Math 338 – Homework Two
Due at the start of class, Wednesday, September 17.

1: A constant heat source $f(x) \equiv 2$ is applied to an insulated rod of length $L = 1$, so that the steady-state temperature distribution (if it exists) is a solution to the differential equation

$$0 = u_{xx} + 2.$$ (1)

For each of the boundary value boundary value problems, either

i) Solve the problem directly (without using Fourier series) and sketch a graph of the solution or

ii) Show that the problem has no solution and explain the lack of solution in physical terms.

1a: The endpoints of the rod have fixed temperature:

$$u(0) = u(1) = 0.$$  

1b: The rod is completely insulated:

$$u_x(0) = u_x(1) = 0.$$  

1c: One end of the rod has fixed temperature, the other is insulated:

$$u(0) = u_x(1) = 0.$$  

(over)
2: Consider the following Neumann problem:

\[-\Delta u = f \quad x \in \Omega\]
\[\langle \nabla u, n \rangle = 0 \quad x \in \partial \Omega\]

This represents the steady-state heat distribution \( u \) in a region \( \Omega \) subject to heat source \( f \). The boundary values indicate that the region is perfectly insulated. Show that if this problem admits a solution then we must have \( \int_{\Omega} f \, dx = 0 \) (Hint: Use the divergence theorem). Explain the meaning of this result in terms of heat flow and apply this result to one of the boundary value problems from problem 1.

3a: Find the Fourier series

\[ f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx). \]

for \( f(x) = e^x \) on \([-\pi, \pi]\). Write your solution in a relatively compact form using the identities

\[ e^\pi - e^{-\pi} = 2 \sinh(\pi) \quad \text{cos}(k\pi) = (-1)^k \]

3b: Without performing any additional integrations, find the Fourier series for \( \cosh(x) \) and \( \sinh(x) \) on the same interval. Provide at least one sentence justifying your solutions (the series themselves are not a complete solution).