Cardinality and the Schroeder-Bernstein Theorem.

Mathematics 328

The cardinality of finite sets is a relatively simple matter. If two finite sets X and Y have the same cardinality, then there is a one-to-one correspondence between the elements of X and those of Y. In fact if X and Y each have n elements, where n is a whole number, then there are $n!$ distinct one-to-one correspondences between the sets. Note that for finite sets with the same number of elements it is not possible to put all of X into one-to-one correspondence with a proper subset of Y, or vice-versa.

In the case of infinite sets the matter is much more complicated. To discuss cardinality at all, we have to be more precise about its definition. It is no longer enough to say " 'X and Y have the same cardinality' means that they have the same number of elements." When both of them have infinitely many elements, it is not clear what we mean by saying that the number of elements in each set is the same. We could decide to lump all infinite sets together and that they all have the same number of elements, namely infinitely many. However, there are several reasons why that is not the way to proceed. In the first place, it is not very satisfactory to say of the set $\mathbf{R}$ of real numbers and the set of $\mathbf{N}$ of natural numbers that they have the same number of elements. Intuitively, it seems to us that the former has more elements than the latter. Another reason for rejecting this definition of "having the same number of elements" is that it is easy to find pairs of sets both of which are infinite, but for which it is not possible to put one into one-to-one correspondence with the other. In fact, the simplest example consists of the pair of sets $\mathbf{R}$ and $\mathbf{N}$. Proving that something is not possible is usually more difficult than that something is possible. We will discuss in class the impossibility of a one-to-one correspondence between $\mathbf{R}$ and $\mathbf{N}$.

The solution to the difficulty of discussing the size of an infinite set, is to adopt the existence of a one-to-one correspondence as the definition of "equal cardinality," and then live with the consequences, even if some of them may seem strange. So we have this definition:

Definition: Two sets X and Y have the same (or "equal") cardinality if there exists a one-to-one correspondence between the elements of X and the elements of Y.

We should not expect the one-to-one correspondence to be unique. In fact it never is. Nor is the correspondence necessarily "natural", whatever that means. For example, the set **N** of natural numbers and the set 2**N** of positive even numbers can be put into one-to-one correspondence by means of the bijection $f(n) = 2 \times n$. However, we can create another bijection by mapping 1 to 4 and 2 to 2, and mapping all other numbers the way f does.

Even though this definition of equal cardinality is more satisfactory than the simple minded alternative rejected earlier, it does have some surprising properties. The first surprise is that a set can have the same cardinality as a subset of itself! In fact, we saw an example of this in the preceding paragraph, where it was pointed out that **N** and 2**N** have the same cardinality.

When two infinite sets X and Y are given, it is often very difficult to determine whether they do or do not have the same cardinality. An extremely useful device is presented in the following theorem.

Schroeder-Bernstein Theorem. If X and Y are two sets each of which has the same cardinality as a subset of the other, then X and Y have the same cardinality.

Note that the theorem in fact says the following: If there exist a one-to-one function f from X onto a subset $f(X)$ of Y and a one-to-one function g from Y onto a subset $g(Y)$ of X, then there exists a one-to-one function F from X **onto** Y.

The following proof is essentially due to Birkhoff and MacLane. It is found in "Introduction Topology and Modern Analysis" by G. F. Simmons. Similar proofs can be found in books on set theory or analysis.

Proof: We assume the existence of the functions f and g, and we will use these to construct F. We may assume that neither f nor g is onto, for if f is onto (as well as one-to-one) we can define F to be f, and if g is onto, we can define F to be g^{-1}. Since both f and g are one-to-one, it is permissible to use the mappings f^{-1} and g^{-1} as long as we apply them only to points in $f(X)$ and $g(Y)$ respectively.

We obtain the mapping F by splitting both X and Y into subsets which we characterize in terms of the ancestry of their elements. Let x be an element of X. We apply g^{-1} to it (if we can - that is, if x lies in the subset $g(Y)$) to get the element $g^{-1}(x)$ in Y. If $g^{-1}(x)$ exists, we call it the first ancestor of

x. The element x itself we call the zeroth ancestor of x. We now apply f^{-1} to $g^{-1}(x)$ (if we can), and call $f^{-1}g^{-1}(x)$ the second ancestor of x. We now apply g^{-1} to $f^{-1}g^{-1}(x)$ (if we can), and call $g^{-1}f^{-1}g^{-1}(x)$ the third ancestor of x. This process continues forever, unless we encounter an ancestor that is contained in X but not in $g(Y)$ or in Y but not in $f(X)$. If either of these occurs we stop, and the number of ancestors of x is finite. Thus, as we continue this process of tracing back the ancestry of x, it becomes apparent that there are three possibilities.

- x has infinitely many ancestors. We denote by X_i the subset of X which consists of all elements with infinitely many ancestors.
- x has an even number of ancestors; this means that x has a last ancestor (that is, one which itself has no first ancestor) in X. We denote by X_e the subset of X consisting of all elements with an even number of ancestors.
- x has an odd number of ancestors; this means that x has a last ancestor in Y. We denote by X_o the subset of X which consists of all elements with an odd number of ancestors.

The three sets X_i, X_e, X_o form a disjoint class whose union is X. We decompose Y in just the same way into three subsets Y_i, Y_e, Y_o. It is easy to see that f maps X_i onto Y_i and X_e onto Y_o , and that g^{-1} maps X_o onto Y_e. Thus we can complete the proof by defining F in the following piecemeal manner:

$$F(x) = \begin{cases} f(x) & \text{if } x \in X_i \bigcup X_e, \\ g^{-1}(x) & \text{if } x \in X_o \end{cases}$$

It follows immediately from what we said about the 'pieces' of F that it is one-to-one and onto all of Y. □