Binary Theta Series and Modular Forms with Complex Multiplication

Ernst Kani[1]

1 Introduction

Let D be a negative discriminant, and let $\Theta(D)$ be the complex vector space generated by the binary theta series ϑ_f attached to the positive definite binary quadratic forms $f(x,y) = ax^2 + bxy + cy^2$ whose discriminant $D(f) = b^2 - 4ac$ equals D/t^2, for some integer t. It is a well-known classical fact that $\Theta(D)$ is a subspace of the space $M_1(|D|, \psi_D)$ of modular forms of weight 1, level $|D|$ and Nebentypus ψ_D, where $\psi_D = (\frac{D}{\cdot})$ is the Kronecker-Legendre character.

The purpose of this paper is to give an *intrinsic interpretation* of $\Theta(D)$ as a subspace of $M_1(|D|, \psi_D)$. More precisely, it turns out that $\Theta(D)$ is precisely the subspace $M_1^{CM}(|D|, \psi_D)$ of modular forms which have *complex multiplication* (CM) by their Nebentypus character ψ_D (in the sense of Ribet[10]):

Theorem 1 *If D is a negative discriminant, then the space $\Theta(D)$ of theta series equals the space of modular forms of weight 1, level $|D|$ and Nebentypus ψ_D with CM by ψ_D, i.e.,*

$$\Theta(D) \;=\; M_1^{CM}(|D|, \psi_D). \tag{1}$$

As a consequence of this result and of its proof, we can analyze the module structure of this space under the Hecke algebra and thus determine its dimension; cf. Theorem 16 and Remark 17 below.

There are two main ingredients for the proof of Theorem 1. The first of these is the precise description given in [8] of how each theta series ϑ_f can be expressed in terms of the (extended) Atkin-Lehner basis of $M_1(N, \psi)$; cf. [8], Theorems 1 and 3. The second ingredient uses the results of Deligne/Serre[4] to give a Galois-theoretic interpretation of newforms of weight 1 with CM; cf. Theorem 9. The following is a special case (part(d)) of that result.

Theorem 2 *Let $f \in S_1(N, \psi)$ be a newform. Then f has CM by ψ if and only if the image of the associated Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is a dihedral group.*

From this, together with a result of Bruckner[2] and the results of [8], the proof of Theorem 1 follows readily; cf. §5.

We note in passing that the above proof also yields a structure theorem (cf. Theorem 15 below) for *all* Galois representations of the above type; this may be viewed as a refinement of the discussion of Serre[13], §7. Moreover, this can be used to count the number of such representations of fixed conductor; cf. Remark 17(c).

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2 Binary theta series

As in the introduction, let $f(x,y) = ax^2 + bxy + cy^2$ be an integral, positive definite binary quadratic form of discriminant $D = b^2 - 4ac < 0$. Thus $D = f_D^2 d_K$, where d_K is the (fundamental) discriminant of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let $\tilde{Q}_D$ denote the set of all such forms, and let Q_D denote the subset consisting of the *primitive* forms, i.e., those for which $\gcd(a,b,c) = 1$. Note that if $f \in \tilde{Q}_D$ has *content* $c(f) := \gcd(a,b,c)$, then $f/c(f)$ is primitive with discriminant $D/c(f)^2$, so $f/c(f) \in Q_{D/c(f)^2}$. Thus $\tilde{Q}_D = \bigcup_{c|f_D} cQ_{D/c^2}$.

The *binary theta series* attached to $f \in \tilde{Q}_D$ is the function on the upper half-plane $\mathfrak{H}$ given by

$$\vartheta_f(z) \;=\; \sum_{x,y\in\mathbb{Z}} e^{2\pi i f(x,y)z} \;=\; \sum_{n=0}^{\infty} r_n(f) e^{2\pi i n z},$$

where $r_n(f) = \#\{(x,y) \in \mathbb{Z}^2 : f(x,y) = n\}$ denotes the number of representations of n by f. We observe that it follows from the definition that

$$\vartheta_f(z) \;=\; \vartheta_{f/c}(cz) \;=:\; V_c(\vartheta_{f/c})(z), \quad \text{if } c = c(f); \tag{2}$$

here the above "dilation operator" V_c is the same as that of Lang[9], p. 108.

We shall use throughout the following important classical fact (cf. Miyake[11], Corollary 4.9.5(3)).

Proposition 3 *If $f \in \tilde{Q}_D$, then ϑ_f is a modular form of weight 1, level $|D|$ and Nebentypus $\psi_D = \left(\frac{D}{\cdot}\right)$. Thus $\tilde{\Theta}_D := \sum_{f\in\tilde{Q}_D} \mathbb{C}\vartheta_f$ is a subspace of $M_1(|D|, \psi_D)$.*

Definition. If $D < 0$ and $D \equiv 0, 1 \pmod 4$, then the *space of all binary theta series of level dividing D* is

$$\Theta(D) \;=\; \sum_{c|f_D} \tilde{\Theta}_{D/c^2} \;\subset\; M_1(|D|, \psi_D).$$

Here the above inclusion follows from Proposition 3 together with the fact that we have the obvious inclusion $M_1(|D/c^2|, \psi_{D/c^2}) \subset M_1(|D|, \psi_D)$, if $c|f_D$.

In what follows, we shall use some of the results that were obtained in [8]. There, however, the focus was on the space $\Theta_D = \sum_{f\in Q_D} \mathbb{C}\vartheta_f$ generated by the theta series attached to the *primitive* forms $f \in Q_D$. The relation between the spaces Θ_D, $\tilde{\Theta}_D$ and $\Theta(D)$ is as follows:

$$\tilde{\Theta}_D \;=\; \sum_{c|f_D} V_c(\Theta_{D/c^2}) \quad \text{and} \quad \Theta(D) \;=\; \sum_{c|f_D} \sum_{c_1|f_D/c} V_{c_1}\Theta_{D/(cc_1)^2}. \tag{3}$$

Indeed, since $\tilde{Q}_D = \bigcup_{c|f_D} cQ_{D/c^2}$, the first formula follows immediately by using (2), and the second follows from the first and the definition of $\Theta(D)$.

We now review some facts proven in [8]. For this, recall that the group $\mathrm{GL}_2(\mathbb{Z})$ acts on binary quadratic forms by change of coordinates, and that this action preserves the sets $\tilde{Q}_D$ and Q_D It is immediate that $r_n(fT) = r_n(f)$, for all $n \geq 0$ and $T \in \mathrm{GL}_2(\mathbb{Z})$, so $\vartheta_{fT} = \vartheta_f$. We can thus index the theta series by the quotient set $\tilde{Q}_D/\mathrm{GL}_2(\mathbb{Z})$.

As was shown in [8], Proposition 7, the set $\{\vartheta_f : f \in Q_D/\mathrm{GL}_2(\mathbb{Z})\}$ is a basis of Θ_D. Moreover, another basis of Θ_D was constructed by using the fact (due to Gauss) that the set $\mathrm{Cl}(D) := Q_D/\mathrm{SL}_2(\mathbb{Z})$ has the structure of an abelian group. This basis is attached to the character group $\mathrm{Cl}(D)^* := \mathrm{Hom}(\mathrm{Cl}(D), \mathbb{C}^\times)$ of $\mathrm{Cl}(D)$ in the following way. If $\chi \in \mathrm{Cl}(D)^*$ is a character, then put

$$\vartheta_\chi := \frac{1}{w_D} \sum_{f \in Cl(D)} \chi(f)\vartheta_f, \tag{4}$$

where $w_D = r_1(1_D)$ and 1_D denotes the principal form. It is easy to see that $\vartheta_{\chi^{-1}} = \vartheta_\chi$, and that the set $\{\vartheta_\chi : \chi \in \mathrm{Cl}(D)^*\}$ is a basis of Θ_D; cf. [8], Proposition 9.

For what follows, it is often useful to identify a character $\chi \in \mathrm{Cl}(D)^*$ with its associated Hecke character $\tilde{\chi} := \chi \circ \lambda_D^{-1} \circ \varphi_D$ via the well-known isomorphisms

$$\lambda_D : \mathrm{Cl}(D) \xrightarrow{\sim} \mathrm{Cl}(\mathfrak{O}_D) \quad \text{and} \quad \varphi_D : I_K(f_D)/P_{K,\mathbb{Z}}(f_D) \xrightarrow{\sim} \mathrm{Cl}(\mathfrak{O}_D).$$

Here, as in [8] or Cox[3], $\mathfrak{O}_D$ is the order of discriminant D (and conductor f_D) in K, and $I_K(f_D)$ is the group of fractional $\mathfrak{O}_K$-ideals, etc. We recall the following result from [8]:

Proposition 4 *Let $\chi \in \mathrm{Cl}(D)^*$ be a character on $\mathrm{Cl}(D)$ with associated Hecke character $\tilde{\chi} = \chi \circ \lambda_D^{-1} \circ \varphi_D$. If p is a prime with $p \nmid D$, then $\vartheta_\chi \in \Theta_D$ is a normalized eigenfunction with respect to the Hecke operator T_p with eigenvalue*

$$a_p(\chi) = \begin{cases} 2\mathrm{Re}(\tilde{\chi}(\mathfrak{p})) & \text{if } \psi_D(p) = 1 \text{ and } p\mathfrak{O}_K = \mathfrak{p}\bar{\mathfrak{p}}, \\ 0 & \text{if } \psi_D(p) = -1. \end{cases} \tag{5}$$

Proof. The fact that ϑ_χ is a normalized T_p-eigenfunction is a special case of [8], Theorem 12. For convenience of the reader we sketch the proof of this fact. By using properties of invertible ideals of the order $\mathfrak{O}_D$ of discriminant D (cf. [8], §6.2), one shows that the Fourier coefficients $a_n(\chi)$ of ϑ_χ are multiplicative. By using Satz 42 of Hecke[6], it thus follows from this (and the fact that ϑ_χ is a modular form of level dividing $|D|$) that ϑ_χ is a T_p-eigenfunction for all primes $p \nmid D$.

Since ϑ_χ is normalized, its T_p-eigenvalue is the p-th Fourier coefficient $a_p(\chi)$ of ϑ_χ. Since $p \nmid D$, we know by [8], formula (13), that $a_p(\chi) = a_p(\tilde{\chi}) := \sum_{\mathfrak{a} \in Id_p(\mathfrak{O}_K)} \tilde{\chi}(\mathfrak{a})$, where, as in [8], $Id_n(\mathfrak{O}_K)$ denotes the set of integral ideals of $\mathfrak{O}_K$ of norm n. Now if $\psi_D(p) = -1$, then $Id_p(\mathfrak{O}_K) = \emptyset$, so $a_p(\chi) = 0$. On the other hand, if $\psi_D(p) = 1$, then $Id_p(\mathfrak{O}_K) = \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, where $p\mathfrak{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ and $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Since $p\mathfrak{O}_K \in P_{K,\mathbb{Z}}(f_D)$, we have $\tilde{\chi}(\overline{\mathfrak{p}}) = \tilde{\chi}(\mathfrak{p})^{-1} = \overline{\tilde{\chi}(\mathfrak{p})}$, and so $a_p(\chi) = \tilde{\chi}(\mathfrak{p}) + \tilde{\chi}(\overline{\mathfrak{p}}) = 2\mathrm{Re}(\tilde{\chi}(\mathfrak{p}))$, which proves (5).

The following result, which plays an important role in the proof of the main theorem (Theorem 1), is also an easy consequence of the results of [8]. For this, we recall from [8], §4, that a character $\chi \in \mathrm{Cl}(D)^*$ is called *primitive* if it is not a lift of any character on $\mathrm{Cl}(D/c^2)$ with $1 \neq c|f_D$ via the canonical surjection $\overline{\pi}_{D,D/c^2} : \mathrm{Cl}(D) \to \mathrm{Cl}(D/c^2)$.

Proposition 5 *Let $\chi \in \mathrm{Cl}(D)^*$ be a primitive character, and let $f \geq 1$. Then $V_t(\vartheta_\chi) \in \Theta(Df^2)$, for all $t|f^2$.*

Proof. We induct on f. The assertion is clear for $f = 1$ because (4) shows that $V_1(\vartheta_\chi) = \vartheta_\chi \in \Theta_D \subset \Theta(D)$.

Now assume that $f > 1$ and that the assertion is true for all $f_1|f$, $1 \leq f_1 < f$. Let $t|f^2$, and consider first the case that $t \neq f^2$. Then there is a prime $p|f$ such that $t|\frac{f^2}{p}$. If $p \nmid t$, then $t|\frac{f^2}{p^2}$, so the induction hypothesis gives $V_t(\vartheta_\chi) \in \Theta(Df^2/p^2) \subset \Theta(D)$, and hence the assertion follows in this case. On the other hand, if $p|t$, then $t = pt_1$ where $t_1|\frac{f^2}{p^2}$, and so by the induction hypothesis we have that $V_{t_1}(\vartheta_\chi) \in \Theta(Df^2/p^2)$, and hence $V_t(\vartheta_\chi) = V_p(V_{t_1}(\vartheta_\chi)) \in V_p(\Theta(Df^2/p^2)) \subset \Theta(D)$ by (3).

Finally, suppose that $t = f^2$, and put $D' = Df^2$ and $\chi' = \chi \circ \overline{\pi}_{D',D} \in \mathrm{Cl}(D')^*$. Then $\vartheta_{\chi'} \in \Theta_{D'} \subset \Theta(D')$. Since χ is primitive, we have that $D_{\chi'} = D$ and $(\chi')_{pr} = \chi$ in the notation of [8], Theorem 3, and hence by that theorem there exist $c_t = c_t(\chi') \in \mathbb{R}$ such that

$$\vartheta_{\chi'} \;=\; \sum_{t|f^2} c_t V_t(\vartheta_\chi) \;=\; fV_{f^2}(\vartheta_\chi) + \vartheta, \quad \text{where } \vartheta = \sum_{t|f^2, t\neq t^2} c_t V_t(\vartheta_\chi).$$

By what was proved above we know that $\vartheta \in \Theta(Df^2)$, and hence it follows that $V_{f^2}(\vartheta_\chi) = \frac{1}{f}(\vartheta_{\chi'} - \vartheta) \in \Theta(D)$, as desired.

3 Modular forms with complex multiplication

Before defining CM-forms, we recall some basic facts about extended newform/Atkin-Lehner theory. Let $M_k(N, \psi)$ be the space of modular forms of weight k, level N and Nebentypus ψ. As in [8], we say that $f \in M_k(N, \psi)$ is a *primitive form* of level $N_f|N$ if either $f \in S_k(N, \psi)$ is a normalized newform of level $N_f|N$ (so f is a primitive (cusp) form in the sense of [11], §4.6) or if $f = f_k(z; \psi_1, \psi_2)$ is one of the Eisenstein series defined on p. 178 of [11]; here $N_f = m_1 m_2$, where m_i is the conductor of ψ_i.

As usual, let $\mathbb{T}(N) = \mathbb{T}(N)_{k,\psi} \subset \mathrm{End}_{\mathbb{C}}(M_k(N, \psi))$ denote the $\mathbb{C}$-algebra generated by the Hecke operators T_n with $(n, N) = 1$. If $\lambda \in \mathbb{T}(N)^* := \mathrm{Hom}_{\mathrm{ring}}(\mathbb{T}(N), \mathbb{C})$ is a character of the algebra $\mathbb{T}(N)$, then there is a unique divisor $N_\lambda|N$ and primitive form

$f_\lambda \in M_k(N_\lambda, \psi)$ such that the λ-*eigenspace* $M_k(N,\psi)[\lambda] := \{f \in M_k(N,\psi) : f|_k T = \lambda(T)f, \forall T \in \mathbb{T}(N)\}$ is given by

$$M_k(N,\psi)[\lambda] \; = \bigoplus_{n|N/N_\lambda} \mathbb{C}V_n(f_\lambda); \tag{6}$$

cf. [11], Corollary 4.6.20 and Theorem 4.7.2. Moreover, it follows from [11], Lemma 4.6.9(3) and Theorem 4.7.2, that we have the (extended Atkin-Lehner) decomposition

$$M_k(N,\psi) \; = \bigoplus_{\lambda\in\mathbb{T}(N)^*} M_k(N,\psi)[\lambda]. \tag{7}$$

For any non-zero $\mathbb{T}(N)$-eigenfunction $f \in M_k(N,\chi)$, let $\lambda_f \in \mathbb{T}(N)^*$ denote the associated character which is defined by $f|_kT = \lambda_f(T)f$, for $T \in \mathbb{T}(N)$. It thus follows from the above results that the rule $f \mapsto \lambda_f$ induces a bijection between the set of primitive forms of $M_k(N,\chi)$ and the set $\mathbb{T}(N)^*$ of characters of $\mathbb{T}(N)$.

Definition. Let θ be a Dirichlet character with conductor m. We say that a character $\lambda \in \mathbb{T}(N)^*$ has *complex multiplication* (CM) *by* θ if

$$\lambda(T_p)\theta(p) \; = \lambda(T_p), \quad \text{for all primes } p \nmid Nm, \tag{8}$$

or, equivalently, if

$$\lambda(T_p) \; = \; 0, \quad \text{for all primes } p \nmid Nm \text{ with } \theta(p) \neq 1. \tag{9}$$

If this is the case, then we write (symbolically) $\lambda = \lambda\theta$. The sum

$$M_k^{CM}(N,\psi;\theta) := \bigoplus_{\lambda=\lambda\theta} M_k(N,\psi)[\lambda]$$

of the $\mathbb{T}(N)$-eigenspaces whose characters $\lambda \in \mathbb{T}(N)^*$ have CM by θ is called the *space of CM-forms by* θ, and its elements are called *CM-forms by* θ. If $\theta = \psi$ is the Nebentypus character, then we write $M_k^{CM}(N,\psi) := M_k^{CM}(N,\psi;\psi)$.

Remark 6 (a) The above definition of a CM-form is a slight extension of that of Ribet[10], who only considers normalized newforms. Note that in that case the above definition coincides with that of Ribet; cf. his Remark 1 on p. 34 of [10].

(b) If $M_k^{CM}(N,\psi;\theta) \neq \{0\}$, then as in Ribet[10], Remark 1 (p. 34), it follows that $\theta^2 = 1$, i.e., that θ is a quadratic character.

In his paper[10], Ribet gives a construction and classification of newforms with CM when the weight $k > 1$. In the next section we shall classify all the CM-forms of weight 1. Here we first note:

Proposition 7 *Let $D < 0$ be a discriminant. If $\chi \in \mathrm{Cl}(D)^*$, then ϑ_χ has complex multiplication by ψ_D. Thus $\Theta(D) \subset M_1^{CM}(|D|, \psi_D)$.*

Proof. By Proposition 4 we know that $\lambda_{\vartheta_\chi}(T_p) = a_p(\chi) = 0$ whenever $\psi_D(p) = -1$, so ϑ_χ has CM by ψ_D, i.e., $\vartheta_\chi \in M_1^{CM}(|D|, \psi_D)$.

Since $\{\vartheta_\chi : \chi \in \mathrm{Cl}(D)^*\}$ generates Θ_D ([8], Proposition 9), it follows that $\Theta_D \subset M_1^{CM}(|D|, \psi_D)$. Replacing D by $D' := \frac{D}{(c_1c)^2}$ with $c_1c|f_D$ yields that $V_{c_1}(\Theta_{D'}) \subset V_{c_1}(M_1^{CM}(|D'|, \psi_{D'})) \subset M_1^{CM}(|D|, \psi_D)$, and so (3) shows that $\Theta(D) \subset M_1^{CM}(|D|, \psi_D)$.

4 Galois representations with complex multiplication

As was mentioned above, Ribet[10] classified the newforms with CM of weight $k > 1$. Here we extend this study to all modular forms of weight 1.

Let us first consider the non-cuspidal case. Here we have:

Proposition 8 *If $f \in M_1(N, \psi)$ is a non-zero $\mathbb{T}(N)$-eigenfunction which is not a cusp form, then there exist two Dirichlet characters ψ_i with conductors m_i such that $m_1m_2|N$, $\psi_1\psi_2 = \psi$ and $\lambda_f(T_p) = \psi_1(p) + \psi_2(p)$, for all primes $p \nmid N$. Moreover, f has CM by a nontrivial quadratic Dirichlet character θ if and only if $\psi_2 = \psi_1\theta$. In particular, $f \in M_1^{CM}(N, \psi)$ if and only if $\psi_1^2 = \psi_2^2 = 1$.*

Proof. The hypothesis implies that f is an Eisenstein series; cf. [8], Lemma 15. Thus, by [11], Theorem 4.7.2, there exist two primitive Dirichlet characters ψ_i with conductors m_i such that $\psi_1\psi_2 = \psi$, $m_1m_2|N$, and $\lambda_f = \lambda_{f_1}$, where $f_1(z) = f_1(z; \psi_1, \psi_2)$ has L-function $L(s, \psi_1)L(s, \psi_2)$. Thus, $\lambda_f(T_p) = a_p(f_1) = \psi_1(p) + \psi_2(p)$, which proves the first assertion.

Now f (or λ_f) has CM by $\theta \Leftrightarrow (\psi_1(p) + \psi_2(p))\theta(p) = \psi_1(p) + \psi_2(p), \forall p \nmid Nm \Leftrightarrow \psi_1\theta + \psi_2\theta = \psi_1 + \psi_2$ (as functions on $(\mathbb{Z}/Nm\mathbb{Z})^\times$). If $\psi_2 = \psi_1\theta$, then $\psi_1 = \psi_2\theta^{-1} = \psi_1\theta$, and so clearly $\psi_1\theta + \psi_2\theta = \psi_1 + \psi_2$. Conversely, if $\psi_1\theta + \psi_2\theta = \psi_1 + \psi_2$, then $\psi_1 \neq \psi_2$, because θ is non-trivial. Thus, by the linear independence of characters we must have that $\psi_1\theta = \psi_1$ or $\psi_1\theta = \psi_2$ and hence that $\psi_1\theta = \psi_2$ because θ is non-trivial.

If $\psi_1^2 = \psi_2^2 = 1$, then $\psi = \psi_1\psi_2$ is also quadratic, and we have $\psi_2 = \psi\psi_1^{-1} = \psi\psi_1$, so f has CM by ψ by what was just shown. Conversely, if f has CM by ψ, then ψ is quadratic by Remark 6(b) and $\psi_2 = \psi_1\psi = \psi_1^2\psi_2$. Thus $\psi_1^2 = 1$ and hence $\psi_2 = \psi\psi_1^{-1}$ is also quadratic.

We now consider the case that $f \in S_1(N, \psi)$ is a cusp form. Here we shall use the classification of newforms in terms of the type of the associated Galois representation.

For this, recall first the following fundamental result of Deligne and Serre [4], Théorème 4.1. If $f \in M_1(N,\psi)$ is a $\mathbb{T}(N)$-eigenfunction with associated eigenvalue character $\lambda_f \in \mathbb{T}(N)^*$, then there is a unique Galois representation

$$\rho_f = \rho_{\lambda_f} : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \ \to \ \mathrm{GL}_2(\mathbb{C})$$

which is unramified outside N such that

$$\mathrm{Tr}(\rho_f(F_p)) \ = \ \lambda_f(T_p) \quad \text{and} \quad \det(\rho_f(F_p)) \ = \ \psi(p), \quad \forall p \nmid N, \tag{10}$$

where $F_p \in G_{\mathbb{Q}}/\mathrm{Ker}(\rho_f)$ is any Frobenius element at p. Furthermore, ρ_f is irreducible if and only if f is a cusp form.

Definition. Let $f \in S_1(N,\psi)$ be a $\mathbb{T}(N)$-eigenfunction and let ρ_f be its associated Galois representation. We say that f (or ρ_f) is *strongly dihedral* if the image of ρ_f is a dihedral group, i.e., if $\mathrm{Im}(\rho_f) \simeq D_n$, for some $n \geq 3$. Moreover, we say (cf. [5], p. 11) that f (or ρ_f) has *dihedral type* if the image of the associated *projective* representation

$$\tilde{\rho}_f : G_{\mathbb{Q}} \ \to \ \mathrm{PGL}_2(\mathbb{C})$$

is a dihedral group, i.e., if $\mathrm{Im}(\tilde{\rho}_f) \simeq D_n$, for some $n \geq 2$.

Note that in Serre[13], Galois representations of dihedral type are called "dihedral". To avoid confusion, the terminology of "strongly dihedral" is introduced here.

The cuspidal CM-forms can be classified as follows.

Theorem 9 *Let $f \in S_1(N,\psi)$ be a $\mathbb{T}(N)$-eigenfunction with associated projective Galois representation $\tilde{\rho}_f$.*

(a) *If θ is a non-trivial quadratic character, then f has CM by θ if and only if ρ_f is induced from G_K, where K is the quadratic field defined by θ.*

(b) *There exists a non-trivial Dirichlet character θ such that f has CM by θ if and only if f has dihedral type.*

(c) *If f has CM by $\theta \neq 1$, then θ is uniquely determined by ρ_f, except in the case that $\mathrm{Im}(\tilde{\rho}_f) \simeq D_2$. In the latter case there are precisely three distinct Dirichlet characters $\theta_1, \theta_2, \theta_3$ such that f has CM by θ_i.*

(d) *f has CM by ψ if and only if f is strongly dihedral.*

Remark 10 (a) If f is a newform of weight $k \geq 2$ with CM by θ, then it follows from Proposition 4.4 of Ribet[10] that θ is uniquely determined by f. Thus, there is no analogue of the phenomenon of Theorem 9(c) for higher weight.

(b) The form $f(z) = \eta(12z)^2 \in S_1(144, \psi_{-144})$ is an explicit example of a weight 1 form with CM by three distinct Dirichlet characters. Indeed, as is explained on p.

243 of Serre[13], we have that $\mathrm{Im}(\rho_f) \simeq D_4$ (and hence that $\mathrm{Im}(\tilde{\rho}_f) \simeq D_2$), and so this assertion follows from Theorem 9(c). More precisely, f has CM by ψ_{-4} ($= \psi_{-144}$), by ψ_{-3} and by $\psi_{12} = \psi_{-4}\psi_{-3}$. This can be seen either directly from the Fourier expansion given for f in [13] or by using Theorem 9(a) and the fact (mentioned in Serre[13]) that ρ_f is induced from G_{K_i}, where $K_1 = \mathbb{Q}(\sqrt{-1})$, $K_2 = \mathbb{Q}(\sqrt{-3})$, and $K_3 = \mathbb{Q}(\sqrt{3})$ are the three quadratic subfields of $\mathrm{Fix}(\mathrm{Ker}(\rho_f))$.

(c) Although we don't need this here, it may be useful to note that every newform f (of weight 1) with CM by the quadratic character θ is of the form $f = f(z;\xi)$, for a some Hecke character ξ on a ray class group of K, where K is the quadratic field corresponding to θ (cf. Theorem 9(a)) and $f(z;\xi)$ is as in [11], §4.8. Indeed, it is clear from the definition that $f(z;\xi)$ has CM by θ. Conversely, if f is a newform with CM by θ, then by Theorem 9(a) we have that $\rho_f \simeq \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi)$, for some linear character $\chi : G_K \to \mathbb{C}^\times$. Now by class field theory χ can be identified with a Hecke character ξ on a suitable ray class group of K, and Hecke's theory (cf. [11], §4.8) shows that we have an associated modular form $f_1 := f(z;\xi) \in S_1(N,\psi)$ (where N and ψ are given by the recipe in [11], §4.8). Since it is easy to check that $\rho_{f_1} \simeq \mathrm{Ind}(\chi) \simeq \rho_f$, it follows that $f = f_1 = f(z;\xi)$.

As we shall show below, the above theorem follows easily once we have verified the following group-theoretical fact.

Proposition 11 *Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a faithful irreducible representation of a finite group G.*

(a) *If $\theta : G \to \mathbb{C}^\times$ is a non-trivial quadratic character and $H = \mathrm{Ker}(\theta)$, then the following conditions are equivalent:*

(i) $\rho \otimes \theta \simeq \rho$; (ii) $\rho_{|H}$ *is reducible;* (iii) H *is abelian;* (iv) $\rho \simeq \mathrm{Ind}_H^G(\chi)$, $\chi \in H^*$.

Moreover, if these conditions hold, then H contains the centre $Z(G)$ of G.

(b) *There exists a non-trivial quadratic character θ satisfying the conditions of* (a) *if and only if $G/Z(G) \simeq D_n$, for some $n \geq 2$.*

(c) *In the situation of* (b), *the character θ is uniquely determined by ρ except when $n = 2$. Furthermore, if $n = 2$, then there are precisely three such characters.*

Proof. (a) (i) $\Rightarrow$ (ii): Although this follows immediately from Clifford's theory (cf. [7], V.17.12), it might be more illuminating to give a direct proof. Suppose that (ii) is false, i.e., that $\rho_{|H}$ is irreducible. Let $\varphi = (\chi_\rho)_{|H}$ denote the H-restriction of the character $\chi_\rho := \mathrm{Tr}(\rho)$ of ρ and let φ^G be the induced character on G. By Frobenius reciprocity we have that $(\chi_\rho, \varphi^G)_G = (\varphi,\varphi)_H = 1$, so $\varphi^G = \chi_\rho + \chi'$, where $\chi' \neq \chi_\rho$ and $\chi'_{|H} = \varphi$. Thus, there exists $g \in G \setminus H$ with $\chi'(g) \neq \chi_\rho(g)$. But since H is normal in G, we have that $\varphi^G(g) = 0$, so $\chi'(g) = -\chi_\rho(g)$, and hence $\chi_\rho(g) \neq 0$. Thus $\chi_{\rho\otimes\theta}(g) = \chi_\rho(g)\theta(g) = -\chi_\rho(g) \neq \chi_\rho(g)$, which contradicts (i).

(ii) $\Rightarrow$ (iii): By hypothesis, $\rho_{|H} \simeq \chi_1 \oplus \chi_2$, where the χ_i's are linear characters of H. Then the commutator subgroup H' of H is contained in $\mathrm{Ker}(\chi_1) \cap \mathrm{Ker}(\chi_2) = \mathrm{Ker}(\rho_{|H}) = \{1\}$, the latter since ρ is faithful. Thus $H' = \{1\}$, i.e., H is abelian.

(iii) $\Rightarrow$ (iv): Since H is abelian, $\rho_{|H} \simeq \chi_1 \oplus \chi_2$ with $\deg(\chi_i) = 1$. Thus, by Frobenius reciprocity $(\rho, \chi_i^G) = (\rho_{|H}, \chi_i) \geq 1$, and so ρ is a constituent of χ_i^G because ρ is irreducible. But since $\deg(\rho) = \deg(\chi_i^G) = 2$, we have that $\rho \simeq \chi_i^G = \mathrm{Ind}_H^G(\chi_i)$. Thus condition (iv) holds.

(iv) $\Rightarrow$ (i): Since H is normal in G and $\rho \simeq \mathrm{Ind}_H^G(\chi)$, it follows that $\chi_\rho(g) = 0$ for $g \notin H$, so clearly $\chi_\rho(g)\theta(g) = 0 = \chi_\rho(g)$ in this case. For $g \in H$ we have that $\theta(g) = 1$, so here $\chi_\rho(g)\theta(g) = \chi_\rho(g)$. Thus $\chi_\rho\theta = \chi_\rho$, and hence $\rho \otimes \theta \simeq \rho$.

This proves that conditions (i) – (iv) are equivalent. Moreover, we have that $Z := Z(G) \leq H := \mathrm{Ker}(\theta)$ because if $z \in Z$, then by Schur's Lemma $\rho(z) = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, so $\chi_\rho(z) = \mathrm{Tr}\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) = 2a \neq 0$, and hence $\theta(z) = 1$ by (i). Thus, $Z \leq H$, as claimed.

(b) Suppose first that $G/Z(G) \simeq D_n$. Then there is a subgroup $H \geq Z(G)$ of index 2 such that $H/Z(G)$ is cyclic, and hence H is abelian by [7], III.7.1. Since $H = \mathrm{Ker}(\theta)$ for a unique quadratic character $\theta : G \to \mathbb{C}^\times$, we see that condition (iii) of (a) holds.

Conversely, suppose that condition (iii) holds for some quadratic character θ. Since $H := \mathrm{Ker}(\theta) \geq Z := Z(G)$, we see that H/Z is an abelian subgroup of index 2 of G/Z. Since $G/Z \leq \mathrm{PGL}_2(\mathbb{C})$, we have that $G/Z \simeq D_n, A_4, S_4$ or A_5 (cf. [13], p. 208). However, none of A_4, S_4 and A_5 has an abelian subgroup of index 2, and hence we must have that $G/Z \simeq D_n$.

(c) If $\{\theta_1, \ldots \theta_k\}$ are distinct quadratic characters satisfying (i)–(iv), then $H_1 := \mathrm{Ker}(\theta_1), \ldots, H_k := \mathrm{Ker}(\theta_k)$ are distinct abelian subgroups of G of index 2. Moreover, since $H_i \geq Z(G)$ by part (a), we have by Lemma 12 below that $k = 1$ when $n \geq 3$.

On the other hand, if $G/Z(G) \simeq D_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then G has precisely three subgroups $H_i \leq G$ of index 2 with $H_i \geq Z(G)$. Since $H_i/Z(G)$ is cyclic (of order 2), we see that each H_i is abelian (by [7], III.7.1 again). Thus, there are three quadratic characters θ_i with $\mathrm{Ker}(\theta_i) = H_i$ which therefore satisfy the conditions of (a).

In the above proof we had used the following simple fact.

Lemma 12 *If $G/Z(G) \simeq D_n$, where $n \geq 3$, then G has a unique abelian subgroup $H \leq G$ of index 2 with $H \geq Z(G)$.*

Proof. Since $G/Z(G)$ is a dihedral group, there is a subgroup $H \leq G$ of index 2 with $H \geq Z(G)$ such that $H/Z(G)$ is cyclic of order n. Thus H is abelian by [7], III.7.1.

Now suppose that H_1 is an abelian subgroup of G of index 2 with $Z(G) \leq H_1$. Thus $H_1 \geq Z(G)G'$. Now if n is odd, then $[D_n : D_n'] = 2$, so $H_1 = Z(G)G' = H$, and hence the assertion is clear. Assume now that n is even, so $[G : Z(G)G'] = [D_n : D_n'] = 4$. If $H_1 \neq H$, then it follows that $H_1 \cap H = Z(G)G'$. Since $[G : Z(G)] = 2n > 4 = [G : H_1 \cap H]$, there exists $h \in H_1 \cap H \setminus Z(G)$. Moreover, since $H_1 \neq H$, there is a $\sigma \in H_1 \setminus H$. Since H_1 is abelian, we have that $\sigma \in C_G(h)$. On the other hand, since

H is abelian, we see that $H \leq C_G(h)$, and so $G = \langle \sigma, H \rangle \leq C_G(h)$, i.e., $h \in Z(G)$, contradiction. Thus $H_1 = H$, and hence H is unique.

We shall also use the following application of Proposition 11. Here we say that a linear character θ on a group G is *odd* if there is an element $\sigma \in G$ with $\sigma^2 = 1$ such that $\theta(\sigma) = -1$. Note that if $G = G_{\mathbb{Q}}$, then this terminology agrees with that of Deligne/Serre[4] because every element $g \in G_{\mathbb{Q}}$ of order 2 is conjugate to the complex conjugation automorphism by a theorem of Artin[1].

Corollary 13 *Let G be a finite group. Then $G \simeq D_n$, for some $n \geq 3$, if and only if G has a faithful irreducible representation ρ of degree 2 such that $\theta := \det(\rho)$ is an* odd *quadratic character whose kernel $\mathrm{Ker}(\theta)$ is abelian. If this is the case, then θ is independent of the choice of ρ and so $\rho \otimes \theta \simeq \rho$ for every faithful irreducible representation ρ of degree 2 of G.*

Proof. Suppose first that $G \simeq D_n$, where $n \geq 3$. Then by the discussion (and notation) of [12], §5.3, we know that there exists a faithful irreducible representation ρ of degree 2; in fact, the representations $\rho = \rho^h$, where $(h, n) = 1$ and $0 < h < \frac{n}{2}$ are precisely all the faithful irreducible representations of G of degree 2. Moreover, we see that $\det(\rho(r^k)) = w^{hk}w^{-hk} = 1$ and $\det(\rho(sr^k)) = -w^{hk}w^{-hk} = -1$, so $\theta = \det(\rho^h)$ is an odd quadratic character which does not depend on the choice of h, and which has cyclic kernel $\mathrm{Ker}(\theta) = \langle r \rangle$. Thus, condition (iii) of Proposition 11(a) holds for $\rho = \rho^h$ and θ, and so it follows from the equivalent condition (i) that $\rho^h \otimes \theta \simeq \rho^h$.

Conversely, suppose that we have a faithful irreducible representation of degree 2 such that $\theta := \det(\rho)$ is an odd quadratic character and such that $H = \mathrm{Ker}(\theta)$ is abelian. Thus, condition (iii) of Proposition 11(a) holds and so it follows from the equivalent condition (iv) that $\rho \simeq \mathrm{Ind}_H^G(\chi)$, for some $\chi \in H^*$, where $H = \mathrm{Ker}(\theta)$. By Mackey we then have that $\rho_{|H} \simeq \chi \oplus \chi^\sigma$, where σ is any element in $G \setminus H$; cf. [12], Proposition 22 (§7.3). Since θ is odd by hypothesis, we can assume that $\sigma^2 = 1$. Moreover, since $1 = \det(\rho)_{|H} = \det(\rho_{|H}) = \chi\chi^\sigma$, we see that $\chi^\sigma = \chi^{-1}$. Thus, $\mathrm{Ker}(\chi^\sigma) = \mathrm{Ker}(\chi)$, and so $\mathrm{Ker}(\chi) = \mathrm{Ker}(\chi^\sigma) \cap \mathrm{Ker}(\chi) = \mathrm{Ker}(\rho_{|H}) = \{1\}$ (because ρ is faithful), so χ is faithful and hence $H \simeq \mathrm{Im}(\chi)$ is cyclic of order $n := |H|$.

Write $H = \langle r \rangle$. Since $\chi^\sigma = \chi^{-1}$, we have that $\chi(\sigma^{-1}r\sigma) = \chi^\sigma(r) = \chi(r^{-1})$, and so $\sigma^{-1}r\sigma = r^{-1}$. Thus $G = \langle r, \sigma : r^n = \sigma^2 = 1, \sigma^{-1}r\sigma = r^{-1} \rangle \simeq D_n$ is a dihedral group of order $2n$. Note that we must have $n \geq 3$, for otherwise D_n is abelian and hence does not have any irreducible representations of degree 2.

Proof of Theorem 9. Let $\theta : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Via the canonical identification $\sigma_m : (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, we can identify θ with (the lift of) the character $\theta^* = \theta \circ \sigma_m^{-1}$ on $G_{\mathbb{Q}}$, and we have that $\theta^*(F_p) = \theta(p)$, for $p \nmid m$. Thus, if $\rho_f \otimes \theta^*$ is the 2-dimensional Galois representation defined by $(\rho_f \otimes \theta^*)(g) = \rho_f(g)\theta^*(g)$ for $g \in G_{\mathbb{Q}}$, then

$$f \text{ has CM by } \theta \quad \Leftrightarrow \quad \rho_f \otimes \theta^* \simeq \rho_f. \tag{11}$$

Indeed, by basic representation theory we have that $\rho_f \otimes \theta^* \simeq \rho_f \Leftrightarrow \mathrm{Tr}(\rho_f \otimes \theta^*(g)) = \mathrm{Tr}(\rho_f(g)), \forall g \in G_\mathbb{Q} \Leftrightarrow \mathrm{Tr}(\rho_f(g))\theta^*(g) = \mathrm{Tr}(\rho_f(g)), \forall g \in G_\mathbb{Q} \Leftrightarrow \mathrm{Tr}(\rho_f(F_p))\theta(p) = \mathrm{Tr}(\rho_f(F_p))$, for all $p \nmid Nm$, where the last equivalence follows from the Chebotarev density theorem. From this and (10) the assertion (11) clearly follows.

(a) Since f is a cusp form, ρ_f is irreducible by Deligne/Serre[4]. Clearly, ρ_f is the lift of a (unique) representation ρ of the finite group $G = G_\mathbb{Q}/\mathrm{Ker}(\rho_f)$. Note that ρ is faithful and irreducible. Furthermore, we may assume that θ^* is the lift of a character θ^*_G of G, for if $\rho_f \otimes \theta^* \simeq \rho_f$, then $\mathrm{Ker}(\rho_f) \leq \mathrm{Ker}(\theta^*)$ (because $g \in \mathrm{Ker}(\rho_f) \Rightarrow \mathrm{Tr}(\rho_f(g)) = 2$, so $\mathrm{Tr}(\rho_f(g))\theta^*(g) = \mathrm{Tr}(\rho_f(g)) \Rightarrow \theta^*(g) = 1$).

Thus, by Proposition 11(a) and (11) we see that f has CM by $\theta \Leftrightarrow \rho \simeq \mathrm{Ind}_H^G(\chi)$, for some linear character χ on $H = \mathrm{Ker}(\theta^*)/\mathrm{Ker}(\rho_f)$. By lifting χ to a character on $G_K = \mathrm{Ker}(\theta^*)$, the latter condition is equivalent to $\rho_f \simeq \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\chi)$, and so (a) follows.

(b) By the proof of part (a), we see that ρ_f has CM by θ if and only if ρ and θ^*_G satisfy the conditions of Proposition 11(a). It is thus clear that the assertion follows from Proposition 11(b).

(c) By the same argument as in part (b), it is clear that the assertion follows from Proposition 11(c).

(d) As in part (a), ρ_f gives rise to a faithful, irreducible representation ρ on $G = G_\mathbb{Q}/\mathrm{Ker}(\rho_f)$ with $\deg(\rho) = 2$. Moreover, by (10) we know that $\det(\rho) = \psi^*_G$, where ψ^*_G is the character on G whose lift to $G_\mathbb{Q}$ is ψ^*. Note that we have that $\psi(-1) = -1$ (because $S_1(N, \psi) \neq \{0\}$), and so it follows that $\psi^*(c) = -1$, where $c \in G_\mathbb{Q}$ denotes complex conjugation, and hence ψ^* and ψ^*_G are odd characters in the sense of Corollary 13. Thus, if f has CM by ψ, then by (11) and Proposition 11(a) we know that $\mathrm{Ker}(\psi^*_G)$ is abelian, and hence it follows from Corollary 13 that $G \simeq D_n$, for some $n \geq 3$, i.e., that f is strongly dihedral.

Conversely, if f is strongly dihedral, then by the last assertion of Corollary 13 we know that $\det(\rho) = \psi^*_G$ and that $\rho \otimes \psi^*_G \simeq \rho$. It thus follows from (11) that f has CM by ψ.

5 The structure of $M_1^{CM}(N, \psi)$

We now apply the results of the previous sections to determine the structure of the space $M_1^{CM}(N, \psi)$. The key step for this is the following result which classifies the primitive forms (cf. §3) which have CM by their Nebentypus character.

Theorem 14 *Let $f \in M_1(N, \psi)$ be a primitive Hecke eigenfunction of level N. Then f has CM by ψ if and only if $-N \equiv 0, 1 \pmod 4$ and there exists a primitive character $\chi \in \mathrm{Cl}(-N)^*$ such that $f = \vartheta_\chi$. If this is the case, then $\psi = \psi_{-N}$.*

One direction of this follows easily from Proposition 7. The converse will be deduced from the following result which is also of independent interest.

Theorem 15 *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ be a two-dimensional Galois representation with odd determinant and with dihedral image, i.e.,* $\mathrm{Im}(\rho) \simeq D_n$*, for some $n \geq 2$. Then:*

(a) *The fixed field of* $\det(\rho)$ *is an imaginary quadratic field K, and the conductor of ρ has the form* $\mathrm{cond}(\rho) = |d_K| f^2$*, for some integer f.*

(b) *There exists a primitive character χ of order $n := \frac{1}{2}|\mathrm{Im}(\rho)|$ on the class group* $\mathrm{Cl}(D)$*, where $D := d_K f^2$, such that the Artin L-function of ρ is the Hecke L-function attached to χ, i.e., $L(s, \rho) = L(s, \tilde{\chi})$, where $\tilde{\chi}$ is the Hecke character associated to χ.*

(c) *There is a unique modular form $\vartheta_\chi \in M_1(|D|, \psi_D)$ such that its L-function is the above Artin L-function, i.e.,*

$$L(s, \rho) \ = \ L(s, \tilde{\chi}) \ = \ L(s, \vartheta_\chi),$$

and ϑ_χ can be written explicitly as a linear combination of theta series ϑ_f with $f \in$ $\mathrm{Cl}(D)$*; in particular, $\vartheta_\chi \in \Theta_D \subset M_1^{CM}(|S|, \psi_D)$. Moreover, ϑ_χ is a cusp form if and only if $n \geq 3$. If this is the case, then ϑ_χ is a newform of level $|D| = \mathrm{cond}(\rho)$.*

Proof. (a) First note that $\psi := \det(\rho)$ is a quadratic character. Indeed, if $n \geq 3$, then this follows from Corollary 13, and if $n = 2$, then $\mathrm{Im}(\rho) \simeq C_2 \times C_2$, so $\rho \simeq \chi_1 \oplus \chi_2$, with χ_i quadratic, and hence $\det(\rho) = \chi_1\chi_2$ is also quadratic.

By hypothesis, $\psi(c) = -1$, where c is (as before) complex conjugation, so $K = \mathrm{Fix}(\psi)$ is an imaginary quadratic field. (Thus $\psi = \psi^*_{d_K}$, in the notation of the proof of Theorem 9.) This proves the first assertion of (a). The second will be established after the proof of part (b).

(b) Put $G := \mathrm{Im}(\rho) \simeq D_n$, and let ρ_G and ψ_G be the representations on G whose lifts to $G_{\mathbb{Q}}$ are ρ and ψ, respectively. Then by the proof of Corollary 13 (and its extension to the case $n = 2$), we see that $H := \mathrm{Ker}(\psi_G)$ is cyclic and that $\rho = \mathrm{Ind}_H^G(\chi_G)$, for some (faithful) character χ_G on H of order n.

Let $L = \mathrm{Fix}(\mathrm{Ker}(\chi_G)) = \mathrm{Fix}(\mathrm{Ker}(\rho))$, so $\mathrm{Gal}(L/\mathbb{Q}) \simeq G \simeq D_n$. Thus, $L/\mathbb{Q}$ is a dihedral extension containing $K = \mathbb{Q}(\sqrt{d_K})$, and hence, by a result of Bruckner[2] (cf. [3], p. 191), L is contained in some ring class field $F = F_f$; i.e., F/K is unramified outside f and $\mathrm{Gal}(F/K) \simeq \mathrm{Cl}(d_K f^2)$ (via the Artin map). Thus, there is a character χ on $\mathrm{Cl}(d_K f^2)$ such that

$$\chi_G(F_{\mathfrak{p}}) \ = \ \tilde{\chi}(\mathfrak{p}), \quad \text{for all primes } \mathfrak{p} \nmid f, \tag{12}$$

where $F_{\mathfrak{p}} \in H$ denotes the Frobenius element at a prime $\mathfrak{p}$ of K and $\tilde{\chi}$ is the Hecke character associated to χ.

If we choose f to be minimal with the property that $L \subset F_f$, then it follows that χ cannot be lifted from $\mathrm{Cl}(d_K f_1^2)$ for any $f_1 | f$ with $f_1 \neq f$. Thus, χ is primitive and so by [8], Theorem 23, we have that the conductor of $\tilde{\chi}$ is $\mathrm{cond}(\tilde{\chi}) = f\mathfrak{O}_K$.

By Artin, we know that the Artin and Hecke L-functions coincide, i.e., that $L(s, \chi_G) = L(s, \tilde{\chi})$, and that $\mathrm{cond}(\chi_G) = \mathrm{cond}(\tilde{\chi}) = f\mathfrak{O}_K$. Moreover, since $\rho_G =$

$\mathrm{Ind}_H^G(\chi_G)$, we also have that $L(s,\rho) = L(s,\rho_G) = L(s,\chi_G)$, and so (b) follows. In addition, we obtain from the conductor formula (cf. Serre[13], (7.3.1)) that $\mathrm{cond}(\rho) = \mathrm{cond}(\rho_G) = |d_K|N_{K/\mathbb{Q}}(\mathrm{cond}(\chi_G)) = |d_K|N_{K/\mathbb{Q}}(f\mathfrak{O}_K) = |d_K|f^2$, which proves the second assertion of part (a).

(c) Let $\vartheta_\chi = \frac{1}{w_D}\sum_{f\in Cl(D)}\chi(f)\vartheta_f \in \Theta_D$ be the theta series attached to $\chi \in \mathrm{Cl}(D)^*$; cf. (4). Since $\chi = \chi_{pr}$ is primitive, we have by Theorem 3(b) of [8] that $L(s,\vartheta_\chi) = L(s,\tilde{\chi})$. This proves the first assertion of (c). Moreover, by Theorem 1 of [8] we know that ϑ_χ is a cusp form $\Leftrightarrow \chi^2 \neq 1 \Leftrightarrow n = \mathrm{ord}(\chi) \geq 3$. The last assertion is a special case of [8], Theorem 3(b).

Proof of Theorem 14. If $D = -N$ is a negative discriminant and if $f = \vartheta_\chi$, where $\chi \in \mathrm{Cl}(D)^*$, then we know by Proposition 7 that $f \in M_1(N,\psi_D)$ has CM by $\psi = \psi_D$.

To prove the converse, assume first that f is not a cusp form. Since f is primitive, we have by definition that $f = f_1(z;\psi_1,\psi_2)$ for some Dirichlet characters ψ_i such that $\psi_1\psi_2 = \psi$ and $d_1d_2 = N$, where d_i is the conductor of ψ_i. Since f has CM by ψ, we have by Proposition 8 that $\psi_1^2 = \psi_2^2 = \psi^2 = 1$. Thus, since $\psi(-1) = -1$ (because $M_1(N,\psi) \neq \{0\}$), it follows that $\psi = \psi_{d_K}$, for some imaginary quadratic field K.

If $\psi_1 = 1$ is the trivial character, then $\psi = \psi_2$ has conductor d_2, and hence $N = d_2 = |d_K|$ is (up to sign) a fundamental discriminant. Thus $f = \vartheta_{\chi_{d_K,0}}$ is the theta series associated to the trivial character $\chi_{d_K,0}$ on $\mathrm{Cl}(d_K)$; cf. [8], Example 35(a).

Assume now that $\psi_i \neq 1$, for $i = 1,2$. Let ψ_i^* denote (as above) the 1-dimensional Galois representation associated to ψ_i, and put $\rho = \rho_f = \psi_1^* \oplus \psi_2^*$. Thus, ρ is a 2-dimensional Galois representation with $\det(\rho) = \psi_1^*\psi_2^* = \psi^*$. Note that $\psi^*(c) = \psi(-1) = -1$, so $\det(\rho)$ is odd and $\psi_1^* \neq \psi_2^*$.

Since $\mathrm{Ker}(\rho) = \mathrm{Ker}(\psi_1^*)\cap\mathrm{Ker}(\psi_2^*)$, we see immediately that $\mathrm{Im}(\rho) \simeq C_2\times C_2 = D_2$ is dihedral. Thus, by Theorem 15 there is a primitive quadratic character $\chi \in \mathrm{Cl}(d_Kt^2)$ (for some $t \geq 1$) such that $L(s,\rho) = L(s,\vartheta_\chi)$. But since $\rho = \psi_1^*\oplus\psi_2^*$, we have that $L(s,\rho) = L(s,\psi_1^*)L(s,\psi_2^*) = L(s,\psi_1)L(s,\psi_2) = L(s,f_1(\cdot;\psi_1,\psi_2)) = L(s,f)$. Thus, ϑ_χ and f are two modular forms of weight 1 which have the same L-function, and so $\vartheta_\chi = f$.

Now assume that f is a cusp form, so f is a newform of level N. If $\rho = \rho_f$ is its associated Galois representation, then $\det(\rho) = \psi^*$ is odd and so by Theorem 9(d) we see that ρ satisfies the hypotheses of Theorem 15. Thus, there is a discriminant $D < 0$ and a primitive character $\chi \in \mathrm{Cl}(D)$ such that $L(s,\rho) = L(s,\vartheta_\chi)$. Since $N = \mathrm{cond}(\rho)$ and $L(s,\rho) = L(s,f)$ by [4], Théorème 4.6, it thus follows from Theorem 15 that $N = |D|$ and that $f = \vartheta_\chi$.

From Theorem 14 we can deduce the following structure theorem for the space $M_1^{CM}(|D|,\psi_D)$. To state this result, we first recall some notation and terminology from [8]. If $\chi \in \mathrm{Cl}(D)^*$, then $f_\chi|f_D$ denotes conductor of χ and $\chi_{pr} \in \mathrm{Cl}(D/\overline{f}_\chi^2)$ denotes the associated primitive character, where $\overline{f}_\chi = f_D/f_\chi$. Furthermore, $\overline{\mathrm{Cl}(D)}^* =$

$\mathrm{Cl}(D)^*/(\chi \mapsto \overline{\chi})$ denotes the set of equivalence classes of characters under the action $\chi \mapsto \overline{\chi} = \chi^{-1}$ on $\mathrm{Cl}(D)^*$.

Theorem 16 *If D is a negative discriminant, then the map $\chi \mapsto \lambda_{\vartheta_\chi}$ induces a bijection between the set $\overline{\mathrm{Cl}(D)^*}$ and the set of characters of the Hecke algebra $\mathbb{T}(|D|)_{1,\psi_D}$ which have CM by ψ_D. Thus*

$$(13) \qquad M_1^{CM}(|D|,\psi_D) \;=\; \bigoplus_{\chi\in\overline{Cl(D)}^*} M_1(|D|,\psi_D)[\lambda_{\vartheta_\chi}] \;=\; \bigoplus_{\chi\in\overline{Cl(D)}^*}\bigoplus_{t|\overline{f}_\chi^2} \mathbb{C}V_t(\vartheta_{\chi_{pr}}).$$

Proof. By Propositions 4 and 7 we know that $\vartheta_\chi = \vartheta_{\overline{\chi}} \in M_1(|D|,\psi_D)$ is a $\mathbb{T}(|D|)$-eigenfunction with CM by ψ_D, so λ_{ϑ_χ} is a character on $\mathbb{T}(|D|)$ with CM by ψ_D.

Since $\Theta_D \subset M_1^{CM}(|D|,\psi_D)$ has multiplicity one by [8], Theorem 1, we see that the map $\chi \mapsto \lambda_{\vartheta_\chi}$ is injective. Moreover, Theorem 14 shows that the map is surjective. Indeed, if $\lambda \in \mathbb{T}(|D|)^*$ is a character which has CM by ψ_D, then by extended Atkin-Lehner theory (cf. §3) there exists a divisor $N_\lambda|N$ and a primitive form $f_\lambda \in M_1(N_\lambda,\psi_\lambda) \subset M_1(|D|,\psi_D)$ such that $\lambda = \lambda_{f_\lambda}$. Since f_λ has CM by ψ_D (and hence by ψ_λ), it follows from Theorem 14 that $D_\lambda := -N_\lambda$ is a negative discriminant, that $\psi_\lambda = \psi_{D_\lambda}$, and that there is a primitive character $\chi_{pr} \in \mathrm{Cl}(D_\lambda)^*$ such that $f_\lambda = \vartheta_{\chi_{pr}}$. Now since ψ_D is a lift of ψ_{D_λ}, it follows that $D = D_\lambda f^2$, for some $f \geq 1$, and so $\chi = \chi_{pr} \circ \overline{\pi}_{D,D_\lambda} \in \mathrm{Cl}(D)^*$. Thus $\lambda = \lambda_{f_\lambda} = \lambda_{\vartheta_{\chi_{pr}}} = \lambda_{\vartheta_\chi}$ (the latter by [8], Theorem 3(b)), and so the map is surjective and hence bijective.

The first equality of (13) follows directly from the first assertion and the definition of $M_1^{CM}(|D|,\psi_D)$, and the second from (6) and the fact that $\vartheta_{\chi_{pr}}$ is a primitive form of level $|D_\chi| = |D|/\overline{f}_\chi^2$; cf. [8], Corollary 25.

From the above results the main result of this paper follows readily, as we shall now see.

Proof of Theorem 1. The inclusion $\Theta(D) \subset M_1^{CM}(|D|,\psi_D)$ was established in Proposition 7, and the opposite inclusion follows from (13) and Proposition 5.

Remark 17 (a) It follows immediately from (1) and (13) that

$$(14) \qquad \dim\Theta(D) \;=\; \dim M_1^{CM}(|D|,\psi_D) \;=\; \sum_{\chi\in\overline{Cl(D)}^*} d(\overline{f}_\chi^2) \;=\; \sum_{f|f_D} d(f^2)\overline{h}_{D/f^2}^{pr},$$

where $d(f)$ denotes the number of divisors of f and $\overline{h}_D^{pr}$ denotes the number of equivalence classes of primitive characters in $\overline{\mathrm{Cl}(D)}^*$. (Note that $\overline{h}_D^{pr}$ equals the number of primitive forms of level $|D|$ in $\Theta(D)$; cf. [8], Theorem 3(b).) Since

$$\overline{h}_D \;:=\; |\overline{\mathrm{Cl}(D)}^*| \;=\; \sum_{f|f_D}\overline{h}_{D/f^2}^{pr}, \quad \text{and} \quad \overline{h}_D^{pr} \;=\; \sum_{f|f_D}\mu(f)\overline{h}_{D/f^2}$$

(the latter by Moebius inversion), it follows from (14) together with the fact that

$$\sum_{f|n} \mu(n/f)d(f^2) = 2^{\omega(n)},$$

where $\omega(n)$ denotes the number of distinct prime divisors of n, that we also have

$$\dim \Theta(D) \;=\; \dim M_1^{CM}(|D|, \psi_D) \;=\; \sum_{f|f_D} 2^{\omega(f)} \overline{h}_{D/f^2}. \tag{15}$$

Thus, $\dim \Theta(D) \leq |Q(D)/\mathrm{GL}_2(\mathbb{Z})| = \sum_{f|f_D} d(f)\overline{h}_{D/f^2}$, and equality holds if and only if f_D is squarefree. Since the space $\Theta(D)$ is generated by the set $\{\vartheta_f : f \in Q(D)/\mathrm{GL}_2(\mathbb{Z})\}$, we thus see that

$$\{\vartheta_f : f \in Q(D)/\mathrm{GL}_2(\mathbb{Z})\} \text{ is a basis of } \Theta(D) \quad \Leftrightarrow \quad f_D \text{ is squarefree.} \tag{16}$$

(b) Since we know by Theorem 1 of [8] that ϑ_χ is a cusp form if and only if $\chi^2 \neq 1$, it follows from Theorem 16 that the Eisenstein part and the cusp space part of $M_1^{CM}(|D|, \psi_D)$ are given by

$$E_1^{CM}(|D|, \psi_D) = \bigoplus_{\chi^2=1} M_1(|D|, \psi_D)[\lambda_{\vartheta_\chi}] \text{ and } S_1^{CM}(|D|, \psi_D) = \bigoplus_{\chi^2\neq1} M_1(|D|, \psi_D)[\lambda_{\vartheta_\chi}],$$

respectively. Thus, since $g_D := [\mathrm{Cl}(D) : \mathrm{Cl}(D)^2]$ is the number of quadratic characters in $\mathrm{Cl}(D)^*$ (and in $\overline{\mathrm{Cl}(D)}^*$), and since $\overline{h}_D = \frac{1}{2}(h_D + g_D)$, where $h_D = |\mathrm{Cl}(D)|$ is the class number of D (cf. [8], (5)), we obtain that

$$\dim E_1^{CM}(|D|, \psi_D) \;=\; \sum_{\chi^2=1} d(\overline{f}_\chi^2) \;=\; \sum_{f|f_D} 2^{\omega(f)} g_{D/f^2}, \tag{17}$$

$$\dim S_1^{CM}(|D|, \psi_D) \;=\; \sum_{\chi^2\neq1} d(\overline{f}_\chi^2) \;=\; \tfrac{1}{2}\sum_{f|f_D} 2^{\omega(f)} (h_{D/f^2} - g_{D/f^2}). \tag{18}$$

(c) It follows from Theorem 15 that number r_N of odd, two-dimensional Galois representations of conductor $N \geq 1$ with dihedral image is

$$r_N \;=\; \begin{cases} \overline{h}_{-N}^{pr} & \text{if } N \equiv 0, 3 \pmod 4, \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

where $\overline{h}_{-N}^{pr}$ is as in part (a). (In this count, we view $1^* \oplus \psi_{d_K}^*$ as a (degenerate) dihedral Galois representation of conductor $|d_K|$ because its image is $C_2 = D_1$.) Thus

$$r_{|D|}^{ab} \;=\; \sum_{f|f_D} \mu(f) g_{D/f^2} \quad \text{and} \quad r_{|D|}^{n-ab} \;=\; \tfrac{1}{2}\sum_{f|f_D} \mu(f)(h_{D/f^2} - g_{D/f^2}). \tag{20}$$

is the number of such representations with abelian and non-abelian dihedral image, respectively.

(d) If $D = d_K$ is a fundamental discriminant, then $\Theta(D) = \Theta_D$, and so in this case $M_1^{CM}(|D|, \psi_D) = \Theta_D$ has multiplicity one by [8], Theorem 1. Moreover, the above formulae (17), (18) and (20) reduce to

$$\dim E_1^{CM}(|D|, \psi_D) = r_{|D|}^{ab} = g_D \quad \text{and} \quad \dim S_1^{CM}(|D|, \psi_D) = r_{|D|}^{n-ab} = \frac{1}{2}(h_D - g_D).$$

In particular, if $D = -p$ is a prime discriminant (i.e., $p \equiv 3 \pmod 4$ is a prime), then $\dim S_1^{CM}(p, \psi_{-p}) = \frac{1}{2}(h_{-p} - 1)$, and hence there are $\frac{1}{2}(h_{-p} - 1)$ non-isomorphic, (strongly) dihedral representations with conductor p; this agrees with the formula of Serre[13], p. 245.

Actually, Serre's formula is slightly stronger because he counts all representations of dihedral type; here he uses the observation that every representation of dihedral type with prime conductor is strongly dihedral; cf. [13], p. 241, 244.

(e) In connection with the last observation of Serre, it should be mentioned that its analogue does not hold for non-prime (fundamental) discriminants, i.e., the space $S_1(|D|, \psi_D)$ may have newforms of dihedral type which are not strongly dihedral or, equivalently, $S_1(|D|, \psi_D)$ may have newforms which have CM but do not have CM by ψ_D.

For example, if $D = d_K = -2212 = -4 \cdot 7 \cdot 79$, then such a newform exists. To see this, consider $K_0 = \mathbb{Q}(\sqrt{79})$. Since $7\mathfrak{O}_{K_0} = \mathfrak{p}\overline{\mathfrak{p}}$, and since the image of the fundamental unit $\varepsilon = 80 + 9\sqrt{79}$ has order 3 in $(\mathfrak{O}_{K_0}/\mathfrak{p})^\times$, one can show that there is a Hecke character ξ on K_0 with signature $+, -$ at infinity which has order $6 = 2h_{K_0}$ and conductor $\mathfrak{p}$. We then have by [11], Theorem 4.8.3, that $f := f(\cdot; \xi) \in S_1(|D|, \psi_D)$ is a newform, and by [13], p. 238, we know that $\rho_f = \mathrm{Ind}_{G_{K_0}}^{G_\mathbb{Q}}(\xi)$ is of dihedral type. Thus, by Theorem 9(a) we see that f has CM by $\theta = \psi_{d_{K_0}} = \psi_{316}$. Now if ρ_f were strongly dihedral, then by Theorem 9(d) it would also have CM by $\psi_D \neq \theta$. Thus $\mathrm{Im}(\tilde{\rho}_f) \simeq D_2$ by Theorem 9(c), and so $\mathrm{Im}(\rho_f) \simeq D_4$. But this is impossible because $\mathrm{ord}(\xi) = 6$, and so f cannot be strongly dihedral.

We remark without proof that the above example can be generalized to show that there are infinitely many fundamental discriminants D_i (which do not have a common odd prime factor) such that $S_1(|D_i|, \psi_{D_i})$ contains a newform f_i of dihedral type which is not strongly dihedral.

We conclude this paper with the following example which illustrates the structure theorem (Theorem 16).

Example 18 $D = -144 = -4 \cdot 6^2$.

We have $\mathrm{Cl}(D) = \{[1, 0, 36], [4, 0, 9], [5, 4, 8], [5, -4, 8]\} = \langle [5, 4, 8] \rangle \simeq \mathbb{Z}/4\mathbb{Z}$, where $[a, b, c]$ denotes the equivalence class of the form $ax^2 + bxy + cy^2$. Let χ be a generator of $\mathrm{Cl}(D)^* \simeq \mathrm{Cl}(D)$. Since $h_{D/2^2} = h_{-36} = 2$ and $h_{D/3^2} = h_{-16} = 1$, we see that χ is primitive (so $f_\chi = 6$) and that χ^2 has conductor $f_{\chi^2} = 3$. Thus, by [8], Theorems 1 and 3, ϑ_χ is a newform of level 144 and hence by Remark 17(b) and (6) we obtain

$$S_1^{CM}(144, \psi_{-144}) \;=\; \mathbb{C}\vartheta_\chi.$$

Note that it follows from the discussion of Serre[13], p. 243, that $\vartheta_\chi(z) = \eta(12z)^2$, where $\eta(z)$ is the Dedekind eta-function.

The primitive forms associated to $\chi_{D,0} = \chi^4$ and χ^2 are $\vartheta_K := f_1(\,\cdot\,; 1, \psi_{-4}) \in E_1(4, \psi_{-4})$ and $\vartheta_{36} := f_1(\,\cdot\,; \psi_{-3}, \psi_{12}) \in E_1(36, \psi_{-36})$, respectively. Thus

$$M_1(144, \psi_{-144})[\lambda_{\vartheta_{\chi^4}}] = \langle V_t\vartheta_K : t|36\rangle \quad \text{and} \quad M_1(144, \psi_{-144})[\lambda_{\vartheta_{\chi^2}}] = \langle \vartheta_{36}, V_2\vartheta_{36}, V_4\vartheta_{36}\rangle,$$

and $E_1^{CM}(144, \psi_{-144})$ is the direct sum of these two spaces. In particular, we see that $\dim E_1^{CM}(144, \psi_{-144}) = d(36) + d(4) = 9 + 3 = 12$, and so $\dim M_1^{CM}(144, \psi_{-144}) = 13$.

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