Math 338 – Homework Five
Due at the start of class, Wednesday, October 8.

1: Let \( \kappa \) and \( \gamma \) be strictly positive real numbers. Find all solutions to the following Sturm-Liouville problem on \([0, L]\):

\[
\kappa X''(x) + \gamma X(x) = \lambda X
\]

\[
X(0) = 0 \quad \quad \quad X(L) = 0
\]

Your solution should show that you have found all of the solutions by ruling out certain values of \( \lambda \) (you may assume that \( \lambda \) is a real number). It may help to divide your cases up according to \( \lambda \)'s relation to \( \gamma \). For example, you might consider \( \lambda > \gamma \), in which case you can write \( \lambda = \gamma + \kappa \omega^2 \) for some nonzero real number \( \omega \). Alternatively, you might reduce this to a problem we have already solved.

2: Consider the one-dimensional wave equation with Dirichlet boundary conditions:

\[
u_{tt} = c^2 u_{xx}
\]

\[
u(t, 0) = 0 \quad \quad \quad \nu(t, L) = 0
\]

\[
u(0, x) = g(x) \quad \quad \quad \nu_t(0, x) = h(x)
\]

The total energy for a solution \( \nu \) at time \( t \) is given by

\[
E(t) = \frac{1}{2} \int_0^L u_t^2 \, dx + \frac{c^2}{2} \int_0^L u_x^2 \, dx
\]

Prove that, regardless of boundary conditions, we have

\[
\frac{d}{dt} E(t) = c^2 u_t u_x|_0^L.
\]

Argue that for our boundary conditions, the energy is conserved, regardless of initial conditions. It may help to note that

\[
\frac{\partial}{\partial x}(u_t u_x) = u_{tx} u_x + u_t u_{xx}
\]
3a: Solve the following partial differential equation using separation of variables:

\[ u_t = \kappa u_{xx} + \gamma u \]

\[ u(t, 0) = 0 \quad u(t, L) = 0 \]

Assume a general initial condition \( u(0, x) = g(x) \). This PDE represents a rod whose thermal conductivity is \( \kappa \) (if \( \kappa \) is large, heat energy moves easily through the rod). The rod is subject to a heat source which is directly proportional to the current temperature (the term \( \gamma u \)). You may assume that \( \kappa \) and \( \gamma \) are positive real numbers.

3b: Find the particular solution for \( g(x) = \sin(\pi x / L) \) and determine the values of \( \kappa \) and \( \gamma \) for which this solution decays to zero; is constant in time; and goes to infinity.