* \Leftrightarrow \mathrm{Cl}(D)^2 = \{1\} \Leftrightarrow h_D = g_D$, which proves (19).

The above results immediately imply the following corollary which generalizes the work of Siegel[21] on theta-series attached to forms in $m = 2k > 4$ variables (cf. [21], particularly pp. 577-581) to the case of binary forms. Note that this gives an alternate proof (for binary forms) of Kitaoka's extension of Siegel's work to the case $m \geq 2$; cf. [13], Lemma 1.

Corollary 18 (a) *Let $f_1, f_2 \in Q_D$ be two primitive forms. Then $\vartheta_{f_1} - \vartheta_{f_2}$ is a cusp form if and only if f_1 and f_2 are* genus-equivalent, *i.e.*

$$\vartheta_{f_1} - \vartheta_{f_2} \in S_1(|D|, \psi_D) \;\Leftrightarrow\; \chi(f_1) = \chi(f_2), \forall \chi \in \mathrm{Cl}(D)^*[2] \;\Leftrightarrow\; f_1^{-1} f_2 \in \mathrm{Cl}(D)^2. \tag{20}$$

(b) *For any $f \in Q_D$ we have that*

$$F(z, f) := \frac{g_D}{h_D} \sum_{f_1 \in Cl(D)^2} \vartheta_{f f_1}(z) \;=\; \frac{w_D}{h_D} \sum_{\chi \in Cl(D)^*[2]} \chi(f) \vartheta_\chi(z) \;\in\; \Theta_D^E. \tag{21}$$

Thus, $F(z, f) \in \Theta_D^E$ is the Eisenstein component of $\vartheta_f(z)$, and $\vartheta_f(z) - F(z, f) \in \Theta_D^S$ is its cuspidal component.

Proof. (a) By (8) we have $\vartheta_{f_1} - \vartheta_{f_2} = \frac{h_D}{w_D}\sum_{\chi\in\overline{Cl(D)}^*} w(\chi)\mathrm{Re}(\chi(f_1)-\chi(f_2))\vartheta_\chi$, and so it follows from Remark 17(a) that $\vartheta_{f_1} - \vartheta_{f_2} \in S_1(|D|,\psi_D) \Leftrightarrow \mathrm{Re}(\chi(f_1)-\chi(f_2)) = 0, \forall\chi \in \mathrm{Cl}(D)^*[2] \Leftrightarrow \chi(f_1) = \chi(f_2), \forall\chi \in \mathrm{Cl}(D)^*[2]$ because all $\chi \in \mathrm{Cl}(D)^*[2]$ are real-valued. This proves the proves the first equivalence of (20) and hence also (20) because the second equivalence is obvious.

(b) By (8) we have that $F(z,f) = \frac{g_D w_D}{h_D^2}\sum_{\chi\in Cl(D)^*} c(f,\chi)\vartheta_\chi(z)$ with

$$c(f,\chi) = \sum_{f_1\in Cl(D)^2} \overline{\chi}(ff_1)) = \overline{\chi}(f)\sum_{f_1\in Cl(D)^2}\overline{\chi}(f_1)) = \begin{cases} \overline{\chi}(f)\frac{h_D}{g_D} & \text{if } \chi^2 = 1, \\ 0 & \text{otherwise} \end{cases}$$

where the last identity follows from a suitable orthogonality relation and the fact that $|\mathrm{Cl}(D)^2| = \frac{h_D}{g_D}$. From this the asserted identity (21) follows immediately.

By (21) and Theorem 14 it is clear that $F(z,f) \in \Theta_D^E$. Moreover, by (8) and (21) we have that $\vartheta_f(z) - F(z,f) = \frac{w_D}{h_D}\sum_{\chi^2\neq 1}\overline{\chi}(f)\vartheta_\chi(z) \in \Theta_D^S$, the latter by Theorem 14 again. This proves the last assertion.

Remark 19 The Fourier coefficients $a_n(F(z,f))$ of the Eisenstein series $F(z,f)$ are easily determined, at least when $(n,D) = 1$. Indeed, we have

$$a_n(F(z,f)) = \frac{w_D g_D}{h_D}\varepsilon(f,n)\#Id_n(\mathfrak{O}_K), \tag{22}$$

where $\varepsilon(f,n) = 1$ if $n \equiv f(x,y) \pmod{|D|}$, for some $x,y \in \mathbb{Z}$, and $\varepsilon(f,n) = 0$ otherwise. To see this, note first that $a_n(F(z,f)) = \frac{w_D}{h_D}\sum_{\chi\in Cl(D)^*[2]}\chi(f)a_n(\chi)$ by (21). Thus $a_n(F(z,f)) = 0 = \varepsilon(f,n)$, if $Id_n(\mathfrak{O}_K) = \emptyset$ (cf. (9)), so assume that $\exists\mathfrak{a}_n \in Id_n(\mathfrak{O}_K)$. If $f_n \in Q_D$ is such that $\lambda_D(f_n) = \varphi_D(\mathfrak{a}_n)$, then there exist $x,y \in \mathbb{Z}$ such that $f(x,y) = n$. Now for any $\chi \in \mathrm{Cl}(D)^*[2]$ we have by (16) that $a_n(\chi) = \tilde{\chi}(\mathfrak{a}_n)\#Id_n(\mathfrak{O}_K) = \chi(f_n)\#Id_n(\mathfrak{O}_K)$, so we obtain that

$$a_n(F(z,f)) = \frac{w_D}{h_D}\#Id_n(\mathfrak{O}_K)\sum_{\chi\in Cl(D)^*[2]}\chi(ff_n).$$

By an orthogonality relation we have that $\sum_{\chi^2=1}\chi(ff_n) = |\mathrm{Cl}(D)^*[2]| = g_D$ if $ff_n \in \mathrm{Cl}(D)^2$ and equals 0 otherwise. By genus theory (cf. [3], §3.B), this means that this sum equals $g_D\varepsilon(f,n)$, and so (22) follows.

As another application of the above results we compute the *trace* $tr(T_n|V)$ of the Hecke operator T_n on the vector spaces $V = \Theta_D^E$ and $V = \Theta_D^S$.

Corollary 20 *If $(n, f_D) = 1$, then the traces of the Hecke operator T_n on the spaces Θ_D^E and Θ_E^S are given by*

$$tr(T_n|\Theta_D^E) = \frac{h_D}{w_D}a_n(F(z,1_D)) \quad and \quad tr(T_n|\Theta_D^S) = \frac{h_D}{2w_D}(r_n(1_D) - a_n(F(z,1_D))). \tag{23}$$

Proof. By Remark 17(a), Theorem 12, and (21) we have that

$$tr(T_n|\Theta_D^E) = \sum_{\chi \in Cl(D)^*[2]} a_n(\chi) = \frac{h_D}{w_D} a_n(F(z, 1_D)).$$

This proves the first equation of (23). To prove the second equation, put $X = \{\chi \in \overline{\mathrm{Cl}(D)}^* : \chi^2 \neq 1\}$. Then by Remark 17(a) and Theorem 12 we have

$$tr(T_n|\Theta_D^S) = \sum_{\chi \in X} a_n(\chi) = \frac{1}{2} \sum_{\chi^2 \neq 1} a_n(\chi) = \frac{1}{2} \sum_{\chi} a_n(\chi) - \frac{1}{2} \sum_{\chi^2 = 1} a_n(\chi).$$

Since the first sum equals $\frac{h_D}{w_D} r_n(1_D)$ by (8) and since (as above) the second sum equals $\frac{h_D}{w_D} a_n(F(z, 1_D))$ by (21), the formula (23) follows.

4 Primitive characters and primitive forms

We now want to study the relation between the theta-series ϑ_χ and the modular form $f(z; \tilde{\chi})$ attached to the associated Hecke character $\tilde{\chi} = \chi \circ \lambda_D^{-1} \circ \varphi_D$ in more detail. For this, it is useful to introduce the following terminology.

Definition. If $\chi \in \mathrm{Cl}(D)^*$ is a character on the class group $\mathrm{Cl}(D)$, then we say that χ is *primitive* if we have that $\mathrm{Ker}(\overline{\pi}_{D,D/c^2}) \not\subset \mathrm{Ker}(\chi)$, for all divisors $c|f_D$ with $c > 1$. Here $\overline{\pi}_{D,D/c^2} : \mathrm{Cl}(D) \to \mathrm{Cl}(D/c^2)$ is the homomorphism induced by the map $\overline{\pi}_{\mathfrak{O}_D, \mathfrak{O}_{D/c^2}} : \mathrm{Cl}(\mathfrak{O}_D) \to \mathrm{Cl}(\mathfrak{O}_{D/c^2})$ which is defined in Remark 39 of the Appendix; in other words, $\overline{\pi}_{D,D'} = \lambda_{D'}^{-1} \circ \overline{\pi}_{\mathfrak{O}_D, \mathfrak{O}_{D'}} \circ \lambda_D$, for $D' = D/c^2$.

Moreover, the *conductor* f_χ of $\chi \in \mathrm{Cl}(D)^*$ is defined by

$$f_\chi = \gcd(f : f|f_D \text{ and } \mathrm{Ker}(\overline{\pi}_{D,f^2 d_K}) \subset \mathrm{Ker}(\chi)).$$

Thus, if $\chi \in \mathrm{Cl}(D)^*$ is primitive, then clearly $f_\chi = f_D$. Moreover, the converse is also true, as the following result shows.

Proposition 21 *Let $\chi \in \mathrm{Cl}(D)^*$ be a character with conductor f_χ, and let $f|f_D$. Then*

$$\mathrm{Ker}(\overline{\pi}_{D,f^2 d_K}) \subset \mathrm{Ker}(\chi) \Leftrightarrow f_\chi | f. \tag{24}$$

Thus, if $D_\chi := f_\chi^2 d_K$, then $\chi = \chi_{pr} \circ \overline{\pi}_{D,D_\chi}$, for a unique character $\chi_{pr} \in \mathrm{Cl}(D_\chi)^*$, and χ_{pr} is primitive.

Proof. From the definition of D_χ and (52) we have that $\mathfrak{O}_{D_\chi} = \prod_f \mathfrak{O}_{f^2 d_K}$, where the product is over all $f|f_D$ with $\mathrm{Ker}(\overline{\pi}_{D,f^2 d_K}) \subset \mathrm{Ker}(\chi)$. It thus follows from Corollary 41 that $\mathrm{Ker}(\overline{\pi}_{D,D_\chi}) \subset \mathrm{Ker}(\chi)$, and so the assertion (24) is obvious. Moreover, since $\overline{\pi}_{D,D_\chi} : \mathrm{Cl}(D) \to \mathrm{Cl}(D_\chi)$ is surjective (cf. Proposition 38), there is a unique character χ_{pr} on $\mathrm{Cl}(D_\chi)$ such that $\chi = \chi_{pr} \circ \overline{\pi}_{D,D_\chi}$. Finally, χ_{pr} is primitive because if $\mathrm{Ker}(\overline{\pi}_{D_\chi,D_\chi/c^2}) \subset \mathrm{Ker}(\chi_{pr})$, for some $c|f_{D_\chi}$, then $\mathrm{Ker}(\overline{\pi}_{D,(f_\chi/c)^2 d_K}) = \overline{\pi}^{-1}_{D,D_\chi}(\mathrm{Ker}(\overline{\pi}_{D_\chi,D_\chi/c^2})) \subset \overline{\pi}^{-1}_{D,D_\chi}(\mathrm{Ker}(\chi_{pr})) = \mathrm{Ker}(\chi)$, so $f_\chi|\frac{f_\chi}{c}$ by (24) and hence $c=1$, i.e. χ_{pr} is primitive.

Using the results of the Appendix, we can now compute the Fourier coefficients $a_n(\chi)$ of ϑ_χ in the case that $n|f_D^2$.

Proposition 22 *Let χ be a character on $\mathrm{Cl}(D)$ with conductor f_χ, and put $\bar{f}_\chi = f_D/f_\chi$. If $n|f_D^2$, then*

$$a_n(\chi) = \begin{cases} c\prod_{p|c}\left(1 - \frac{1}{p}\psi_{D/c^2}(p)\right) & \text{if } n = c^2 \text{ and } c|\bar{f}_\chi, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Proof. If n isn't a square, then $Id_n(\mathfrak{O}_D) = \emptyset$ by Proposition 48, so by (9) we have that $a_n(\chi) = 0$ in this case. Thus, assume that $n = c^2$, and put $\chi^* = \chi \circ \lambda_D^{-1}$. Then it follows from (9) and Proposition 48 that

$$a_n(\chi) \;=\; \sum_{\mathfrak{a}\in c\mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D/c^2}})} \chi^*(\mathfrak{a}) \;=\; \sum_{\mathfrak{a}\in \mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D/c^2}})} \chi^*(\mathfrak{a}).$$

Now from (55) and (24) we see that $\mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D/c^2}})P(\mathfrak{O}_D) \le \mathrm{Ker}(\chi^*) \Leftrightarrow f_\chi|\frac{f_D}{c} \Leftrightarrow c|\bar{f}_\chi$. If this is not the case, i.e. if χ^* is non-trivial on $\mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D/c^2}})$, then this sum equals 0 by an orhogonality relation. On the other hand, if $c|\bar{f}_\chi$, then we have that $a_n(\chi) = |\mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D/c^2}})|$ and so the assertion follows by using (57).

We now turn our attention to the modular form $f(z;\tilde{\chi})$, where, as before, $\tilde{\chi} = \chi \circ \lambda_D^{-1} \circ \varphi_D$. To determine its properties, we need to know the *conductor* $\mathrm{cond}(\tilde{\chi})$ of the Hecke character $\tilde{\chi}$; here $\mathrm{cond}(\tilde{\chi}) \in Id(\mathfrak{O}_K)$ is as defined on p. 91 of [16].

Theorem 23 *The conductor of the Hecke character $\tilde{\chi} = \chi \circ \lambda_D^{-1} \circ \varphi_D$ is $\mathrm{cond}(\tilde{\chi}) = f_\chi \mathfrak{O}_K$. In particular, χ is primitive if and only if $\tilde{\chi}$ is a primitive Hecke character.*

Proof. As usual, we view $\tilde{\chi}$ as a homomorphism $\tilde{\chi} : I_K(f) \to \mathbb{C}^\times$ with $P_{K,\mathbb{Z}}(f) \subset \mathrm{Ker}(\tilde{\chi})$, where $f = f_D$. By definition (cf. [16], p. 91), $\mathfrak{m}_{\tilde{\chi}} := \mathrm{cond}(\tilde{\chi})$ is the largest $\mathfrak{O}_K$-ideal $\mathfrak{m}$ such that $P_{K,1}(\mathfrak{m}) \cap I_K(f) \subset \mathrm{Ker}(\tilde{\chi})$. Here $P_{K,1}(\mathfrak{m}) = \{\alpha\mathfrak{O}_K : \alpha \in K_1(\mathfrak{m})\}$, where $K_1(\mathfrak{m}) = 1 + \mathfrak{m}\mathfrak{O}_\mathfrak{m}$ and $\mathfrak{O}_\mathfrak{m} = \{\lambda \in K : v_\mathfrak{p}(\lambda) \ge 0, \forall \mathfrak{p}|\mathfrak{m}\}$ denotes the semi-local ring of K defined by the prime divisors of $\mathfrak{m}$. (Note that it follows from the Strong Approximation Theorem that $\mathfrak{O}_\mathfrak{m} = (\mathfrak{O}_K)_{1+\mathfrak{m}}$ is the localization of $\mathfrak{O}_K$ with respect to the multiplicative set $1+\mathfrak{m}$, and so we see that $K_1(\mathfrak{m}) = \langle 1+\mathfrak{m}\rangle$ is the group generated by $1+\mathfrak{m}$. Thus, this definition of $P_{K,1}(\mathfrak{m})$ coincides with that of [3], p. 160.)

We first observe that $f_\chi\mathfrak{O}_K \subset \mathfrak{m}_{\tilde{\chi}}$. Indeed, since χ is the lift of $\chi_{pr} \in \mathrm{Cl}(D_\chi)$, we have that $\mathrm{Ker}(\tilde{\chi}) = \mathrm{Ker}(\tilde{\chi}_{pr}) \cap I_K(f) \supset P_{K,\mathbb{Z}}(f_\chi) \cap I_K(f) \supset P_{K,1}(f_\chi) \cap I_K(f)$, and so $f_\chi\mathfrak{O}_K \subset \mathfrak{m}_{\tilde{\chi}}$ by definiton.

We next verify that $f_\chi\mathfrak{O}_K = \mathfrak{m}_{\tilde{\chi}}$. For this, we may assume without loss of generality that χ is primitive, i.e. that $f_\chi = f$, because if we replace χ by χ_{pr} (using Proposition 21), then $f_\chi = f_{\chi_{pr}}$ and $\mathrm{cond}(\tilde{\chi}) = \mathrm{cond}(\tilde{\chi}_{pr})$.

Thus, assume that $f_\chi = f$. If $\mathfrak{m}_{\tilde{\chi}} \neq \mathfrak{m} := f\mathfrak{O}_K$, then there is a prime ideal $\mathfrak{p}|\mathfrak{m}$ such that $\mathfrak{m} \subset \mathfrak{p}\mathfrak{m}_{\tilde{\chi}}$. This means that $\mathrm{Ker}(\tilde{\chi}) \supset P_{K,1}(\mathfrak{m}_\chi) \cap I_K(f) \supset P_{K,1}(\mathfrak{p}^{-1}\mathfrak{m}) \cap I_K(f)$. We observe that if $P_K^{\mathbb{Z}}(f) := \{x\mathfrak{O}_K : x \in \mathbb{Q}_f^\times\}$, where $\mathbb{Q}_f^\times = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } (ab, f) = 1\}$, then $P_K^{\mathbb{Z}}(f) \subset P_{K,\mathbb{Z}}(f)$, and so $P_K^{\mathbb{Z}}(f)(P_{K,1}(\mathfrak{p}^{-1}\mathfrak{m}) \cap I_K(f)) \subset \mathrm{Ker}(\tilde{\chi})$. But since

$$P_{K,\mathbb{Z}}(fp^{-1}) \cap I_K(f) \;\subset\; P_K^{\mathbb{Z}}(f)(P_{K,1}(f\mathfrak{p}^{-1}) \cap I_K(f)), \quad \forall \mathfrak{p} \,|\, p \,|\, f, \tag{26}$$

as will be shown below, and since it is easy to see that

$$P_{K,\mathbb{Z}}(m) \cap I_K(f) \;=\; P_K^{\mathbb{Z}}(f)(P_{K,1}(m\mathfrak{O}_K) \cap I_K(f)), \quad \text{if } m|f, \tag{27}$$

we obtain that $P_{K,\mathbb{Z}}(\frac{f}{p}) \cap I_K(f) = P_K^{\mathbb{Z}}(f)(P_{K,1}(\frac{f}{p}\mathfrak{O}_K) \cap I_K(f)) \subset P_K^{\mathbb{Z}}(f)(P_{K,1}(f\mathfrak{p}^{-1}) \cap I_K(f)) \subset \mathrm{Ker}(\tilde{\chi})$. But this means that $f = f_\chi | \frac{f}{p}$, contradiction.

It remains to verify (26). If p is inert in K, then $fp^{-1}\mathfrak{O}_K = f\mathfrak{p}^{-1}$, and so (26) follows from (27). Thus, assume that p splits or is ramified in K. Then we have that

$$\mathbb{Q}_f^\times K_1(f\mathfrak{p}^{-1}) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times \;=\; \mathbb{Q}_f^\times K_1(fp^{-1}\mathfrak{p}) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times. \tag{28}$$

Indeed, if $p\mathfrak{O}_K = \mathfrak{p}^2$ is ramified, then this is clear because $p^{-1}\mathfrak{p} = \mathfrak{p}^{-1}$, so assume that $p\mathfrak{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, i.e. that $p^{-1}\mathfrak{p} = \overline{\mathfrak{p}}^{-1}$. If $\lambda \in \mathbb{Q}_f^\times K_1(fp^{-1}\mathfrak{p}) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times$, then $N(\lambda) = \lambda\overline{\lambda} \in \mathbb{Q}_f^\times$ and $\overline{\lambda} \in \mathbb{Q}_f^\times K_1(fp^{-1}\overline{\mathfrak{p}}) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times$, so $\lambda = N(\lambda)\overline{\lambda}^{-1} \in \mathbb{Q}_f^\times K_1(fp^{-1}\overline{\mathfrak{p}}) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times$. Thus, by symmetry, $\mathbb{Q}_f^\times K_1(fp^{-1}\mathfrak{p}) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times = \mathbb{Q}_f^\times K_1(fp^{-1}\overline{\mathfrak{p}}) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times$, and so (28) follows.

From this, (26) follows readily. Indeed, let $\mathfrak{a} \in P_{K,1}(fp^{-1}\mathfrak{O}_K) \cap I_K(f)$. Then $\mathfrak{a} = \lambda\mathfrak{O}_K$ with $\lambda \in K_1(fp^{-1}\mathfrak{O}_K) \cap \mathfrak{O}_{f\mathfrak{O}_K}^\times$, and so by (28) we see that $\mathfrak{a} \in P_K^{\mathbb{Z}}(f)(P_{K,1}(f\mathfrak{p}^{-1}) \cap I_K(f))$. Thus $P_{K,1}(fp^{-1}\mathfrak{O}_K) \cap I_K(f) \subset P_K^{\mathbb{Z}}(f)(P_{K,1}(f\mathfrak{p}^{-1}) \cap I_K(f))$, and so (26) follows in view of (27).

From the above theorem it follows easily that if $\chi \in \mathrm{Cl}(D)^*$ is a primitive character, then $f(z; \tilde{\chi})$ is a *primitive form* in the sense of the following definition.

Definition. A modular form $f \in M_k(N, \psi)$ is called a *primitive form* if either $f \in S_k(N, \psi)$ is a normalized newform of some level $M|N$ (so f is a primitive form in the sense of [16], §4.6) or if $f = f_k(z; \psi_1, \psi_2)$ is one of the Eisenstein series defined on p. 178 of [16].

Remark 24 It follows from the theory of newforms (cf. [16], §4.6) and the theory of Eisenstein forms (cf. [16], §4.7) that each $\mathbb{T}(N)$-eigenspace of $M_k(N, \psi)$ contains a unique primitive form. Thus, there is natural bijection between primitive forms and $\mathbb{T}(N)$-eigenspaces of $M_k(N, \psi)$.

Corollary 25 *If $\chi \in \mathrm{Cl}(D)^*$ is a primitive character, then $f(z;\tilde{\chi})$ is a primitive form of level $|D|$.*

Proof. Suppose first that $\chi^2 \neq 1$. Then from the proof of Theorem 14 we know that $\tilde{\chi} \neq \chi' \circ N_K$ for all Dirichlet characters χ', and by Theorem 23 we know that $\tilde{\chi}$ is a primitive Hecke character. It thus follows from Theorem 4.8.2 of [16] that $f(z;\tilde{\chi})$ is a primitive cusp form (of level $|D|$).

Now suppose that $\chi^2 = 1$. Since $\tilde{\chi}$ is a primitive Hecke character by Theorem 23 and since $f(z;\tilde{\chi})$ is not a cusp form by (the proof of) Theorem 14, it follows from the last part of the proof of Theorem 4.8.2 of [16] that there exist Dirichlet characters χ_1, χ_2 with $\chi_1\chi_2 = \psi_D$ such that $L(s, f(z;\tilde{\chi})) = L(s,\chi_1)L(s,\chi_2)$. Since the latter equals $L(s, f_1(z;\chi_1,\chi_2))$ by [16], Theorem 4.7.1, we conclude that $f(z;\tilde{\chi}) = f_1(z;\chi_1,\chi_2)$, and hence $f(z;\tilde{\chi})$ is primitive of level $|D|$.

Remark 26 If $\chi \in \mathrm{Cl}(D)^*[2]$ is a primitive *quadratic* character, then by combining the above proof with that of Theorem 14, we obtain the formula

$$f(z;\tilde{\chi}) \;=\; f_1(z;\psi_{D_1},\psi_{D_2}), \quad \text{where } \tilde{\chi} = \psi_{D_1} \circ N_K = \psi_{D_2} \circ N_K, \tag{29}$$

and $D = D_1D_2$ is a suitable factorization of D into fundamental discriminants. Indeed, by the proof of Corollary 25 we know that (29) holds for some pair of characters χ_1, χ_2, and the proof of Theorem 14, particularly equations (15) and (16), show that $\chi_i = \psi_{D_i}$ for $i = 1, 2$, where $D = D_1D_2$ is the fundamental factorization associated to χ.

We can now prove Theorem 3 of the introduction.

Proof of Theorem 3. (a) This follows immediately from Proposition 21.

(b) Since $\chi_{pr} \in \mathrm{Cl}(D_\chi)^*$ is primitive by part (a), it follows from Corollary 25 that $f(z;\tilde{\chi}_{pr}) \in M_1(|D_\chi|, \psi_{D_\chi})$ is a primitive form of level $|D_\chi|$. By Theorem 12 and (14) we know that $\vartheta_{\chi_{pr}}$ is a normalized $\mathbb{T}(D_\chi)$-eigenfunction which has the same eigenvalues as $f(z;\tilde{\chi}_{pr})$. Since $f(z;\tilde{\chi}_{pr})$ is primitive of level $|D_\chi|$, its associated eigenspace in $M_1(|D_\chi|, \psi_{D_\chi})$ is one-dimensional (cf. [16], Theorems 4.6.12 and 4.7.2) and so

$$\vartheta_{\chi_{pr}}(z) \;=\; f(z;\tilde{\chi}_{pr}) \tag{30}$$

because both forms are normalized. Thus $\vartheta_{\chi_{pr}}$ is primitive and $L(s,\vartheta_{pr}) = L(s,\tilde{\chi}_{pr})$.

Now by Theorem 12 we know that ϑ_χ is a $\mathbb{T}(D)$-eigenfunction which lies in the eigenspace defined by the primitive form $\vartheta_{\chi_{pr}}(z) = f(z;\tilde{\chi}_{pr})$ because by (14) we have that $\vartheta_\chi \sim_{f_D} f(z;\tilde{\chi}) \sim_{f_D} f(z;\tilde{\chi}_{pr})$. Since $D/D_\chi = (f_D/f_\chi)^2 = \bar{f}_\chi^2$, it follows from the description of the $\mathbb{T}(D)$-eigenspaces of $M_1(|D|,\psi_D)$ given in Corollary 4.6.20 and (implicitly) in Theorem 4.7.2 of [16], that there exist constants $c_n(\chi) \in \mathbb{C}$ with $n|\bar{f}_\chi^2$ such that (2) holds. Thus, if we put $c_n(\chi) = 0$ when $n \nmid \bar{f}_\chi^2$, then we have

$$a_m(\chi) \;=\; \sum_{\substack{n|\bar{f}_\chi^2 \\ n|m}} c_n(\chi)a_{m/n}(\chi_{pr}) \;=\; \sum_{n|m} c_n(\chi)a_{m/n}(\chi_{pr}), \quad \text{for all } m \geq 1, \tag{31}$$

and so it follows that $L(s,\vartheta_\chi) = (\sum_{n\ge1} c_n(\chi)n^{-s})L(s,\vartheta_{\chi_{pr}})$; cf. [7], Theorem 284. Thus (3) holds. Moreover, it follows that $n \mapsto c_n(\tilde\chi)$ is multiplicative because both $L(s,\vartheta_\chi)$ and $L(s,\vartheta_{\chi_{pr}})$ have Euler products by Theorem 12, and hence so does

$$C(s,\chi) \;:=\; \sum_{n\ge1}\frac{c_n(\chi)}{n^s} \;=\; \frac{L(s,\vartheta_\chi)}{L(s,\vartheta_{\chi_{pr}})} \;=\; \frac{L(s,\vartheta_\chi)}{L(s,\tilde\chi_{pr})}. \tag{32}$$

This means that $n \mapsto c_n(\chi)$ is multiplicative.

Finally, we note that it follows from (32) that all the $c_n(\chi)$ are real because the Fourier coefficients $a_n(\chi)$ and $a_n(\chi_{pr})$ are real; cf. (7).

5 The Dirichlet series $C(s,\chi)$

In the previous section we had seen that each theta-series ϑ_χ can be expressed as a linear combination of "shifted" modular forms $f(nz;\tilde\chi_{pr}) = \vartheta_{\chi_{pr}}(nz)$ associated to the primitive Hecke character $\tilde\chi_{pr}$; cf. Theorem 3. We now want to obtain precise formulae for the coefficients $c_n(\chi)$ of this linear combination. We thus study the (finite) Dirichlet series $C(s,\chi)$ of equation (32) in more detail.

As in Hardy/Wright[7], §17.4, we let $F_p(s)$ denote the *p-Euler factor* of a Dirichlet series $F(s)$ at the prime p, i.e.

$$F_p(s) \;=\; \sum_{k\ge0} a_{p^k}p^{-ks}, \quad \text{if} \quad F(s) \;=\; \sum_{n\ge1} a_n n^{-s}.$$

The p-Euler factors $C_p(s,\chi)$ and $L_p(s,\chi)$ of the Dirichlet series $C(s,\chi)$ and $L(s,\chi) := L(s,\vartheta_\chi)$ are related as follows.

Proposition 27 *If $\chi \in \mathrm{Cl}(D)^*$, then for any prime p we have that*

$$C_p(s,\chi) \;=\; L_p(s,\chi)(1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}). \tag{33}$$

Proof. By taking $m = p^k$ (for $k \ge 0$) in (31) we see that $L_p(s,\chi) = C_p(s,\chi)L_p(s,\chi_{pr})$, and so (33) follows because

$$L_p(s,\chi_{pr}) \;=\; \left(1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}\right)^{-1}. \tag{34}$$

Indeed, since $\vartheta_{\chi_{pr}} = f(z;\tilde\chi_{pr})$ is a primitive form of level $|D_\chi|$ by Theorem 3(b), equation (34) follows from Corollary 4.6.22 and (4.7.16) of [16].

We are now ready to determine $C(s,\chi)$ explicitly.

Theorem 28 *Let $\chi \in \mathrm{Cl}(D)^*$ be a character of conductor $f_\chi | f_D$, and let $\chi_{pr} \in \mathrm{Cl}(D_\chi)$ be the associated primitive character on $\mathrm{Cl}(D_\chi)$, where $D_\chi = f_\chi^2 d_K$. Then*

$$C(s,\chi) \;=\; \prod_{p|\bar f_\chi} C_p(s,\chi), \tag{35}$$

where $\bar f_\chi = f_D/f_\chi$. Moreover, if $p|\bar f_\chi$ and if $p^{\bar e_p}||\bar f_\chi$ is the highest power of p dividing

$\bar{f}_\chi$, *then*

$$(36)\, C_p(s,\chi) = \frac{1-p^{(1-2s)\bar{e}_p}}{1-p^{1-2s}}\left(1-a_p(\chi_{pr})p^{-s}+\psi_{D_\chi}(p)p^{-2s}\right)+\left(1-\tfrac{1}{p}\psi_{D_\chi}(p)\right)p^{(1-2s)\bar{e}_p};$$

in other words, we have for $k \geq 1$ *that*

$$(37) \qquad c_{p^k}(\chi) \;=\; \begin{cases} -a_p(\chi_{pr})p^{(k-1)/2} & \textit{if } k \equiv 1 \,(\mathrm{mod}\ 2) \textit{ and } k < 2\bar{e}_p, \\ p^{k/2}+p^{k/2-1}\psi_{D_\chi}(p) & \textit{if } k \equiv 0 \,(\mathrm{mod}\ 2) \textit{ and } k < 2\bar{e}_p, \\ p^{\bar{e}_p} & \textit{if } k = 2\bar{e}_p \\ 0 & \textit{if } k > 2\bar{e}_p \end{cases}$$

Proof. The first assertion (35) follows immediately from the multiplicativity of the $c_n(\chi)$'s; cf. Theorem 3(b). To prove (36), we first observe that if $0 \leq k \leq 2\bar{e}_p$, then we obtain from (25) that

$$(38) \qquad a_{p^k}(\chi) \;=\; \begin{cases} 0 & \text{if } k \equiv 1 \,(\mathrm{mod}\ 2), \\ p^{k/2} & \text{if } k \equiv 0 \,(\mathrm{mod}\ 2) \text{ and } k < 2\bar{e}_p, \\ p^{\bar{e}_p}(1-\frac{1}{p}\psi_{D_\chi}(p)) & \text{if } k = 2\bar{e}_p. \end{cases}$$

Indeed, if k is odd, then this is clear, so assume that k is even. Then (25) yields that $a_{p^k}(\chi) = p^{k/2}(1-\frac{1}{p}\psi_{D/p^k}(p))$. Now if $k < 2\bar{e}_p$, then $p|\frac{D}{p^k}$ because $p^{2\bar{e}_p}|D$. Thus $\psi_{D/p^k}(p) = 0$, and hence (38) holds in this case. On the other hand, if $k = 2\bar{e}_p$, then $\frac{D}{p^k} = D_\chi f_1^2$, where $f_1 = \bar{f}_\chi p^{-\bar{e}_p}$. Since $(f_1, p) = 1$, we have that $\psi_{D/p^k}(p) = \psi_{D_\chi}$, and so (38) holds in all cases.

Note that by using the identity $\sum_{k=0}^{\bar{e}_p-1} p^k X^2 = \frac{1-(pX^2)^{\bar{e}_p}}{1-pX^2}$ we can re-write (38) in the form

$$(39) \qquad \sum_{k=0}^{2\bar{e}_p} a_{p^k}(\chi)X^k \;=\; \frac{1-(pX^2)^{\bar{e}_p}}{1-pX^2} + (1-\tfrac{1}{p}\psi_{D_\chi}(p))(pX^2)^{\bar{e}_p}.$$

Thus, viewing (33) as an identity of power series in $X = p^{-s}$, it follows from (33) and (39) that

$$C_p(s,\chi) \equiv \frac{1-(pX^2)^{\bar{e}_p}}{1-pX^2}(1-a_p(\chi_{pr})X+\psi_{D_\chi}(p)X^2)+(1-\tfrac{1}{p}\psi_{D_\chi}(p))(pX^2)^{\bar{e}_p} \pmod{X^{2\bar{e}_p+1}}.$$

Now since $C_p(s,\chi)$ is a polynomial of degree $\leq 2\bar{e}_p$ (because $c_{p^k}(\chi) = 0$ by definition when $k > 2\bar{e}_p$), and since the same is true for the right hand side, it follows that these two polynomials are equal, and so (36) follows (by replacing X by p^{-s}). The last assertion (37) follows immediately from (36) by comparing the coefficients in $X = p^{-s}$.

Corollary 29 *In the situation of* Theorem 28 *we have that*

$$c_{\bar{f}_\chi^2}(\chi) \;=\bar{f}_\chi. \tag{40}$$

Thus, ϑ_χ has exact level $|D|$, and $C(s,\chi) = 1$ if and only if χ is primitive. In particular, ϑ_χ is a primitive form if and only if χ is a primitive character.

Proof. Since $\bar{f}_\chi = \prod_{p|\bar{f}_\chi} p^{\bar{e}_p}$ and $c_n(\chi)$ is multiplicative, we have by (37) that

$$c_{\bar{f}_\chi^2}(\chi) = \prod_{p|\bar{f}_\chi} c_{p^{2\bar{e}_p}}(\chi) = \prod_{p|\bar{f}_\chi} p^{\bar{e}_p} = \bar{f}_\chi,$$

which proves (40). From this and (2) it follows that ϑ_χ has exact level $|D|$.

If χ is primitive, i.e. if $\bar{f}_\chi = 1$, then clearly $C(s,\chi) = 1$ by definition (cf. (3)). Conversely, if $C(s,\chi) = 1$, i.e., if $c_n(\chi) = 0$ for all $n > 1$, then (40) forces that $\bar{f}_\chi = 1$, which means that χ is primitive.

To prove the last assertion, recall first that if $\chi = \chi_{pr}$ is a primitive character, then ϑ_χ is a primitive form by Theorem 3(b). Conversely, suppose that ϑ_χ is a primitive form. Then $\vartheta_\chi = \vartheta_{\chi_{pr}}$ because both are in the same $\mathbb{T}(D)$-eigenspace of $M_1(|D|,\psi_D)$, and so $C(s,\chi) = L(s,\chi)/L(s,\chi_{pr}) = L(s,\vartheta_\chi)/L(s,\vartheta_{\chi_{pr}}) = 1$, which means that χ is primitive.

Remark 30 (a) It follows from (8) and (3) that if $f \in Q_D$, then its (Epstein) zeta-function Z_f of Remark 6(b) can be written as a sum of the L-functions $L(s,\chi) := L(s,\vartheta_\chi)$ and the Hecke L-functions $L(s,\chi_{pr}) = L(s,\tilde{\chi}_{pr})$ in the following way:

$$Z_f(s) \;=\; \frac{w_D}{h_D} \sum_{\chi\in Cl(D)^*} \overline{\chi(f)} L(s,\chi) \;=\; \frac{w_D}{h_D} \sum_{\chi} \overline{\chi(f)} C(s,\chi) L(s,\chi_{pr}); \tag{41}$$

this generalizes the well-known relation (cf. [4]) for Z_f when $D = d_k$ is a fundamental discriminant. However, the above relation and Corollary 29 show that if $D \neq d_K$ is not a fundamental discriminant, then Z_f is *never* a linear combination of the associated Hecke L-functions because in that case the factor $C(s,1)$ is not a constant since the trivial character $\chi = 1$ is not primitive; cf. Example 35 below.

(b) Similarly (and equivalently), by combining (8) with (2), we obtain in view of (35) and (38) the following explicit expression of the theta series ϑ_f in terms of the (extended) Atkin-Lehner basis:

$$\vartheta_f(z) \;=\; \frac{w_D}{h_D} \sum_{\chi\in Cl(D)^*} \overline{\chi(f)} \vartheta_\chi(z) \;=\; \frac{w_D}{h_D} \sum_{\chi} \overline{\chi(f)} \sum_{n|\bar{f}_\chi} \vartheta_{\chi_{pr}}(nz). \tag{42}$$

As was mentioned in the introduction, Theorem 28 yields immediately a formula for the L-function $L(s,\chi) = L(s,\vartheta_\chi)$.

Corollary 31 *If $\chi \in \mathrm{Cl}(D)^*$, then the L-function $L(s,\chi)$ of ϑ_χ has the Euler product*

$$L(s,\chi) \;=\; \prod_p L_p(s,\chi) \tag{43}$$

where for $p \nmid \bar{f}_\chi$ the p-Euler factor $L_p(s,\chi)$ is

$$L_p(s,\chi) \;=\; \left(1 - a_p(\chi)p^{-s} + \psi_D(p)p^{-2s}\right)^{-1} = \left(1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}\right)^{-1}, \tag{44}$$

whereas for $p \mid \bar{f}_\chi$ it is given by

$$L_p(s,\chi) \;=\; \frac{1 - p^{(1-2s)\bar{e}_p}}{1 - p^{1-2s}} + \frac{\left(1 - \frac{1}{p}\psi_{D_\chi}(p)\right) p^{(1-2s)\bar{e}_p}}{1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}}. \tag{45}$$

Proof. Since the $a_n(\chi)$'s are multiplicative (cf. Theorem 12), it follows that $L(s,\chi)$ has an Euler product (43). Since $C_p(s,\chi) = 1$ when $p \nmid \bar{f}_\chi$, it follows from (3) that $L_p(s,\chi) = L_p(s,\chi_{pr})$, so in particular $a_p(\chi) = a_p(\chi_{pr})$. Moreover, since $D = D_\chi \bar{f}_\chi^2$, we see that $\psi_D(p) = \psi_{D_\chi}(p)$, so the second equality of (44) holds, and hence the first equality follows from (34).

The formula (45) follows immediately from (33) and (36).

Remark 32 (a) The above Corollary 31 shows that the second identity of the equation on the bottom of p. 280 of Lang[15] is incorrect when χ is not a primitive character.

(b) An alternate way of writing formula (45) is follows: if $p^{\bar{e}_p} || \bar{f}_\chi$, then

$$a_{p^k}(\chi) = \begin{cases} 0 & \text{if } k \equiv 1 \,(\mathrm{mod}\ 2) \text{ and } k < 2\bar{e}_p \\ p^{k/2} & \text{if } k \equiv 0 \,(\mathrm{mod}\ 2) \text{ and } k < 2\bar{e}_p \\ p^{e_p}\left(1 - \frac{1}{p}\psi_{D_\chi}(p)\right) a_{p^{k-2\bar{e}_p}}(\chi_{pr}) & \text{if } k \geq 2e_p \end{cases}$$

(c) Corollary 31 can be viewed as a partial generalization of some of the main results of the articles of Sun and Williams[22], [23]. Indeed, for a negative discriminant $D < 0$, the function $F(A,n)$ of [22] is essentially the same as the function $a_n(\chi)$ above. More precisely, if we put $\chi_A(K) = e^{2\pi i[K,A]}$, where $[K,A]$ (which depends on a choice of a "basis" of $\mathrm{Cl}(D)$) is as defined on p. 143 of [22], then a comparison of formula (6) with the formula for $F(A,n)$ on p. 144 of [22] shows that

$$a_n(\chi_A) = F(A,n), \quad \forall A \in \mathrm{Cl}(D),\ n \geq 1.$$

In particular, Corollary 31 generalizes Theorem 5.3 of [23] (for $D < 0$) to the non-cyclic case, and gives a succinct general formula for it (and also for the many special cases discussed in §8 of [22]). In addition, Corollary 31 generalizes (for $D < 0$) Theorem 4.3 of [22] because $a_n(\chi_{D,0}) = N(n,D)/w(D)$ in the notation of Example 35(b) and that of [22].

As another application of Theorem 28, we determine when ϑ_χ is an eigenfunction under the *full* Hecke algebra $\mathbb{T}_{|D|} = \langle T_n : n \geq 1 \rangle$ of level $|D|$.

Corollary 33 *If $\chi \in \mathrm{Cl}(D)^*$, then ϑ_χ is a $\mathbb{T}_{|D|}$-eigenfunction if and only if χ is primitive.*

Proof. If $\chi = \chi_{pr}$ is primitive, then by (34) the L-function of ϑ_χ has the Euler product

$$L(s,\chi) = \prod_p (1 - a_p(\chi)p^{-1} + \psi_D(p)p^{-2s})^{-1}. \tag{46}$$

Thus, by a theorem of Hecke (cf. [16], Theorem 4.5.16), it follows that ϑ_χ is a $\mathbb{T}_{|D|}$-eigenfunction.

Conversely, suppose that ϑ_χ is a $\mathbb{T}_{|D|}$-eigenfunction. Then by Hecke's Theorem again, we know that its L-function $L(s,\chi)$ has an Euler product (46). If χ is not primitive, then $\bar{f}_\chi = f_D/f_\chi > 1$, so there exists a prime $p|\bar{f}_\chi$. Then by (38) we know that $a_p(\chi) = 0$, so the p-Euler factor of ϑ_χ is trivial, i.e. $L_p(s,\chi) = 1$. (Recall that $p|\bar{f}_\chi|D$.) Thus by (33) we see that $C_p(s,\chi) = 1 - a_p(\chi_{pr})p^{-1} + \psi_{D_\chi}(p)p^{-2s}$, i.e. that

$$c_p(\chi) = -a_p(\chi), \quad c_{p^2}(\chi) = \psi_{D_\chi}(p) \quad \text{and} \quad c_{p^k}(\chi) = 0,\ \forall k > 2.$$

Comparing this to (37) shows that $\bar{e}_p \leq 1$. But if $\bar{e}_p = 1$, then by (37) $c_{p^2}(\chi) = p \neq \psi_{D_\chi}(p)$, contradiction. Thus $\bar{f}_\chi = 1$, so χ is primitive.

The above corollary allows us to determine when the spaces Θ_D, Θ_D^E and Θ_D^S are $\mathbb{T}_{|D|}$-modules.

Proposition 34 (a) *The space Θ_D is a $\mathbb{T}_{|D|}$-module if and only if its Eisenstein space Θ_D^E is a $\mathbb{T}_{|D|}$-module if and only if D is a fundamental discriminant.*

(b) *The space Θ_D^S of cusp forms is a $\mathbb{T}_{|D|}$-module if and only if every proper discriminental divisor D' of D idoneal, i.e. iff D satisfies the following condition:*

$$g_{D/p^2} = h_{D/p^2}, \quad \textit{for all primes } p \textit{ such that } p^2|D \textit{ and } D/p^2 \equiv 0,1 \pmod 4. \tag{47}$$

Proof. (a) If D is a fundamental discriminant, then every $\chi \in \mathrm{Cl}(D)^*$ is primitive, and hence by Corollary 33 (and Proposition 9) Θ_D has a basis consisting of $\mathbb{T}_{|D|}$-eigenforms and hence is a $\mathbb{T}_{|D|}$-module.

Next, suppose that Θ_D is a $\mathbb{T}_{|D|}$-module. Then we have:

$$\vartheta_\chi \text{ is a } \mathbb{T}_{|D|}\text{-eigenfunction}, \quad \forall \chi \in \mathrm{Cl}(D)^* \tag{48}$$

To see this, note first $f := \vartheta_\chi$ is a $\mathbb{T}(D)$-eigenfunction by Theorem 12. Consider $n \geq 1$ and put $f_n := f|T_n$. By hypothesis, $f_n \in \Theta_D$. Since T_n commutes with the operators in $\mathbb{T}(D)$, f_n is the same $\mathbb{T}(D)$-eigenspace as f. By multiplicity 1 (cf. Theorem 1), it follows that $f_n = c_n f$, for some $c_n \in \mathbb{C}$. Thus $f = \vartheta_\chi$ is a $\mathbb{T}_{|D|}$-eigenfunction, as claimed.

Thus, if Θ_D is a $\mathbb{T}_{|D|}$-eigenspace, then it follows from (48) that $\Theta_D^E = \sum_{\chi^2=1} \mathbb{C}\vartheta_\chi$ is also a $\mathbb{T}_{|D|}$-eigenspace.

Now assume that Θ_D^E is a $\mathbb{T}_{|D|}$-eigenspace. Then the above proof of (48) yields that for each quadratic χ, the series ϑ_χ is a $\mathbb{T}_{|D|}$-eigenfunction, and hence by Corollary 33 we obtain that all quadratic characters are primitive. In particular, the the trivial character $\chi = 1$ is primitive, which can happen only when D is fundamental; cf. Example 35 below.

(b) By the same argument as in part (a) we see that Θ_D^S is a $\mathbb{T}_{|D|}$-module if and only if every non-quadratic $\chi \in \mathrm{Cl}(D)^*$ is primitive. But this means that for each prime p as in (47) we have that $\mathrm{Cl}(D/p^2)^*$ has only quadratic characters, and so $\mathrm{Cl}(D/p^2)$ is an elementary abelian 2-group, i.e. $g_{D/p^2} = h_{D/p^2}$. Thus (47) holds (and conversely).

We now illustrate the main results of this paper by working out some special cases. The first example examines ϑ_χ when $\chi = 1 \in \mathrm{Cl}(D)^*$ is the trivial character.

Example 35 (a) If $D = d_K < 0$ is a *fundamental* discriminant, then the theta function of associated to the trivial character $\chi_{d_K,0} = 1 \in \mathrm{Cl}(d_K)^*$ is

$$\vartheta_K(z) \;:=\; \vartheta_{\chi_{d_K,0}}(z) \;=\; f_1(z;1,\psi_{d_K}) \;\text{ with }\; L(s,\vartheta_K) = \zeta(s)L(s,\psi_{d_K}) = \zeta_K(s);$$

i.e. its associated L-function is just the Dedekind ζ-function of $K = \mathbb{Q}(\sqrt{D})$. In particular, the T_p-eigenvalue of ϑ_K is $a_p(\vartheta_K) = 1 + \left(\frac{d_K}{p}\right)$, for all primes p.

(b) If $D = d_K f_D^2$ is any negative discriminant, then the trivial character $\chi_{D,0} = 1$ on $\mathrm{Cl}(D)$ has conductor $f_{\chi_{D,0}} = 1$, and its associated primitive character is $(\chi_{D,0})_{pr} = \chi_{d_K,0} = 1 \in \mathrm{Cl}(d_K)^*$. Thus, by (2), (3) and part (a) we have that

$$\vartheta_{\chi_{D,0}}(z) \;=\; \sum_{n|f_D^2} c_n(\chi_{D,0})\vartheta_K(nz) \quad\text{and}\quad L(s,\vartheta_{\chi_{D,0}}) \;=\; C(s,\chi_{D,0})\zeta_K(s),$$

where the coefficients $c_n(\chi_{D,0})$ of the finite Dirichlet series $C(s,\chi_{D,0})$ are given explicitly by Theorem 28.

We thus obtain an expression for $a_n(\chi_{0,D}) = \#Id_n(\mathfrak{O}_D)$ for all $n \geq 1$. In particular, the formula of Remark 32(b) complements the expression obtained in the Appendix (cf. Proposition 48). Thus, if $p^{e_p}||f_D$, then we have

$$\#I_{p^k}(\mathcal{O}_D) = p^{e_p}\left(1 - \tfrac{1}{p}\psi_{d_K}(p)\right)\#I_{p^{k-2e_p}}(\mathfrak{O}_K) = p^{e_p}\left(1 - \tfrac{1}{p}\psi_{d_K}(p)\right)\sum_{i=0}^{k-2e_p}\psi_{d_K}(p)^k,$$

if $k \geq 2e_p$. (For $k < 2e_p$ we have the expressions $\#I_{p^k}(\mathcal{O}_D) = 0$ and $\#I_{p^k}(\mathcal{O}_D) = p^{k/2}$, for k odd and even, respectively, which are obtained in the Appendix §6.2.)

We now illustrate what happens when the class number h_D is small.

Example 36 (a) $h_D = 1$.
In this case $\mathrm{Cl}(D) = \{1_D\}$ and $\mathrm{Cl}(D)^* = \{\chi_{D,0}\}$. Thus, by (6) we have that $\vartheta_{1_D} = w_D\vartheta_{\chi_{D,0}}$. If $D = d_K$ is a fundamental discriminant (i.e. if $-D = 3, 4, 7, 8, 11, 19, 43, 67, 163$), then we have by Example 35(a) that

$$\vartheta_{\chi_{D,0}} = \vartheta_K \quad \text{and} \quad L(s, \chi_{D,0}) = \zeta_K(s).$$

In the remaining cases (i.e. $-D = 12, 16, 27, 28$) this formula is no longer true. Indeed, since $f = f_D \in \{2, 3\}$ is here a prime, we obtain from Theorem 28 that $C(s, \chi_{D,0}) = 1 - a_D f^{-s} + f^{1-2s}$, where $a_D := a_f(\chi_{d_K,0}) = 1 + \left(\frac{d_K}{f}\right)$, (i.e. $a_D = 0, 1, 1, 2$, respectively), and so

$$\vartheta_{\chi_{D,0}}(z) = \vartheta_K(z) - a_D\vartheta_K(fz) + f\vartheta_K(f^2z), \quad L(s, \vartheta_{\chi_{D,0}}) = (1 - a_Df^{-s} + f^{1-2s})\zeta_K(s).$$

(b) $h_D = 2$.
Here $\mathrm{Cl}(D) = \{1_D, cl(f)\}$ and $\mathrm{Cl}(D)^* = \{\chi_{D,0}, \chi\}$. Since $D < -4$, we have that $w_D = 2$ and hence it follows from (6) that

$$\vartheta_{\chi_{D,0}} = \tfrac{1}{2}(\vartheta_{1_D} + \vartheta_f) \quad \text{and} \quad \vartheta_\chi = \tfrac{1}{2}(\vartheta_{1_D} - \vartheta_f).$$

Since $g_D = h_D = 2$, we know by genus theory that $D = D_1D_2c^2$, where $D_1 < 0$ and $D_2 > 0$ are fundamental discriminants (with $(D_1, D_2) = 1$) and $c \geq 1$, and that the associated Hecke character $\tilde{\chi}$ satisfies $\tilde{\chi}(\mathfrak{p}) = \psi_{D_i}(N_K(\mathfrak{p}))$, for $i = 1, 2$ and $\mathfrak{p} \nmid D$. Thus, $f_\chi = D_1D_2$ and $\bar{f}_\chi = c$, and

$$\vartheta_{\chi_{pr}}(z) \;=\; f_1(z; \psi_{D_1}, \psi_{D_2}) \quad \text{and} \quad L(s, \vartheta_{\chi_{pr}}) \;=\; L(s, \psi_{D_1})L(s, \psi_{D_2}).$$

In particular, the T_p-eigenvalue of ϑ_χ is $a_p(\chi) = \left(\frac{D_1}{p}\right) + \left(\frac{D_2}{p}\right)$, for $p \nmid D$. Note that if χ is primitive, i.e. if $c = 1$, then $\vartheta_\chi = \vartheta_{\chi_{pr}}$. If $c > 1$, then a check of all 29 D's with $h_D = 2$ shows that necessarily $D = -60$, and then ϑ_χ is worked out in Example 37(b).

(c) $h_D = 3$.
Here $\mathrm{Cl}(D) = \{1_D, cl(f), cl(f)^{-1}\}$ and $\mathrm{Cl}(D)^* = \{\chi_{D,0}, \chi, \chi^{-1}\}$. Since $\mathrm{Re}(\chi(f)) = \mathrm{Re}(\frac{-1\pm\sqrt{-3}}{2}) = -\frac{1}{2}$, we have by (7) that

$$\vartheta_{\chi_{D,0}} = \tfrac{1}{2}(\vartheta_{1_D} + 2\vartheta_f) \quad \text{and} \quad \vartheta_\chi = \vartheta_{\chi^{-1}} = \tfrac{1}{2}(\vartheta_{1_D} - \vartheta_f).$$

By Remark 17(a) we know that $\Theta_D^E = \mathbb{C}\vartheta_{\chi_{D,0}}$ and that $\Theta_D^S = \mathbb{C}\vartheta_\chi$; in particular, ϑ_χ is a cusp form and a $\mathbb{T}(D)$-eigenfunction; cf. Theorem 1. Moreover, Corollary 29 shows that ϑ_χ is a newform of level $|D|$ if and only if χ is primitive, i.e. if and only there is no $c > 1$ such that $h_{D/c^2} = 3$. This condition holds for 23 of the cases with $h_D = 3$ (i.e. for $-D = 23$, 31, 44, 59, 76, 83, 92, 107, 108, 124, 139, 172, 211, 243, 268, 283, 307, 331, 379, 499, 547, 643, 652, 883, 907) but fails for the two cases $-D = 92 = 2^2 \cdot 23$ and $-D = 124 = 2^2 \cdot 31$.

We complement the above discussion by working out two numerical examples.

Example 37 (a) $D = -23$.
Here $\mathrm{Cl}(D) = \{[1,1,6],[2,1,3],[2,-1,3]\}$, where $[a,b,c]$ denotes the equivalence class of the form $ax^2+bxy+cy^2$, and so $h_D = 3$. Since D is a fundamental discriminant, we have by Examples 35(a) and 36(c) that $\vartheta_{\chi_{D,0}} = \vartheta_K = f_1(\,\cdot\,;1,\psi_{-23})$ and ϑ_χ are primitive forms of level 23, and that ϑ_χ is a cusp form. (In fact, $N = |D| = 23$ is the smallest level for which $\Theta_D^S \neq \{0\}$.) Moreover,

$$\vartheta_{\chi_{D,0}} = \vartheta_K = \tfrac{1}{2}(\vartheta_{[1,1,6]} + 2\vartheta_{[2,1,3]}) \quad \text{and} \quad \vartheta_\chi = \tfrac{1}{2}(\vartheta_{[1,1,6]} - \vartheta_{[2,1,3]}),$$

and from this (or from (8)) we obtain that

$$\vartheta_{[1,1,6]} = \tfrac{2}{3}(\vartheta_K + 2\vartheta_\chi) \quad \text{and} \quad \vartheta_{[2,1,3]} = \tfrac{2}{3}(\vartheta_K - \vartheta_\chi).$$

(b) $D = -60 = -15\cdot 2^2$.
Here $\mathrm{Cl}(D) = \{[1,0,15],[3,0,5]\}$, so $h_D = 2$ and $\mathrm{Cl}(D)^* = \{\chi_{D,0},\chi\}$. Since $h_{d_K} = h_{-15} = 2$, we see that $f_{\chi_{D,0}} = f_\chi = 1$ and so both $\chi_{D,0}$ and χ are imprimitive with $\bar{f}_{\chi_{D,0}} = \bar{f}_\chi = 2$. The theta-functions of the associated primitive characters are $\vartheta_K = f_1(\,\cdot\,;1,\psi_{-15})$ and $\vartheta_{\chi_{pr}} = f_1(\,\cdot\,;\psi_{-3},\psi_5)$, respectively. Since $a_2(\vartheta_K) = 1 + \left(\frac{-15}{2}\right) = 2$ and $a_2(\chi_{pr}) = \left(\frac{-3}{2}\right) + \left(\frac{5}{2}\right) = -2$, it follows from Theorem 28 that $C(s,\chi_{D,0}) = 1 - 2\cdot 2^{-s} + 2\cdot 2^{-2s}$ and that $C(s,\chi) = 1 + 2\cdot 2^{-s} + 2\cdot 2^{-2s}$. Thus,

$$\vartheta_{\chi_{D,0}}(z) = \vartheta_K(z) - 2\vartheta_K(2z) + 2\vartheta_K(4z) \quad \text{and} \quad \vartheta_\chi(z) = \vartheta_{\chi_{pr}}(z) + 2\vartheta_{\chi_{pr}}(2z) + 2\vartheta_{\chi_{pr}}(4z),$$

and the associated L-functions are

$$L(s,\chi_{D,0}) = (1 - 2^{1-s} + 2^{1-2s})\zeta_K(s) \quad \text{and} \quad L(s,\chi) = (1 + 2^{1-s} + 2^{1-2s})L(s,\chi_{pr}),$$

where $L(s,\chi_{pr}) = L(s,\psi_{-3})L(s,\psi_5)$.

6 Appendix: Ideals of Quadratic Orders

The purpose of this appendix is to collect some well-known results about ideals in an order $\mathfrak{O}_D$ of an imaginary quadratic field K, and to extend these to obtain the results which were used in this article.

6.1 Lattices in an imaginary quadratic field

Let $K = \mathbb{Q}(\sqrt{d_K})$ be an imaginary quadratic field of discriminant $d_K < 0$, and let $\mathfrak{O}_K = \mathbb{Z} + \omega_K\mathbb{Z}$ denote its ring of integers, where $\omega_K = \frac{1}{2}(d_K + \sqrt{d_K})$. For any $f \geq 1$, put $D = f^2 d_K$ and

$$\mathfrak{O}_D \;=\; \mathbb{Z} + f\omega_K\mathbb{Z} \;=\; \mathbb{Z} + \omega_D\mathbb{Z}, \quad \text{where } \omega_D = \tfrac{1}{2}(D + \sqrt{D}).$$

It is immediate that $\mathfrak{O}_D$ is a subring of $\mathfrak{O}_K$ of index $f_{\mathfrak{O}_D} := [\mathfrak{O}_K : \mathfrak{O}_D] = f$. The ring $\mathfrak{O}_D$ is called the *order of discriminant* D (or *of conductor* f), and it is well-known that every subring R of $\mathfrak{O}_K$ with quotient field K is of this form, i.e. $R = \mathfrak{O}_D$ with $D = f_R^2 d_K$, where $f_R = [\mathfrak{O}_K : R]$. (For such and other basic facts about orders see [26], §90-113, [1], §II.7, [15], §8.1.)

Let $\mathrm{Lat}(K)$ denote the set of all *lattices* in K, i.e. the set of all finitely generated $\mathbb{Z}$-modules which contain some $\mathbb{Q}$-basis of K.

If $L \in \mathrm{Lat}(K)$ is a lattice, then its *associated order* is $\mathfrak{O}(L) = \{\lambda \in K : \lambda L \subset L\}$. It is easy to see that $\mathfrak{O}(L)$ is an *order* of K, i.e. $\mathfrak{O}(L) = \mathfrak{O}_D$, for some $D = f^2 d_K$.

The *norm* of the lattice L is by defined by $N(L) = |\det(T)|$, where $T \in \mathrm{GL}_2(\mathbb{Q})$ is such that $T(\mathfrak{O}(L)) = L$; cf. [1], §II.6. Note that if $L \subset \mathfrak{O}_K$, then

$$(49) \qquad [\mathfrak{O}_K : L] = [O_K : \mathfrak{O}(L)]N(L).$$

If $L_1, L_2 \in \mathrm{Lat}(K)$ are two lattices, then the product (module) L_1L_2 is again a lattice. By [1], §II.7, Ex. 6 and 10, its order and norm are given by the formulae

$$(50) \qquad \mathfrak{O}(L_1L_2) = \mathfrak{O}(L_1)\mathfrak{O}(L_2) \quad \text{and} \quad N(L_1L_2) = N(L_1)N(L_2).$$

In particular, we see that the set $I(\mathfrak{O}_D) = \{L \in \mathrm{Lat}(K) : \mathfrak{O}(L) = \mathfrak{O}_D\}$ is closed under multiplication. In fact, $I(\mathfrak{O}_D)$ is a group with unit $\mathfrak{O}_D$: it can be identified with the group of invertible $\mathfrak{O}_D$-submodules of K; cf. [15], p. 91. Thus, if $P(\mathfrak{O}_D) = \{\lambda\mathfrak{O}_D : \lambda \in K^\times\}$ denotes the subgroup of principal $\mathfrak{O}_D$-modules, then the quotient

$$\mathrm{Cl}(\mathfrak{O}_D) = \mathrm{Pic}(\mathfrak{O}_D) = I(\mathfrak{O}_D)/P(\mathfrak{O}_D)$$

is called the *class group* (or *Picard group*) of the order $\mathfrak{O}_D$, and its (group) order $h_D := |\mathrm{Cl}(\mathfrak{O}_D)|$ is called the *class number* of $\mathfrak{O}_D$.

If $\mathfrak{O}_{D_i}$ are two orders of K with discriminants $D_i = f_{D_i}^2 d_K$, then

$$(51) \qquad \mathfrak{O}_{D_1} \supset \mathfrak{O}_{D_2} \quad \text{if and only if} \quad f_{D_1} \mid f_{D_2},$$

and hence

$$(52) \qquad \mathfrak{O}_{D_1} \cap \mathfrak{O}_{D_2} = \mathfrak{O}_{\mathrm{lcm}(f_{D_1}, f_{D_2})^2 d_K} \quad \text{and} \quad \mathfrak{O}_{D_1}\mathfrak{O}_{D_2} = \mathfrak{O}_{\gcd(f_{D_1}, f_{D_2})^2 d_K}.$$

Proposition 38 *If R and R' are two orders of K with $R \subset R'$, then the rule $L \mapsto LR'$ defines a surjective homomorphism $\pi_{R,R'} : I(R) \to I(R')$ with kernel*

$$(53) \qquad \mathrm{Ker}(\pi_{R,R'}) = \{L \in I(R) : L \subset R' \textit{ and } [R' : L] = [R' : R]\}.$$

Proof. If $L \in I(R)$, then by (50) we have $\mathfrak{O}(LR') = \mathfrak{O}(L)\mathfrak{O}(R') = RR' = R'$, so $LR' \in I(R')$. Thus, the rule $L \mapsto LR'$ defines a map $\pi = \pi_{R,R'} : I(R) \to I(R')$. Moreover, π is a homomorphism because $\pi(L_1)\pi(L_2) = L_1R'L_2R' = L_1L_2R'R' = L_1L_2R' = \pi(L_1L_2)$.

To prove (53), let $L \in \mathrm{Ker}(\pi)$. Then $LR' = R'$. Since R' is an order, we have that $N(R') = [\mathfrak{O}(R') : R'] = [R' : R'] = 1$, and so by (50) we have that $N(L) = N(L)N(R') = N(LR') = N(R') = 1$. Thus, since $\mathfrak{O}(L) = R$, we see that $[R' : R] = [R' : R]N(L) = [R' : R][R : L] = [R' : L]$. Moreover, since $L = L \cdot 1 \subset LR' = R'$ it follows that $L \subset R'$, and hence $L \in \mathcal{K} := \{L \in I(R) : L \subset R' \text{ and } [R' : L] = [R' : R]\}$.

Conversely, if $L \in \mathcal{K}$, then $N(L) = [R : L] = [R : L]/[R' : R] = 1$, and hence $N(LR') = N(L)N(R') = 1 \cdot 1 = 1$. Now since $\mathfrak{O}(LR') = \mathfrak{O}(L)\mathfrak{O}(R') = RR' = R'$, we have that $1 = N(LR') = [R' : LR']$. But since $L \subset R'$, we have $LR' \subset R'$ and so this forces $LR' = R'$. Thus $L \in \mathrm{Ker}(\pi)$, which proves (53).

It remains to show that π is surjective. Let $L' \in I(R')$. Then by [15], Theorem 5 (p. 93), there exists $\lambda \in K^\times$ such that $\lambda L' + fR' = R'$, where $f = [\mathfrak{O}_K : R]$. We then have that $L := \lambda L' \cap R \in I(R)$ and that $LR' = \lambda L'$ by applying [15], Theorem 4 (p. 92), to R' and R. (See also Proposition 48 below.) Thus $\pi(\lambda^{-1}L) = (\lambda^{-1}L)R' = L'$, and so π is surjective.

Remark 39 Since $\pi_{R,R'}(\lambda R) = \lambda R'$, for all $\lambda \in K^\times$, it is clear that $\pi_{R,R'}$ induces a surjective homomorphism

(54) $$\overline{\pi}_{R,R'} : \mathrm{Cl}(R) = I(R)/P(R) \ \to\ \mathrm{Cl}(R') = I(R')/P(R').$$

Moreover, it is immediate that

(55) $$\mathrm{Ker}(\overline{\pi}_{R,R'}) \ = \ \mathrm{Ker}(\pi_{R,R'})P(R)/P(R)$$

because if $LP(R) \in \mathrm{Ker}(\overline{\pi}_{R,R'})$, then $L = \lambda R'$, for some $\lambda \in K^\times$, and then $LP(R) = (\lambda^{-1}L)(\lambda R)P(R) \in \mathrm{Ker}(\pi_{R,R'})P(R)/P(R)$, so (55) follows.

Corollary 40 *In the above situation put $\bar{f} = [R' : R]$. Then the map $\lambda \mapsto L_\lambda := \mathbb{Z}\lambda + \bar{f}R'$ induces an exact sequence*

(56) $$0 \longrightarrow (\mathbb{Z}/\bar{f}\mathbb{Z})^\times \longrightarrow (R'/\bar{f}R')^\times \overset{L_{R',\bar{f}}}{\longrightarrow} \mathrm{Ker}(\pi_{R,R'}) \longrightarrow 0.$$

Thus, if D' denotes the discriminant of $R' = \mathfrak{O}_{D'}$ and if $\psi_{R'}(p) = \left(\frac{D'}{p}\right)$ denotes the associated Legendre-Kronecker symbol, then

(57) $$|\mathrm{Ker}(\pi_{R,R'})| \ = \ \bar{f} \prod_{p|\bar{f}} \left(1 - \frac{1}{p}\psi_{R'}(p)\right),$$

Proof. First note that if $\lambda \in R'$, then $L_\lambda/\bar{f}R' \leq R'/\bar{f}R'$ is the cyclic subgroup generated by $\bar{\lambda} = \lambda + \bar{f}R'$, so $L_\lambda = L_{\bar{\lambda}}$ depends only on $\bar{\lambda}$. Moreover, since $\lambda\bar{f}R' = \bar{f}R'$ if $\bar{\lambda} \in (R'/\bar{f}R')^\times$, it is clear that $L_\lambda L_{\lambda'} = L_{\lambda\lambda'}$, and so $\lambda \mapsto L_\lambda$ defines a homomorphism $L_{R',f} : (R'/\bar{f}R')^\times \to \mathrm{Lat}(K)$. To see that $\mathrm{Im}(L_{R',f}) \subset \mathrm{Ker}(\pi_{R,R'})$, note first that L_λ

is clearly an R-module because $R = \mathbb{Z} + \bar{f}R'$. Moreover, since $\lambda\lambda' \equiv 1 \pmod{\bar{f}R'}$, for some $\lambda' \in R'$, we see that $L_\lambda L_{\lambda'} = L_1 = R$, and so $L_\lambda \in I(R)$. In addition, $L_\lambda R' = R'$ because $L_\lambda R'$ is an R'-ideal which contains 1, and so $L_\lambda \in \mathrm{Ker}(\pi_{R,R'})$.

To show that $L_{R',\bar{f}}$ is surjective, let $L \in \mathrm{Ker}(\pi_{R,R'})$. Then by (53) we know that $L \subset R'$ and that $[R' : L] = \bar{f}$, so $L \supset \bar{f}R'$ and $[L : \bar{f}R'] = \frac{[R':\bar{f}R']}{[R':L]} = \frac{\bar{f}^2}{\bar{f}} = \bar{f}$. Since $LR' = R'$, we see that $L \not\subset pR'$, for any prime $p|\bar{f}$, and hence $L/\bar{f}R'$ is cyclic, i.e. $L = \mathbb{Z}\lambda + \bar{f}R'$ for some $\lambda \in R'$. Since $|L/\bar{f}R'| = \bar{f}$, this means that λ has order $\bar{f}$ in $R'/\bar{f}R' \simeq \mathbb{Z}/\bar{f}\mathbb{Z} \times \mathbb{Z}/\bar{f}\mathbb{Z}$. Then $\lambda = a + b\bar{f}\omega_{D'}$ with $a, b \in \mathbb{Z}$ and $(a, b, \bar{f}) = 1$, where D' is the discriminant of $R' = \mathfrak{O}_{D'}$, and so it follows easily that $\lambda \in (R'/\bar{f}R')^\times$. Thus $L = L_\lambda = L_{R',\bar{f}}(\lambda)$, and hence $L_{R,\bar{f}}$ is surjective.

Next we observe that $\bar{\lambda} \in \mathrm{Ker}(L_{R',\bar{f}}) \Leftrightarrow L_{\bar{\lambda}} = R = \mathbb{Z} + \bar{f}R' \Leftrightarrow \bar{\lambda} = n + \bar{f}R'$, for some $n \in \mathbb{Z}$ with $(n, \bar{f}) = 1$. Thus, since the map $\mathbb{Z}/\bar{f}\mathbb{Z} \to R'/\bar{f}R'$ is injective, it follows that the sequence (56) is exact.

It remains to verify (57). Applying (56) with $R' = \mathfrak{O}_K$ (and hence $\bar{f} = f := f_R$) yields that

$$|\mathrm{Ker}(\pi_{R,\mathfrak{O}_K})| = \frac{\phi(f\mathfrak{O}_K)}{\phi(f)} = f\prod_{p|f}\left(1 - \frac{1}{p}\psi_{\mathfrak{O}_K}(p)\right),$$

where the second formula is deduced as in [15], p. 95. (Here, as in [15], $\phi(f\mathfrak{O}_K) = |(\mathfrak{O}_K/f\mathfrak{O}_K)^\times|$.) From this (and the surjectivity of $\pi_{R,R'}$) it follows that for any $R' \supset R$ we have

$$|\mathrm{Ker}(\pi_{R,R'})| = \frac{|\mathrm{Ker}(\pi_{R,\mathfrak{O}_K})|}{|\mathrm{Ker}(\pi_{R',\mathfrak{O}_K})|} = \bar{f}\prod_{p|f_R, p\nmid f_{R'}}\left(1 - \frac{1}{p}\psi_{d_K}(p)\right) = \bar{f}\prod_{p|\bar{f}}\left(1 - \frac{1}{p}\psi_{R'}(p)\right).$$

Corollary 41 *If $R \subset R_1 \cap R_2$ and $R_3 = R_1R_2$, then*

$$\mathrm{Ker}(\pi_{R,R_3}) = \mathrm{Ker}(\pi_{R,R_1}) \cdot \mathrm{Ker}(\pi_{R,R_2}) \quad \textit{and} \quad \mathrm{Ker}(\overline{\pi}_{R,R_3}) = \mathrm{Ker}(\overline{\pi}_{R,R_1}) \cdot \mathrm{Ker}(\overline{\pi}_{R,R_2}).$$

Proof. By (55) it is enough to prove the first assertion. Moreover, since $\mathrm{Ker}(\pi_{R,R_i}) = \pi^{-1}_{R,R_1\cap R_2}(\mathrm{Ker}(\pi_{R_1\cap R_2,R_i}))$, for $i = 1, 2, 3$, it suffices to verify the formula for $R = R_1 \cap R_2$.

For this, write $f = [\mathfrak{O}_K : R]$, $f_i = [\mathfrak{O}_K : R_i]$ and $\bar{f}_i = \frac{f}{f_i} = [R_i : R]$, for $i = 1, 2, 3$. Then by (52) we have that $f_3 = (f_1, f_2)$. Since $R = R_1 \cap R_2$, we have that $f = \frac{f_1f_2}{f_3}$ by (52), so $\bar{f}_3 = \bar{f}_1\bar{f}_2$ and $(\bar{f}_1, \bar{f}_2) = 1$. It thus follows from (57) that $|\mathrm{Ker}(\pi_{R,R_3})| = |\mathrm{Ker}(\pi_{R,R_1})| \cdot |\mathrm{Ker}(\pi_{R,R_2})|$. Thus, the assertion follows once we have shown that $\mathrm{Ker}(\pi_{R,R_1}) \cap \mathrm{Ker}(\pi_{R,R_2}) = \{R\}$. Now if $L \in \mathrm{Ker}(\pi_{R,R_1}) \cap \mathrm{Ker}(\pi_{R,R_2})$, then $L \subset R_1 \cap R_2 = R$. But since $[R : L] = N(L) = 1$ (the latter because $LR_i = R_i$), it follows that $L = R$, and hence $\mathrm{Ker}(\pi_{R,R_1}) \cap \mathrm{Ker}(\pi_{R,R_2}) = \{R\}$, as claimed.

6.2 The ideal theory of $\mathfrak{O}_D$

We now study the ideals of the order $R = \mathfrak{O}_D \subset K$ more closely. For this, we shall use the following general result which is applicable here because $\mathfrak{O}_D$ is a noetherian domain of (Krull) dimension 1.

Proposition 42 *Let A be a noetherian domain of dimension 1, and let $\mathfrak{a} \subset A$ be a non-zero ideal of A. If $\mathfrak{m} \in \max(A)$ is any maximal ideal of A with $\mathfrak{a} \subset \mathfrak{m}$, then $\mathfrak{a}(\mathfrak{m}) := \mathfrak{a}A_\mathfrak{m} \cap A$ is an $\mathfrak{m}$-primary ideal of A. In particular, if $|A/\mathfrak{m}| < \infty$, then $|A/\mathfrak{a}(\mathfrak{m})| = |A/\mathfrak{m}|^s$, for some integer $s = s_\mathfrak{m}(\mathfrak{a}) \geq 1$. Moreover, we have*

$$\mathfrak{a} = \bigcap_{\mathfrak{m}} \mathfrak{a}(\mathfrak{m}) = \prod_{\mathfrak{m}} \mathfrak{a}(\mathfrak{m}). \tag{58}$$

Thus $\mathfrak{a}$ is an invertible ideal if and only $\mathfrak{a}(\mathfrak{m})$ is invertible for all maximal ideals $\mathfrak{m}$.

Proof. Since $A_\mathfrak{m}$ is a local 1-dimensional noetherian domain, every proper ideal of $A_\mathfrak{m}$ is primary, and hence $\mathfrak{a}(\mathfrak{m})$ is primary ideal of A with radical $\mathfrak{m}$. This proves the first assertion. From this the second assertion is an immediate consequence because by [2], p. 265, the A-module $M = A/\mathfrak{a}(\mathfrak{m})$ has a composition series $\{M_i\}$ with $M_i/M_{i+1} \simeq A/\mathfrak{m}$, and so the assertion follows (with $s = \text{length}(M)$).

The first identity of (58) is true in any domain; cf. [2], p. 89. Next, if $\mathfrak{m} \neq \mathfrak{m}' \in \max(A)$ are two distinct maximal ideals, then $\mathfrak{a}(\mathfrak{m}) \not\subset \mathfrak{m}'$ and hence $\mathfrak{a}(\mathfrak{m})A_{\mathfrak{m}'} = A_{\mathfrak{m}'}$. Thus, if we put $\mathfrak{a}' = \prod \mathfrak{a}(\mathfrak{m})$, then $\mathfrak{a}'A_\mathfrak{m} = \mathfrak{a}(\mathfrak{m})A_\mathfrak{m} = \mathfrak{a}A_\mathfrak{m}$, for every $\mathfrak{m} \in \max(A)$, and so $\mathfrak{a}' = \mathfrak{a}$, which proves the second equality of (58).

Finally, if $\mathfrak{a}(\mathfrak{m})$ is invertible for all $\mathfrak{m} \in \max(A)$, then $\mathfrak{a}$ is invertible by (58). Conversely, suppose that $\mathfrak{a}$ is invertible and fix $\mathfrak{m} \in \max(A)$ with $\mathfrak{m} \supset \mathfrak{a}$. Then $\mathfrak{a}(\mathfrak{m})A_\mathfrak{m} = \mathfrak{a}A_\mathfrak{m}$ is principal, and for every $\mathfrak{m}' \neq \mathfrak{m}$ we have $\mathfrak{a}(\mathfrak{m})A_{\mathfrak{m}'} = A_{\mathfrak{m}'}$. Thus $\mathfrak{a}(\mathfrak{m})$ is locally principal and hence invertible; cf. [2], p. 117.

Notation. Let $Id(R) = \{L \in I(R) : L \subset R\}$ denote the set of invertible R-ideals. Moreover, for any integer $n \geq 1$, let $Id_n(R) = \{\mathfrak{a} \in Id(R) : N(\mathfrak{a}) = n\}$ denote the set of invertible R-ideals of norm n, and let $Id(R, n) = \{\mathfrak{a} \in Id(R) : (N(\mathfrak{a}), n) = 1\}$ denote the set of invertible R-ideals whose norm is relatively prime to n.

Corollary 43 *Let $\mathfrak{a} \in Id(R)$ be an invertible R-ideal, and $\mathfrak{m} \in \max(R)$ be a maximal ideal, where $R = \mathfrak{O}_D$. Then $\mathfrak{a}(\mathfrak{m}) \in Id(R)$ is also invertible and $N(\mathfrak{a}(\mathfrak{m})) = p^s$, for some s, where $p|\mathfrak{m}$, i.e. p is the unique prime number with $p \in \mathfrak{m}$. Furthermore,*

$$\mathfrak{a} = \prod_{\mathfrak{m}} \mathfrak{a}(\mathfrak{m}) \quad \textit{and hence} \quad N(\mathfrak{a}) = \prod_{\mathfrak{m}} N(\mathfrak{a}(\mathfrak{m})).$$

In particular, if $\mathfrak{m} \in \text{supp}(\mathfrak{a}) := \{\mathfrak{m} \in \max(R) : \mathfrak{a} \subset \mathfrak{m}\}$, and $p|\mathfrak{m}$, then $p|N(\mathfrak{a})$.

Proof. If $\mathfrak{m} \in \max(R)$, then $R/\mathfrak{m}$ is a finite field of characteristic p, where $p \in \mathfrak{m}$. Thus, since R is a 1-dimensional noetherian ring and since $N(\mathfrak{a}) = [R : \mathfrak{a}]$, if $\mathfrak{a} \in Id(R)$, the corollary follows immediately from Proposition 42 (together with (50)).

We now want to study the set $Id_n(R)$ of invertible ideals of norm n of R more closely. First we note:

Proposition 44 *If $(m, n) = 1$, then the map $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a}\mathfrak{b}$ induces a bijection*

$$Id_m(R) \times Id_n(R) \xrightarrow{\sim} Id_{mn}(R).$$

Proof. If $(\mathfrak{a}, \mathfrak{b}) \in Id_m(R) \times Id_n(R)$, then $\mathfrak{a}\mathfrak{b} \in Id_{mn}(R)$ by (50). Suppose now that $\mathfrak{c} \in Id_{mn}(R)$, and put, for an integer k, $\mathrm{supp}_k(\mathfrak{c}) := \mathrm{supp}(\mathfrak{c} + kR) = \{\mathfrak{m} \in \mathrm{supp}(\mathfrak{c}) : p|\mathfrak{m} \Rightarrow p|k\}$ and $\mathfrak{c}_k = \prod_{\mathfrak{m}\in\mathrm{supp}_k(\mathfrak{c})} \mathfrak{a}(\mathfrak{m})$. Then by Corollary 43 we have $\mathrm{supp}(\mathfrak{c}) = \mathrm{supp}(\mathfrak{c})_m \,\dot\cup\, \mathrm{supp}(\mathfrak{c})_n$, and so $\mathfrak{c} = \mathfrak{c}_m\mathfrak{c}_n$. Since $(N(\mathfrak{c}_m), n) = (N(\mathfrak{c}_n), m) = 1$, we see that $(\mathfrak{c}_m, \mathfrak{c}_n) \in Id_m(R) \times Id_n(R)$, and so the map is surjective. It is injective because $\mathfrak{c}$ is uniquely determined by its local components $\mathfrak{c}(\mathfrak{m})$.

Thus, by the above result, it is enough to study the sets $Id_n(R)$ for prime powers $n = p^r$. If p is prime to the conductor f_R of R, i.e. if $p \nmid f_R = [\mathfrak{O}_K : R]$, then $Id_n(R)$ is essentially the same as $Id_n(\mathfrak{O}_K)$, as the next (well-known) result shows.

Proposition 45 *The rule $\mathfrak{a} \mapsto \mathfrak{a} \cap R$ induces an injection $\tilde\varphi_R : Id(\mathfrak{O}_K, f_R) \hookrightarrow Id(R)$ with image $Id(R, f_R)$, and we have that*

$$\tilde\varphi_R(Id_n(\mathfrak{O}_K)) = Id_n(R), \quad \textit{for all } n \geq 1,\ (n, f_R) = 1. \tag{59}$$

Thus

$$\#Id_n(R) \;=\; \#Id_n(\mathfrak{O}_K) \;=\; \sum_{d|n} \psi_{d_K}(d), \quad \textit{if } (n, f_R) = 1. \tag{60}$$

Proof. The first assertion follows from [15], Theorem 4 (p. 92). Moreover, since that theorem also asserts that $\tilde\varphi_R(\mathfrak{a})\mathfrak{O}_K = \mathfrak{a}$, if $\mathfrak{a} \in Id(\mathfrak{O}_K, f_R)$, it follows from (50) that $N(\tilde\varphi_R(\mathfrak{a})) = N(\tilde\varphi_R(\mathfrak{a}))N(\mathfrak{O}_K) = N(\mathfrak{a})$. Thus (59) holds, and hence the first equality of (60) follows. The second equality of (60) is well-known; cf. Weber[26], p. 345.

We now turn to study the ideals of $R = \mathfrak{O}_D$ with norm dividing the conductor $f_R = f_D$. Here we first prove:

Proposition 46 *If $p|f_D$ is a prime divisor of the conductor $f_D = [\mathfrak{O}_K : \mathfrak{O}_D]$ of $\mathfrak{O}_D$, then $\mathfrak{m}_p := p\mathfrak{O}_{D/p^2}$ is the unique maximal ideal of $\mathfrak{O}_D$ containing p. Moreover, $|\mathfrak{O}_D/\mathfrak{m}_p| = p$ and $\mathfrak{m}_p$ is an $\mathfrak{O}_D$-ideal which is not invertible, i.e. $\mathfrak{m}_p \notin Id(\mathfrak{O}_D)$. In particular, $Id_p(\mathfrak{O}_D) = \emptyset$.*

Proof. Since $\mathfrak{m}_p = p(\mathbb{Z} + (f_D/p)\omega_K\mathbb{Z}) = p\mathbb{Z} + f_D\omega_K\mathbb{Z}$, we see that $\mathfrak{m}_p \subset \mathfrak{O}_D = \mathbb{Z} + f_D\omega_K\mathbb{Z}$ and that $[\mathfrak{O}_D : \mathfrak{m}_p] = p$. Clearly, $\mathfrak{m}_p$ is a principal $\mathfrak{O}_{D/p^2}$-ideal, and hence a fortiori an $\mathfrak{O}_D$-ideal. Thus, $\mathfrak{m}_p$ is a maximal ideal of $\mathfrak{O}_D$ and $\mathfrak{O}(\mathfrak{m}_p) = \mathfrak{O}_{D/p^2}$, so $\mathfrak{m}_p$ cannot be invertible as an $\mathfrak{O}_D$-ideal.

Now suppose that $\mathfrak{m} \supset p\mathfrak{O}_D$ is a maximal ideal. Then (by Cohen-Seidenberg) $\mathfrak{m} = \mathfrak{m}\mathfrak{O}_{D/p^2} \cap \mathfrak{O}_D \supset p\mathfrak{O}_{D/p^2} \cap \mathfrak{O}_D = \mathfrak{m}_p$, and so $\mathfrak{m} = \mathfrak{m}_p$. Thus $\mathfrak{m}_p$ is the unique maximal ideal containing p.

Finally, if $\mathfrak{a} \in Id_p(\mathfrak{O}_D)$, then $[\mathfrak{O}_D : \mathfrak{a}] = N(\mathfrak{a}) = p$, so $\mathfrak{a}$ is a maximal ideal of $\mathfrak{O}_D$ with $p \in \mathfrak{a}$, and hence $\mathfrak{a} = \mathfrak{m}_p$. But $\mathfrak{m} \notin Id(\mathfrak{O}_D)$, contradiction. Thus $Id_p(\mathfrak{O}_D) = \emptyset$.

Corollary 47 *Suppose that $c|f_D = [\mathfrak{O}_K : \mathfrak{O}_D]$. If $\mathfrak{a} \in Id_n(\mathfrak{O}_D)$ and $c^2|n$, then $\mathfrak{a}\mathfrak{O}_{D/c^2} = c\mathfrak{b}$, for some $\mathfrak{b} \in Id_{n/c^2}(\mathfrak{O}_{D/c^2})$.*

Proof. Suppose first that $c = p$. Then by Corollary 43 and Proposition 46 we know that $\mathfrak{a} \subset \mathfrak{m}_p = p\mathfrak{O}_{D/p^2}$, and so $\mathfrak{b}' := \mathfrak{a}\mathfrak{O}_{D/p^2} \subset p\mathfrak{O}_{D/p^2}$. Thus $\mathfrak{b} := \frac{1}{p}\mathfrak{b}' \subset \mathfrak{O}_{D/p^2}$, and hence $\mathfrak{b} \in Id_{n/p^2}(\mathfrak{O}_{D/p^2})$ and $\mathfrak{a}\mathfrak{O}_{D/p^2} = p\mathfrak{b}$. Thus the assertion holds for $c = p$.

To prove the general case, induct on c. Since the assertion is vacuous for $c = 1$, we may assume that there is a prime $p|c$. Put $D' = D/c^2, \bar{D} = D/p^2$ and $\bar{n} = n/p^2$. Then, by what was just shown, $\mathfrak{a}\mathfrak{O}_{\bar{D}} = p\bar{\mathfrak{a}}$ with $\bar{\mathfrak{a}} \in Id_{\bar{n}}(\mathfrak{O}_{\bar{D}})$. Put $\bar{c} = c/p$. Then $\bar{c}|\frac{f}{p} = [\mathfrak{O}_K : \mathfrak{O}_{\bar{D}}]$ and $\bar{c}^2|\bar{n}$, so by induction $\bar{\mathfrak{a}}\mathfrak{O}_{D'} = \bar{c}\mathfrak{b}$ with $\mathfrak{b} \in Id_{\bar{n}/\bar{c}^2}(\mathfrak{O}_D)$. Thus $\mathfrak{a}\mathfrak{O}_{D'} = (p\bar{\mathfrak{a}})\mathfrak{O}_{D'} = p\bar{c}\mathfrak{b} = c\mathfrak{b}$, and so the assertion also holds for c.

As we shall now see, the above results show that when $n|f_D^2$, there is close connection between $Id_n(\mathfrak{O}_D)$ and $\mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D/n}})$.

Proposition 48 *Suppose that $n|f_D^2 = D/d_K$. If n is not a square, then $Id_n(\mathfrak{O}_D) = \emptyset$, whereas for $n = c^2$ we have that*

$$Id_{c^2}(\mathfrak{O}_D) = c\mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D/c^2}}).$$

Proof. Suppose first that n is not a square, i.e. suppose that there is a prime p such that $p^{2r+1} \parallel n$, for some $r \geq 0$. If $\mathfrak{a} \in Id_n(\mathfrak{O}_D)$, then $\mathfrak{a}_p := \mathfrak{a}(\mathfrak{m}_p) \in Id_{p^{2r+1}}(\mathfrak{O}_D)$, and hence, putting $D' = D/p^{2r}$, we have by Corollary 47 that $\mathfrak{a}_p\mathfrak{O}_{D'} = p^r\mathfrak{b}$, with $\mathfrak{b} \in Id_p(\mathfrak{O}_{D'})$. But since $p|[\mathfrak{O}_K : \mathfrak{O}_{D'}]$, this contradicts Proposition 46. Thus $Id_n(\mathfrak{O}_D) = \emptyset$.

Now suppose that $n = c^2$, and put $D' = D/c^2$. If $\mathfrak{a} \in Id_n(\mathfrak{O}_D)$, then by Corollary 47 we have that $\mathfrak{a}\mathfrak{O}_{D'} = c\mathfrak{b}$ with $\mathfrak{b} \in Id_1(\mathfrak{O}_{D'})$, i.e. $\mathfrak{a}\mathfrak{O}_{D'} = c\mathfrak{O}_{D'}$, and hence $\mathfrak{a} = cL$ with $L = \frac{1}{c}\mathfrak{a} \in \mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D'}})$. Thus $Id_n(\mathfrak{O}_D) \subset c\mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D'}})$.

Conversely, if $L \in \mathrm{Ker}(\pi_{\mathfrak{O}_D,\mathfrak{O}_{D'}})$, then $L \subset \mathfrak{O}_{D'}$ (cf. (53)), so $cL \subset c\mathfrak{O}_{D'} \subset \mathfrak{O}_D$, the latter because $c = [\mathfrak{O}_{D'} : \mathfrak{O}_D]$. Thus, $cL \in Id_n(\mathfrak{O}_D)$ because $N(cL) = c^2N(L) = c^2 = n$, and hence $Id_n(\mathfrak{O}_D) = c\mathrm{Ker}(\pi_{D,D'})$, as claimed.

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