The following integration may be useful:

\[ \int_0^L \left( A + \frac{x}{L} (B - A) \right) \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{L}{n\pi} (A - (-1)^n B) \]

1: Suppose we want to solve the heat equation \( u_t = \kappa u_{xx} \) on an interval \([0, L]\) with the following mixed boundary conditions:

\[ u(t, 0) = 0 \quad u_x(t, L) = 0. \]

Assume an initial condition \( u(0, x) = x \). This problem can be solved using the Sturm-Liouville theory that we will cover in November, but here we provide an alternative approach.

1a: Find a function \( g : [0, 2L] \to \mathbb{R} \) with the following properties:

(i) On \([0, L]\), \( g(x) = x \);

(ii) On \([L, 2L]\), the graph of \( g \) is a reflection of the line \( y = x \) over the line \( x = L \).

That is, extend \( u(0, x) \) to the segment \([0, 2L]\) using “even symmetry with respect to \( x = L \).”

1b: Solve the heat equation on \([0, 2L]\) with Dirichlet boundary conditions \( u(t, 0) = u(t, 2L) = 0 \) and initial condition \( u(0, x) = g(x) \). You may leave your Fourier coefficients undetermined but should provide a formula for them.

1c: Prove that your solution from 1b, when restricted to \([0, L]\), solves the original boundary value problem.
2: Consider the heat equation $u_t = u_{xx}$ with boundary conditions

$$u(t, 0) = \sin(t) \quad u(t, 6) = 0$$

Use Duhamel’s principle to solve this equation under the assumption that $u(0, x) = 0$. You must find exact Fourier coefficients whenever needed but may leave the expression

$$\sin \left( \frac{n\pi x}{6} \right) \int_0^t \cos(s) e^{-n^2\pi^2(t-s)/36} \, ds$$

as it stands.

3: Let $u(t, x)$ be as in problem 2. Find a function $w(t, x)$ such that the function $U(t, x) = u(t, x) + w(t, x)$ satisfies the heat equation $U_t = U_{xx}$ with boundary conditions $U(t, 0) = \sin(t) + A$, $U(t, 6) = B$, and $U(0, x) = h(x)$. 