Math 338 – Homework Ten
Due in class on Wednesday, November 19

1a: Using separation of variables, find all nonzero functions \( \varphi : [0, \pi] \times [0, \pi] \to \mathbb{R} \) and real numbers \( \lambda \) for which

\[
\Delta \varphi = \lambda \varphi
\]

and \( \varphi = 0 \) on the boundary of the square \( [0, \pi] \times [0, \pi] \). You should have one function and one real number for each pair of positive integers \((m, n)\) and so it will help to label them as \( \varphi_{m,n} \) and \( \lambda_{m,n} \).

1b: Show that your functions are orthogonal for the following inner-product:

\[
\langle f, g \rangle = \int_0^\pi \int_0^\pi f(x, y)g(x, y)\, dx\, dy
\]

You may use the fact that for \( m \neq n \), one has \( \int_0^\pi \sin(ms)\sin(ns)\, ds = 0 \).

1c: Are your functions orthonormal for this inner product?

2: The set of functions \( \{\varphi_{m,n}\} \) you found in Problem 1 is complete, which means that if \( u : [0, \pi] \times [0, \pi] \to \mathbb{R} \) then \( u \) can be represented as

\[
u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} \varphi_{m,n}(x, y)
\]

for some coefficients \( b_{m,n} \). If \( u \) has a dependence on time, then we can account for this by allowing the coefficients \( b_{m,n} \) to be functions of time, so

(1) \[
u(t, x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}(t) \varphi_{m,n}(x, y)
\]

Using (1), solve the heat equation

\[
u_t = \Delta \nu
\]

with homogeneous boundary conditions \( u(t, x, y) = 0 \) if \( x = 0, y = 0, x = \pi, \) or \( y = \pi \). Assume an initial condition \( u(0, x, y) = g(x, y) \).
3a: Consider the following vectors in $\mathbb{R}^4$:

\[
a = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

These vectors are orthogonal, but not orthonormal. Let $x$ be

\[
x = \begin{pmatrix} 3 \\ -3 \\ 3 \\ 3 \end{pmatrix}
\]

Find the projection $P(x)$ of $x$ on the space spanned by $\{a, b\}$ and the projection $Q(x)$ of $x$ onto the span of $\{c, d\}$.

3b: Consider the space $L^2(-\pi, \pi)$ with the inner-product

\[
\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx.
\]

Find the projection $P(f)$ of the function $f$ defined by $f(x) = x$ onto the subspace spanned by

\[
c_1(x) := \cos(x) \quad c_2(x) := \cos(2x) \quad s_1(x) := \sin(x) \quad s_2(x) := \sin(2x).
\]

\(^1\text{Remember this is the space of piecewise continuous functions on } [-\pi, \pi]\)