Math 338 – Homework Three
Due at the start of class, Wednesday, September 24.

1a: Consider the function \( f(x) = 1 \) on the interval \([0, \pi]\). Sketch the even and odd extensions of \( f \) to the real line. At points of discontinuity, carefully mark the function values to which the Fourier series will converge.

1b: For each extension in 1a decide whether or not the Fourier series will converge uniformly.

2: In Homework Two you found that the solution to \( 0 = u_{xx} + 2 \) on \([0, 1]\) with Dirichlet boundary conditions \( u(0) = u(1) = 0 \) is the quadratic function

\[
u(x) = x - x^2.
\]

This is a special case of the differential equation \(-u_{xx} = f\) and represents the equilibrium distribution of heat in a rod of length \( \ell = 1 \) under the following assumptions:

(i) With the exception of the endpoints, the rod is insulated;
(ii) The endpoints are kept at a constant temperature of zero \((u(0) = u(1) = 0)\);
(iii) A constant heat source is applied uniformly to the rod \((f = 2)\).

Let’s consider this problem from the Fourier series point of view.

2a: Find the Fourier sine series for the function \( f(x) = 2 \) on the interval \([0, 1]\) (if \( u \) is a solution, then this is a Fourier sine series for \(-u_{xx}\)).

2b: Integrate your series to find a Fourier cosine series for \( u_x \). Remember that integration will introduce a constant term. Calculate this constant explicitly.

2c: Integrate once more to find a Fourier series for \( u \), again remembering to calculate the constant of integration.

2d: What is the Fourier sine series for \( f(x) = x - x^2 \) on the interval \([0, 1]\)?
3: With a little care you can integrate term-by-term to get new Fourier series. Differentiation can be problematic, however. This problem is meant to provide you with a nice example of things going wrong.

3a: Find the Fourier series for $g(x) = 1$ on the interval $[-\pi, \pi]$.

3b: The Fourier series for the function $\int_0^x g(s) \, ds = x$ on the interval $[-\pi, \pi]$ is

$$\int_0^x g(s) \, ds \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

Differentiate this term-by-term. Do you recover the Fourier series for $g$?