1: Consider the wave equation on the real line:

\[ u_{tt} = c^2 u_{xx} \]

with initial conditions

\[ u(0, x) = g(x) \quad u_t(0, x) = 0. \]

In class we derived the following d’Alembert solution to this problem:

\[ u(t, x) = \frac{1}{2} g(x + ct) + \frac{1}{2} g(x - ct). \]  

Here we’ll use Fourier transform to derive the same solution. To keep things simple, we assume that \( g \) is a continuously differentiable odd function.

1a: Let \( \hat{u} \) denote the Fourier sine transform of \( u \):

\[ \hat{u}(t, \alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(t, y) \sin(\alpha y) \, dy. \]

Find an expression for \( \hat{u}(t, \alpha) \) in terms of \( \hat{g}(\alpha) \). Hint: First prove that

\[ \frac{\partial^2}{\partial t^2} \hat{u} = -\alpha^2 c^2 \hat{u}. \]

You can assume that

\[ \lim_{x \to \pm\infty} u(t, x) = \lim_{x \to \pm\infty} u_x(t, x) = 0 \]

1b: Use the trig identity \( 2 \sin(A) \cos(B) = \sin(A + B) + \sin(A - B) \) and your expression for \( \hat{u} \) to prove that \( \hat{u}(t, \alpha) \) is the Fourier sine transform of \( \frac{1}{2} g(x + ct) + \frac{1}{2} g(x - ct) \), thus proving formula (1).

2: Consider an infinite region of the plane determined by \( x \geq 0, y \geq 0 \). Find a bounded function \( u \) for which \( \Delta u = 0, u(x, 0) = f(x) \), and \( u(0, y) = g(y) \).
Solve either problem 3 or problem 4.

3: In some courses, students are seeing Fourier transforms defined in terms of complex exponential. For example, the Fourier transform of $f$ can be defined as:

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt.
\]

In our course, we’ve avoided complex numbers and are sticking with sines and cosines. Instead we associate with $f$ a function defined using sines or cosines, for example:

\[
A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\omega t) \, dt.
\]

This problem is meant to suggest that these approaches are essentially the same and that you have the necessary background to sort out the details whenever the need may arise. We’ll focus on the case where $f$ is even so that we need only the Fourier cosine transform. The general case is the same with more details.

3a: Prove that if $f$ is even, then

\[
\hat{f}(\omega) = \sqrt{\frac{\pi}{2}} A(\omega),
\]

where $\hat{f}$ is defined by (2). It will help to remember Euler’s formula: $e^{ix} = \cos(x) + i \sin(x)$.

3b: The original function $f$ can be recovered from (2) through

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega.
\]

Use equation (3) to show that when $f$ is even, (4) reduces down to

\[
f(t) = \int_{0}^{\infty} A(\omega) \cos(\omega t) \, d\omega,
\]

the formula we use in our class.
4: Consider the wave equation

\begin{equation}
    u_{tt} = c^2 u_{xx}
\end{equation}

for \( x \geq 0 \). We impose the homogeneous Dirichlet boundary condition \( u(t, 0) = 0 \) and assume initial conditions

\[
    u(0, x) = g(x) \quad u_t(0, x) = 0.
\]

4a: Up to constant multiples, find all functions \( T_\alpha(t) \) and \( X_\alpha(x) \) such that \( T_\alpha(t)X_\alpha(x) \) is a bounded function which satisfies (5) and for which \( T_\alpha(t)X_\alpha(0) = 0 \).

4b: The solution \( u \) will be a “generalized linear combination” of the form

\[
    u(t, x) = \int_0^\infty C(\alpha)T_\alpha(t)X_\alpha(x) \, dx.
\]

Find \( C(\alpha) \) and write down an integral formula for \( u(t, x) \).