Math 338 – Homework Seven
Due in class on Wednesday, October 29

1: At the start of the semester you proved that the solution to

\[ 0 = u_{xx} + 2 \quad \text{and} \quad u(0) = u(1) = 0 \]

is given by \( u(x) = x - x^2 \) and interpreted the boundary value problem in terms of heat on a rod.

We’ve also looked at how the steady state solution decays if the heat source is turned off:

\[ u_t = u_{xx} \quad \text{and} \quad u(t, 0) = u(t, 1) = 0 \quad \text{and} \quad u(0, x) = x - x^2. \]

In this problem, I want you to look at how the system arrives at steady state from zero initial conditions.

1a: Solve

\[ u_t = u_{xx} + 2 \quad \text{and} \quad u(t, 0) = u(t, 1) = 0 \quad \text{and} \quad u(0, x) = 0. \]

Provide explicit formulas for any Fourier coefficients you use.

1b: Provide a mathematical (not physical) justification for the following limit:

\[ \lim_{t \to \infty} u(t, x) = x - x^2. \]

Then explain this phenomenon physically.

(over)
2: A square metal plate with dimensions is insulated on its broad, flat sides and the narrow edges are kept at the following constant temperatures:

\[ u(x, 0) = 0 \quad u(0, y) = 0 \]
\[ u(x, 1) = x^2 \quad u(1, y) = y^2. \]

Determine the steady state temperature distribution by solving \( \Delta u = 0 \) with the above boundary conditions. The solution looks like this:

![Temperature Distribution](image)

You may leave coefficients in your series solution as (for example) \( b_n \), but you must provide integral formulas for all coefficients. Hint: In class we solved the problem where three sides are kept at temperature zero.

3: Consider a region \( \Omega \) in the plane corresponding to polar coordinates \( 1 \leq r \leq 5 \) and \( 0 \leq \theta \leq \pi \).

Find a function \( \psi \) which satisfies \( \Delta \psi = 0 \) on \( \Omega \), along with boundary conditions

\[ \psi(r, 0) = 0 \quad \psi(r, \pi) = 0 \quad \psi(1, \theta) = 0 \quad \psi(5, \theta) = 5 \sin(\theta). \]

The level sets \( \psi(r, \theta) = c \) for different values of \( c \) are streamlines for an ideal fluid flowing past an semicircular obstacle (under some mild assumptions). Below you can see several of these streamlines:

![Streamlines](image)