Mathematics and Beauty: Kieran Egan's kinds of understanding as a filter for identifying manifestations of beauty in the study of mathematics

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Abstract

I will attempt to classify manifestations of beauty in elementary mathematics, and show how this classification can be connected to the kinds of understanding described in Kieran Egan's *The Educated Mind*. I will suggest that for imaginative mathematics teaching, attention to this beauty is essential.

Most people do not like mathematics. Children's favorite subjects are sports, art, music, literature, but rarely mathematics. When meeting someone for the first time at a party, if I mention that I am a mathematician, the response is usually "I was never much good at mathematics", and sometimes all I get is awkward silence. In nearly all cases the conversation then switches to something of greater human interest.

The situation is not very different among teachers. High school teachers are specialists, so we can expect most of them to have a liking for the subject. Among elementary school teachers this is not usually the case. More often their university degrees are in the humanities, and many of them have found their experience with mathematics unsatisfying. One of the students in my mathematics course for pre-service elementary school teachers, who had taken no mathematics since grade 11, put it kindly on a questionnaire at the beginning of my course:

> "It's not that I don't like math, but I prefer other academics [sic], such as English, etc. that may have more than one answer and require a different type of imagination."

Another student wrote:

> "I really dislike the fact that I feel as though I have just squeaked through math all my life rather than really understanding it. Although I have always gotten good marks in the math courses I have taken, I do not feel comfortable with the subject in any way."

Even though my conversation partner at the party has resigned himself to the fact that he was never good at mathematics and gets along perfectly well without it, when his children go to school, mathematics (along with reading) is the subject most closely watched. Mathematics is deemed important for success in our competitive economy. In what other school subject do so many students take additional courses outside school, even those students whose marks are already very high? An excellent performance in mathematics gives that extra edge. Teachers must find a way to make it palatable to students even if they feel insecure about the subject themselves. Many can only hope that their students' experience will be better than theirs was. Mathematics is important, respected and feared, but few think of it as interesting, let alone beautiful.

The way we teach mathematics sometimes reflects these attitudes. We think of it as medicine. It is good for you, but it is not expected to taste good. So we try to find ways to trick students into liking mathematics. One of the ways in which we encourage students to do mathematics is to find problems centered on applications. For example, I came across a problem in which students are given a partly obscured aerial view of a herd of elephants, and are asked to estimate the total number of elephants in the herd. Now using applied problems is an excellent thing to do, but sometimes one gets the impression that this is the only way we can make mathematics interesting. I cannot imagine a student interested in the size of a herd of elephants in a picture, other than because they enjoy the mathematics involved in estimating it. The fact that it can be approached mathematically makes the question interesting, not the problem itself. I should add that this problem was an incidental add-on to a larger problem about the possible extinction of elephants in Kenya, which obviously does have a significant non-mathematical appeal.

At other times, discouraged by our own and our students' flagging interest in mathematics, we invent "fun activities" for our students. The intention is good, for we want mathematics to be fun. However, it can easily mean that the real purpose of the lesson is lost. In that case our strategy has become a cop-out, betraying a belief that the mathematics itself is not interesting.

A third consequence of teachers' belief that mathematics is not beautiful can be a watering down of the curriculum. Faced with the fact that the students find it hard already, everything is removed but the routine exercises meant to ensure that the students will pass the necessary tests.

Recognizing that this road is one that leads to boredom rather than real understanding, a fourth option is to let the lessons be determined largely by the students' own interests and pace of development with as little teacher input as possible. While sensitivity to students' interests and sense-making is very important, this, too, can skirt the challenge of presenting deep mathematics in a way that appeals to the students.

Mathematics and Beauty

How has it come about that we have this schizophrenic relationship with mathematics? I think it is because we associate mathematics (and the same goes for much of science) with control. Mathematics is something to be wielded, not something to be contemplated, received, enjoyed, and played with.

It has not always been so. For the ancient Greeks, mathematics was considered key to our understanding of the world, perhaps even more than it is today. However, their understanding was closer to a contemplative, religious understanding, and to their concept of beauty. In the 5th Century B.C., the Pythagorean Philolaus wrote:

> "Were it not for number and its nature, nothing that exists would be clear to anybody either in itself or in its relation to other things — You can observe the power of numbers exercising itself not only in the affairs of demons and gods but in all the acts and the thoughts of men in handicraft and music." (quoted from Kline [6])

For the Pythagoreans the qualities of mathematics were closely connected to their understanding of beauty, which they described as proportion, order, balance and harmony. Key to this connection is their observation that pleasure in music is produced by simple mathematical relations between the lengths of strings or pipes producing the notes. This mathematical/aesthetic understanding seems to extend to much of their science. For example, in Pythagorean astronomy, planets revolved around a fixed earth, moving more rapidly if they were more distant from the center. These distances were thought to vary in a way that produced a harmonic combination of sounds, the "music of the spheres", entirely accounted for by simple mathematical relations (Kline [6]).

This identification of beauty as proportion, together with the central role ascribed to mathematics in their understanding of the world led the Pythagoreans to identify beauty with being itself. To be was to participate in order and harmony (as opposed to chaos), and thus to be was to be beautiful. This same motif is central in pre-classical Greek religion, which centered on the victory of order over chaos (Farley [4], pages 18, 19)

Later, Plato demythologized the Olympic creation myths, but kept their framework. Instead of the god Eros warring the forces of chaos (as in the poetry of Hesiod), for Plato there is a hungry and thirsty Eros of the soul that can be satisfied only by what is highest and most beautiful (Farley, [4] page 19). Lest we forget that mathematics is the prime candidate among things that are high and beautiful, the inscription above the entrance to Plato's academy read: "Let no one ignorant of geometry enter here."

The Greek understanding of beauty continued, with modifications, for about 2000 years. The historian of aesthetics Tatarkiewicz calls it the "Great Theory of Beauty" ([9]). While proportion, order, balance and harmony continued to be the key words describing what beauty is (notice that these are all words with mathematical reference), the Hellenic understanding of its origin in the victory of order over chaos, is replaced in the Middle Ages by God's creative act (Farley [4], page 21) understood, once again, as the establishment of order when before there was only chaos. God's beauty draws all things to itself, producing the general harmony of the universe. Beauty is the 'resplendence of form' ([2])

The importance of mathematics diminished over this period of time, especially during the early middle ages. Because of its association with astrology it even became an object of suspicion, as in the following quote of St. Augustine:

> The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of hell.

Rationalism

Beginning around the 17th Century, with the rise of rationalism, mathematics regained its central importance, this time at the expense of beauty. Mathematics became the centre of the search for certain knowledge and the associated attempts to control our environment. At the same time, beauty lost its important place, and was increasingly seen to be in the eye of the beholder. This process accelerated in the modern era, with the growth of industrial power, commerce and urbanization. Mathematics lurks at the heart of the technology and the commerce that drive the modern industrial state, and beauty is one of its first victims. Mathematics brought us the theory of relativity, quantum mechanics and the atom bomb. Mathematics governs the stock market and the bottom lines of multinational corporations. Mathematics has become useful, economically important. In the meantime, beauty has increasingly become a matter of personal preference.

Mathematicians themselves continue to find their subject beautiful (Sinclair [8]), but sometimes in ways that suggest that its beauty is disconnected from the rest of life. In fact, the field has now been divided into applied mathematics, the useful things that you want your children to know, and pure mathematics, a luxury indulged in by the few. Applied mathematics is mechanical but important; pure mathematics is beautiful if you can understand it, but not to be taken too seriously if you want to get ahead in the world. One reason for the division is that we are now much less inclined to describe the world as harmonious, ordered and balanced. If mathematics is to be thought of as beautiful, and if beauty means order, harmony and balance, then the connection between that kind of mathematics and the "real" world is tenuous. Listen to this quote by Bertrand Russell (as quoted by Kline [6]):

> "Mathematics, rightly viewed, possesses ... supreme beauty - a beauty cold and austere, like that of a sculpture, without appeal to any part of our weaker nature. ··· The true spirit of delight, the exaltation, the sense of being more than man, ··· is

> to be found in mathematics as surely as in poetry. ⋯ Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home and where one, at least, of our nobler impulses can escape from the dreary exile of the actual world."

This is clearly an appeal to beauty, and like the Greeks and the medieval scholars, Russell associates beauty with order. However, this beauty implies an escape from the chaos of the world. It is entirely a creation of the mind, and perhaps more than ever, this ordered beauty is devoid of any dynamic or organic elements. It is static and absolute. There is no humanity in this beauty. Russell associates it with his sense of being "more than man". This description of mathematics will repel most people, I suspect. It repels me, and I cannot imagine it as the basis of mathematics education.

Postmodernism

In our postmodern world, we are suspicious of quotes such as Russell's. They sound elitist. Russell tried to find absolute mathematical certainty by locating mathematics in logic. We have come to distrust absolutes. My truth is your oppression. Even beauty is suspect, for what I find beautiful may well be the very thing that makes you feel an outsider. If mathematics gave us certainties at one time, it does so no longer. For the postmodern thinker, mathematics is now also, along with beauty, considered to be in the eye of the beholder.

Even education theory is influenced by this postmodern way of speaking. For a modern thinker such as Bertrand Russell, the fundamental entities under consideration in mathematics are clearly given to us in our mathematical intuition using the processes of abstraction and idealization (Zwier [11]). From the postmodern point of view, all we have is language games. Though for working mathematicians it is very difficult, impossible even, to accept it in its most blatant radical form, the philosophy of mathematics most widely accepted among education theorists is Paul Ernest's social constructivism. Though not quite described with the degree of arbitrariness suggested by the term "language game", the object of mathematics, for a social constructivist, is the socially accepted language convention of the time. "A mathematical community in a culture learns a proposition or procedure that is part of a language game by social construction when the community decides to accept it on the basis of cultural conversations, questioning sessions, and arbitration using a variety of heuristics and computational devices and any other methods it decides are important. The conversation and arbitration are done in terms of the rules of the language game that the individual has learned or by reformulating new rules that seem appropriate (Zwier [11])."

As a corrective to educational practices that are top-down, authoritarian, traditionalist, or based on rote learning, the constructivist view is important, for it emphasizes the need to connect new ideas to things already understood by the learner. Constructivism values the role of social interaction in the learning process, and acknowledges the alienation that can occur when a concept is not understood. However, as a complete description of what mathematics is, the social constructivist position is difficult to accept for most research mathematicians. They tend to think of their work as discovery.

Nor can it inspire us, I believe, as teachers of mathematics. I teach because I love the things I have come to know, I would like to share that pleasure with others, and I find that by and large it has been possible to do that. In 1860, Walt Whitman wrote a wonderful short poem, "To You", that captures beautifully the courage needed to be a teacher:

> Stranger, if you passing meet me and desire to speak to me, why should you not speak to me? And why should I not speak to you?

When we teach we should have something to tell, some direction to lead, some imagination to invite into, some beauty to share. A radically postmodern position seems to undermine that.

Recovering Beauty

So how do we recover the power of mathematics as a discipline that (however imperfectly and however partially) really describes some aspect of the world as it is? And how do we teach it imaginatively without resorting to cute superficialities? How do we do this without creating a program accessible only to some sort of elite group of students? And how do we get away from the understanding that mathematics is the embodiment of all that is inhuman, but is essential to success in our corporate society?

I suggest that one of the main ingredients in such a move is a reaffirmation of the role of beauty in mathematics, in place of utility, and that the way forward is to connect beauty to both reference and playfulness. However, in order to avoid the static and absolutist approach of some traditional mathematics on the one hand, and to avoid the arbitrariness of postmodern language games on the other, I want to accomplish this by extending our understanding of beauty, disconnecting reference from certainty and putting playfulness in a context of trust.

By thinking about the beauty of mathematics in terms of reference I mean to say that mathematics (like other disciplines) refers in some generally comprehensible way to the world around us and that it participates of the beauty of the world in a way that makes it worthy of our attention. Mathematics is more that just the theoretical constructs it produces. It encompasses the wonderfully imaginative ways in which these theories can be used to model the world and to speak of it. In fact, it is astounding that these theoretical objects constructed by mental activity fit the physical world as well as they do.

By disconnecting reference from certainty I mean to say that this beauty is present and recognizable and attractive, even when we do not completely comprehend it and thus it does not give us complete certainty or control. We need an understanding of beauty that allows for change, for partial clarity, and even chaos. I want to call a perfect storm beautiful. In her wonderful book, "Pilgrim at Tinker Creek", Annie Dillard writes beautifully about the extravagant, chaotic fecundity of nature:

> "I don't know what it is about fecundity that so appalls. I suppose it is the teeming evidence that birth and growth, which we value, are ubiquitous and blind, that life itself is so astonishingly cheap, that nature is as careless as it is bountiful, and that with extravagance goes a crushing waste that will one day include our own cheap lives" ... ([1], page 160).

We find the world beautiful, in spite of all its chaos. It is essential for an adequate understanding of beauty that it is recognized even in the chaotic, where it is received as intricacy and elicits surprise, and in the grotesque and overwhelming, which elicit awe and even fear, and where beauty is received as transcendence.

Both surprise and transcendence imply an absence of control. The former is characteristic of situations where our language proves inadequate to the task of forming a comprehensive picture of the world, but leaves open the possibility of finding a better model. The latter is characteristic of situations that move even beyond that, that suggest that complete understanding is impossible, that may even question our very self-hood, our personal viability. Our understanding of beauty should be broad enough to include these dimensions. In particular, not all beauty can be spoken of in terms of proportion, balance, order, or harmony. We need to include other concepts such as surprise, intricacy, wonder and transcendence.

Finally, I suggested we should try to embed playfulness in a context of trust. Postmodern thought makes a great deal of playfulness. The possibility of play in the language structures we use to batten down the hatches of our precarious understanding of the world first led to the postmodern position. The postmodern thinker takes her insights to imply that we should abandon ship, or at least take an ironic position close to the lifeboats. Indeed, if our chief goal is to run a tight rationalistic ship for the purpose of arriving in port on schedule, we should be worried. Perhaps, though, the postmodern position is too much a reaction to the modern rationalist understanding of our place in the world. Perhaps we need not be quite so concerned to secure ourselves. Perhaps we were never meant to feel safe in the way we thought we were. What if this deep, rolling, dangerous sea is in fact beautiful? What if it carries us surprisingly safely to a destination we cannot entirely know on a timetable that is not of our making? Can we trust it nevertheless? It seems to me that postmodern playfulness is not really playfulness at all, but has in it an element of deep-seated fear and despair. It is not playful in the sense in which we understand the play of children. The playfulness of children implies trust. Not the kind of trust that assumes total control, but the kind of trust that accepts the world as imbued with meaning the child is hungry to know.

Kinds of beauty in mathematics

Mathematics is a particular, focused way of looking at the world, and much beauty will be missed if we look only through mathematical spectacles. Nevertheless, contrary to the common supposition that Mathematics is austere, it is in fact a good place to look for a rich variety of types of beauty, including not only "static" forms of beauty (proportion, order, balance and harmony) but also surprising, chaotic and unsettling beauty. I will attempt a taxonomy of beauty in mathematics, that makes use of Kieran Egan's kinds of understanding (see[3]). I will show examples of each kind of understanding in mathematics. I will try to describe instances of the experience of beauty available when mathematics is approached with each of these forms of understanding.

Somatic understanding

Mathematics begins very early in a child's life, before the child even learns to speak. This most basic understanding is associated with body rhythms, heartbeat, walking, and music. This pre-linguistic body understanding is termed "somatic" by Egan. Before a child learns how to count, she learns the rhythmic recital of numbers. At first the sounds have no meaning, giving only physical pleasure and the pleasure of pleasing a parent. Children's poems, books and educational television programs continue to make use of rhythm long after the child has learned how to speak. Rote learning is essentially an acknowledgment of the beauty and power of rhythm as somatic understanding to support other kinds of learning.

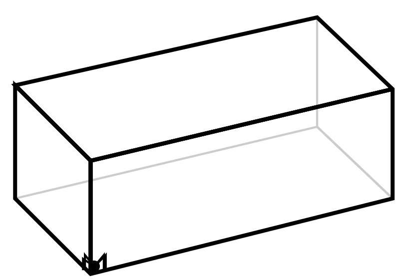
Mythic understanding

The simplest level of understanding Egan associates with language is the mythic. (The others, in order of complexity, are romantic, philosophic and ironic.) Egan associates mythic understanding with oral language and describes it in terms of binary opposites (hot/cold, safe/dangerous, good/evil). Classifying by means of binary distinctions is a first step in making sense of the world. Binary opposition is essential to learning how to count. Once the child has learned to say the numbers, the second step in the process of learning how to count is the establishment of one-to-one correspondence between the recited numbers and the objects of a set. Here the essential skill is to place the objects in the set in opposition to each other so they can all be seen as individuals and matched to the numbers without repetition.

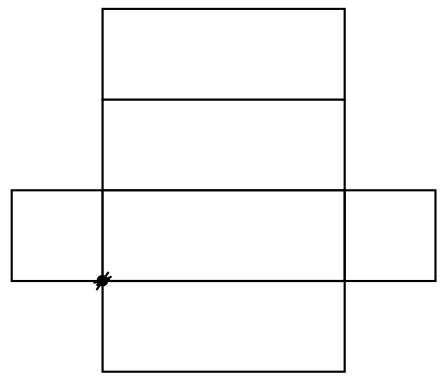
Pattern is another characteristic of mythic understanding in mathematics. My enrichment programme for students in grades 7 and 8 includes an extensive study of patterned number sequences as well as geometric patterns. Each of these is associated with mythic elements. In fact the fascination with patterns in mathematics is such that beauty in mathematics is often recognized primarily in this aspect, and mathematics is described as the science of patterns.

Narrative is another element in Egan's discussion of mythic understanding. Over the years I have come to appreciate the effectiveness of stories as contexts for problems. Before asking students to estimate the sum of successive powers of two, I tell them the ancient story of the poor soldier and the king. The King wants to reward the modest soldier, who then asks that one penny be placed on the first square of a chess board, two on the next, four on the third square, eight on the fourth, and so on.

Rather than give students an exercise using Pythagoras's Theorem to find a distance along the walls of a rectangular room, I tell them a story of a spider and a fly. The spider is located in one corner of the room, and the fly wants to position itself as far from the spider as it can.



In each case the problem is transformed by the story. The story connects the problem to elements in the students' experience, even if it is a fantasy experience. The beauty revealed by embedding the problem in a story is in the surprise that mathematics provides an unexpected solution. The very solution requires an act of imagination. For example, to see the shortest path taken by the spider, the room has to be "cut open along its edges and laid flat."



At the same time the story seems to act as a (relatively) concrete, though complex, object for carrying the meaning inherent in the associated mathematics problem. This is especially true for the chessboard problem, where the calculation

$$1 + 2 + 2^2 + 2^3 \cdots + 2^{63} = 2^{64} - 1$$

can be carried out within the story by placing an extra penny on top of the penny in the first square and then successively placing each pile of pennies on top of the pennies in the following square until we are left with a pile of 2^{64} pennies in the last square. The functioning of the story is analogous to the role of play in younger children. According to Vygotsky, a younger child playing horse with a broom transfers meaning to the broom through her words and gestures, and thereby readies herself for a more full-fledged use of symbols in writing (Vygotsky [10]). In a similar way, the older child prepares herself for philosophic understanding and the associated abstract symbolization by attaching structured mathematical meaning to the story.

The classical understanding of beauty as proportion, order, balance and harmony is closely associated with mythic understanding. Much of the focus in elementary school mathematics teaching is on mythic elements in mathematics. Even in high school, the understanding of students often does not move far beyond that. You can see this from the fact that the public perceives mathematics as received wisdom, a subject that is not alive. Even university students will often express surprise when they hear that people actually do research in mathematics. It seems a contradiction in terms.

Romantic understanding

One of the important ways to help students experience the beauty of surprise and wonder in mathematics is to expose them to some of its more outrageous aspects. As Egan has pointed out, before children reach middle school they become intrigued by the extreme, the surprising, elements in their world. This exploration of boundaries he refers to as "romantic understanding". Mathematics is full of delightful examples of this. Quite early on, children become fascinated by infinity. Children around age 8 will realize that numbers go on for ever and ever, and will be intrigued by that.

I expose my grade 7 and 8 students to the counterintuitive behavior of infinite sets is by telling them the story of Hilbert's Infinite Motel. In this problem, which is well-known to most mathematicians, I ask the students to imagine a motel with infinitely many rooms, numbered $1, 2, 3\cdots$. and an infinite bus, with seats numbered $1, 2, 3\cdots$. A simple instruction to the bus passengers will fit them nicely into the infinite motel, and the motel will be full. But what if at the last minute a Volkswagen bug pulls up with a single driver, or worse still, a second infinite bus? Can the motel manager modify the arrangements so as to accommodate everyone?

As another example of romantic understanding, after a unit on prime numbers, I tell them about the largest know Mersenne prime. Until recently this was a number discovered by a student from Owen Sound, Ontario. The current record holder is one less than two to the power 24,036,583:

$$2^{24,036,583} - 1$$

I challenge the students to estimate the number of digits in this number (over seven million). The kind of beauty elicited in these examples is the kind that elicits surprise and wonder - the sense that the world is so much bigger and more wonderful than first thought, so big in fact that we will probably never understand it all.

Philosophic understanding

The next level of understanding in Egan's classification, his philosophic understanding, constitutes a kind of reaction to the surprised wonder of the romantic thinker. This time the focus is on finding a more complex, intricate harmony that can encompass all the extraordinary phenomena that so intrigue us. The construction of this more complex understanding is possible only through systems of thought constructed by communal cultural activity. Philosophic understanding is the type of understanding usually associated with more advanced mathematics. The number system, with its rules for operations with negative numbers and fractions, and the even stranger irrational numbers belong to this phase in mathematics; Algebra with its complicated rules of formal operations; Geometry with its axioms, proofs and theorems.

John Mighton, the creator of the very successful JUMP programme, a math tutoring programme for floundering elementary school students, has observed that it is precisely when students are first confronted with fractions, and thus with the need to step up to this kind of understanding, that many of them begin to feel that mathematics is impossible for them. Not everyone makes the transition easily, and one of the important questions in the pedagogy of mathematics is how this transition should be facilitated. The prevailing wisdom, inspired by Piaget, and adopted by the National Council of Teachers of Mathematics, is that no theoretical structures should be taught until the developing child is ready to connect them securely to her experience. In particular, rote learning should only take place once understanding is secure. Paradoxically, John Mighton's very successful program appears to fly in the face of this prevailing constructivist doctrine by concentrating on the mechanics first, rather than the understanding (see [7]). His finding is that the mechanics rather than the understanding is the obstacle, and that understanding follows easily once the mechanical manipulation of fractions is mastered. Mighton finds that the beauty of even these very simple arithmetic patterns is a source of pleasure for these children, and a key to their success in learning mathematics. Perhaps the constructivist metaphor should be adapted to allow for the inclusion of prefabricated units in the building process.

In any case, this encounter with structured bodies of mathematics is both the most frustrating and the most beautiful part of mathematics. This is the heart of the subject. Consider the following mysteries:

- Adding successive odd integers always produces a perfect square.

$$1 + 3 + 5 + 7 + 9 = 5^2 .$$

- The sum of successive powers of two is always one less than the next power of two.

$$1 + 2 + 2^2 + 2^3 + 2^4 = 2^5 - 1 .$$

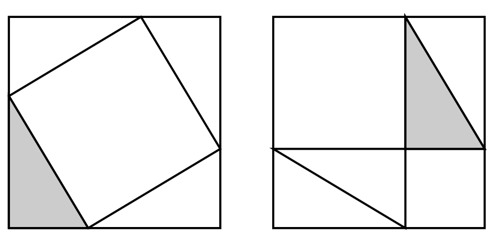
- When you enter 1. followed by a row of zeros, with another 1 in the very last place on your calculator display, and press the square button, all that happens is the last 1 becomes a 2.

$$1.00000001^2 = 1.00000002\,?$$

These and many other surprises cry out for explanation. When children reach middle school they begin to believe that there is a method to the intricate madness of the world, and it begins to give them pleasure to look for it and to find the connections. Unfortunately, the middle school curriculum does not usually include much to encourage teachers to expose students to the idea that there are explanations for the patterns. The fact that successive odd integers add up to a perfect square is not merely to be taken note of the way you accept the fact that your brother is taller than you or that the weather always seems worse on weekends, but it is something that can be explained (proved) by examining the very structure of the numbers involved. The explanation of the fact that successive odd integers add up to perfect squares is contained in the following diagram.

⋆	○	⋆	○	⋆	○
○	○	⋆	○	⋆	○
⋆	⋆	⋆	○	⋆	○
○	○	○	○	⋆	○
⋆	⋆	⋆	⋆	⋆	○
○	○	○	○	○	○

Explanations, or proofs, such as the one implicit in this diagram are themselves things of beauty. A mathematician considers a proof beautiful if it makes use of a surprising line of thought resulting in an elegant and economical argument. The beauty of a proof is in the elegance and economy of thought. Here is one of many proofs of the Theorem of Pythagoras:



This beauty of harmony, of fitting together, of resolved complexity, of balance, is the essence of the Great Theory of Beauty that began with Pythagoras and dominated western aesthetics for 2000 years. The aesthetic pleasure at seeing the connections is so intense that it has led to the suggestion that mathematics, at least in principle, provides certainty about the world that is absolute. The Pythagoreans responded to it in that way, and the same ascription of autonomy to mathematical thought remained dominant throughout the modern era.

Ironic understanding

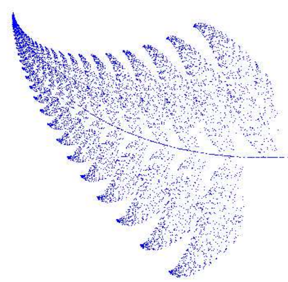
Egan's final level of understanding, that of ironic understanding, raises questions about this confident creation of theories. Mathematical theories that were once thought to apply directly and without question to the world around us have have in some cases been shown not to do that. The best-known example is Euclidean Geometry. Kant considered it a synthetic a priori truth that our physical space was a three-dimensional Euclidean space. The discovery of non-Euclidean geometries made that less certain, and the advent of general relativity has forced us to re-examine what we mean by it. It is not that Euclidean geometry is now less important, for in fact most modeling of the space around us continues to use it with great reliability. The Euclidean model, which was once thought the only model possible, is now seen as no more, but also no less, than the correct local, small-scale version of each of the other models. Its importance is not diminished, it has simply been put in a broader context. Nevertheless, these events indicate that our construction of mathematical models for the world is never completely finished.

By emphasizing understanding in the classroom, and by including mathematical modeling as an important part of the curriculum, we prepare children for the realization that mathematics does not apply seamlessly to the world around them.

I can think of two distinct ways in which mathematical theory itself points to ironic understanding: Chaos Theory and the Incompleteness Theorem of Gödel. The first of these indicates that apparently simple mathematical systems can lead to surprising complexity. The second demonstrates the necessary inadequacy of mathematical theory to give a complete account of mathematical phenomena. The surprising fact is that both of these understandings about mathematics can be detected from within mathematics itself.

No area of mathematics has attracted more public attention over the last decade than chaos theory. Mathematicians have known for a long time that even simple mathematical systems can display very complicated chaotic behaviour. It took a meteorologist, Lorenz (see Gleick [5]), to point out the remarkable implications this has for science. He was interested in the persistent unpredictability of weather, and was surprised to discover that even if one simplifies the mathematical formulas describing weather to an enormous extent, the unpredictability persists. It is unsettling to think that mathematics, for centuries considered the ultimate justification for viewing the world as mechanically predictable, should be instrumental in betraying that dream. But look what chaos gives us in return. Notice the intricate beauty evident in the following picture.

0	0	0	0	.16	0	.01
.85	.04	-.04	.85	0	1.6	.85
.2	-.26	.23	.22	0	1.6	.07
-.15	.28	.26	.24	0	.44	.07



The numbers in the accompanying table seem completely uninteresting, while the picture is organic, and beautiful. Yet, the numbers produce the picture! Without chaos, mathematics would not be able to produce that kind of intricacy.

If mathematical theories seem mechanical in their precision, and thereby suggest that the events of the world can be seen to evolve deterministically once we have the right formulas in place, the chaotic behaviour of even some of the simplest mathematical equations indicates that there is no reason at all to suggest that through mathematics a clear and complete understanding of the world is possible. Our failure to achieve complete control of our environment is not the result of insufficient information. The chaos is inherent in the mathematics itself. The world would not behave deterministically, even if there were a mathematical model to explain all phenomena.

In fact, however, there will never be a mathematical theory that does that, for during the last century, mathematics has also revealed necessary restrictions on the reach of theoretical language. In 1931 Kurt Gödel published a famous paper in which he demonstrated that within any given branch of mathematics, existing or yet to be developed, there are some propositions that cannot be proven either true or false using the rules and axioms of that mathematical branch itself. You might be able to prove every conceivable statement about numbers within a system by going outside the system in order to come up with new rules and axioms, but by doing so you'll only create a larger system with its own unprovable statements. The implication is that all logical systems of any complexity are, by definition, incomplete; each of them contains, at any given time, more true statements than it can possibly prove according to its own defining set of rules.

This is one of the most remarkable instances of ironic understanding I know of. Somehow, this irony is thrust upon the mathematician by the very structure of his theories. It is a beautiful instance of modernism deconstructing itself. Here the beauty is best described as transcendence, for we cannot use our theories to capture all that is true of the world.

Conclusion

Mathematics educators ought not to let their agendas be dominated by utilitarian concerns. They should not let the usefulness of the subject obscure its beauty. A better strategy is to revel in its imaginative fecundity and to marvel, after the fact, that something so beautiful is also useful in understanding our world.

It should come as no surprise that the various kinds of beauty evident in mathematics correlate with our cognitive tools - our ways of understanding. Just as our understanding of mathematics education must move beyond the learning of received wisdom, so our understanding of beauty in mathematics must grow beyond the traditional description of beauty as proportion, order, balance and harmony to include other more dynamic concepts such as surprise, intricacy, wonder and transcendence.

References

[1] Annie Dillard, *Pilgrim at Tinker Creek*, New York, Harper's Magazine Press, 1974.

[2] Umberto Eco, *The Aesthetics of Thomas Aquinas*, Harvard University Press, 2002.

[3] Kieran Egan (1997), *The Educated Mind: How cognitive tools shape our understanding*, Chicago: University of Chicago Press, 1997.

[4] Edward Farley, *Faith and Beauty: A Theological Aesthetic*, Ashgate Publishing Company, Burlington VT, USA, 2001.

[5] James Gleick (1988), *Chaos: Making a New Science*, Penguin Books, 1988.

[6] Kline (1966), *Mathematics and Western Culture*

[7] John Mighton (2003). *The Myth of Ability.* Anansi 2003.

[8] Nathalie Sinclair (2001). Ph.D. thesis, Queen's University.

[9] Wladyslaw Tatarkiewicz (1972). *The Great Theory of Beauty and its Decline.* The Journal of Aesthetics and Art Criticism, Vol. 31, No. 2, (1972) pp. 165-180

[10] L. S. Vygotsky (1978), *Mind in Society: the development of higher psychological processes* Cambridge: Harvard University Press, 1978.

[11] Paul Zwier (2001). Mathematical Truth: Static or Changing? In Howell, Russell, & Bradley, James, (editors). *Mathematics in a Postmodern Age, A Christian Perspective.* Eerdman's 2001.