The space of closed and bounded subsets of $\mathbf{R}^n$

Mathematics 328

The definition of the space

Let (H, d_H) be the space consisting of all closed and bounded subsets of $\mathbf{R}^n$, together with the Hausdorff metric,

$$d_H(S,T) =$$

$$inf\{\,\varepsilon\,|\,\forall x \in S\,\exists\,y \in T \text{ with } d(x,y) < \varepsilon, \text{ and } \forall y \in T\,\exists\,x \in S \text{ with } d(x,y) < \varepsilon\}$$

We begin by checking that d_H satisfies the axioms of a metric, and thus that (H, d_H) is a metric space. It helps to reformulate d_H first.

Suppose we define, for a subset S of $\mathbf{R}^n$,

$$N_\varepsilon(S) = \{y \in \mathbf{R}^n \mid \parallel y - x \parallel < \varepsilon \text{ for some } x \in S\}\,,$$

and $B_\varepsilon(x)$ as the open ball containing x:

$$B_\varepsilon(x) = N_\varepsilon(\{x\}) = \{y \in \mathbf{R}^n \mid \parallel y - x \parallel < \varepsilon\}\,.$$

It is easy to see that $N_\varepsilon(S)$ is then an open set containing S.

Lemma 1. $N_\varepsilon(S) = \bigcup_{x \in S} B_\varepsilon(x)$.

Proof. The proof is left as an exercise.

Lemma 2. $N_\delta(N_\varepsilon(S)) = N_{\varepsilon+\delta}(S)$.

Proof. Suppose $y \in N_\delta(N_\varepsilon(S))$. Then there is a point $z \in N_\varepsilon(S)$ such that $\parallel y - z \parallel < \delta$. But since $z \in N_\varepsilon(S)$, there is a point $x \in S$ such that $\parallel z - x \parallel < \varepsilon$. By the triangle inequality this implies that $\parallel y - x \parallel < \varepsilon + \delta$.

Now suppose $y \in N_{\varepsilon+\delta}(S)$. Then there is an $x \in S$ such that $\| y - x \| < \varepsilon + \delta$. Now choose z on the line segment joining x and y so that z divides that segment in the ratio $\varepsilon : \delta$. Then $z \in N_\varepsilon(x)$ and $y \in B_\delta(z)$. In other words, $z \in N_\varepsilon(S)$ and $y \in N_\delta(N_\varepsilon(S))$.

Lemma 3. If S is a closed set then $\bigcap_{\varepsilon>0} N_\varepsilon(S) = S$.

Proof. Certainly $S \subset \bigcap_{\varepsilon>0} N_\varepsilon(S)$. On the other hand, if $x \in \bigcap_{\varepsilon>0} N_\varepsilon(S)$, then we can take a monotone decreasing sequence (ε_n) with limit equal to 0, and for each ε_n choose a point $x_n \in S$ such that $\| x - x_n \| < \varepsilon_n$. Then x is the limit of a sequence (x_n) of points in the closed set S. But for a closed set this implies that $x \in S$.

□

Lemma 4. The proposition

$$\forall x \in A \, \exists \, y \in B \text{ with } d(x,y) < \varepsilon$$

is equivalent to the proposition

$$A \subset N_\varepsilon(B) \, .$$

The proof is obvious from the definition of $N_\varepsilon(B)$.

Lemma 5.

$$d_H(S,T) = \inf\{\varepsilon > 0 \,|\, S \subset N_\varepsilon(T) \text{ and } T \subset N_\varepsilon(S)\} \, .$$

This is an immediate consequence of Lemma 4.

If we let

$$E(S,T) = \{\varepsilon > 0 \,|\, S \subset N_\varepsilon(T) \text{ and } T \subset N_\varepsilon(S)\} \, .$$

so that $d_H(S,T) = \inf E(S,T)$, then it is clear that $E(S,T)$ is an interval of the form (a, ∞) or $[a, \infty)$ where $a = d_H(S,T)$. In fact, because the sets S

and T are closed and bounded, it can be shown that the second of these alternatives is not possible, but we do not need that here.

Proposition: (H, d_H) is a metric space.

Proof:
(i) It is clear from the definition of d_H that $d_H(S,T)$ is always positive.
(ii) If $S = T$ then clearly every positive ε is in $E(S,T)$ since we can choose $x = y$ in the definition of $d_H(S,T)$. Thus $d_H(S,T) = \inf E(S,T) = 0$. Conversely, suppose $d_H(S,T) = 0$. Then $S \subset N_\varepsilon(T)$ and $T \subset N_\varepsilon(S)$ for every positive ε. By Lemma 3 this implies that $S \subset T$ and $T \subset S$ respectively.
(iii) It is clear from the definition of the Hausdorff metric that $d_H(S,T) = d_H(T,S)$.
(iv) Suppose $d_H(S,T) = a$ and $d_H(T,U) = b$. Let $\varepsilon > 0$. Then $S \subset N_{a+\varepsilon}(T)$ and $T \subset N_{b+\varepsilon}(U)$. Therefore $S \subset N_{a+\varepsilon}(N_{b+\varepsilon}(U)) = N_{a+b+2\varepsilon}(U)$. By reversing the pairs (S,T) and (T,U) we also get $U \subset N_{a+b+2\varepsilon}(S)$. That is, $a + b + 2\varepsilon \in E(S,U)$. Since ε is arbitrary this means that $\inf E(S,U) \leq a + b$. That is, $d_H(S,U) \leq a + b$.

□

The completeness of the space (H, d_H)

Theorem. (H, d_H) *is a complete metric space.*

Proof. Suppose that $\{A_k\}$ is a Cauchy sequence in (H, d_H). We must show that there is an $A \in H$ to which $\{A_k\}$ converges in the metric d_H.
We begin by defining A. We will then show that $A \in H$ and that the sequence $\{A_k\}$ really converges to A. So let

$$A = \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} A_k}\,.$$

We must first show that this set A is really a point in the space H. That is, we must show that A is closed and bounded. To show that A is closed, note that by construction $\overline{\bigcup_{k \geq n} A_k}$ is closed for each value of n. Since an intersection of closed sets is closed, it follows that A is closed.

To show that A is bounded, pick a number ε ($\varepsilon = 1$ will do.) Now choose a number N such that if $m, n \geq N$ then $d_H(A_m, A_N) < \varepsilon$. Note that the sets

$$\overline{\bigcup_{k \geq n} A_k}$$

are nested. Thus it is also true that

$$A = \bigcap_{n \geq N} \overline{\bigcup_{k \geq n} A_k}\,.$$

But here each of the sets A_k satisfies $d_H(A_k, A_N) < \varepsilon$. In particular, this means that each of these sets A_k is contained in the set B, where

$$B = \{x \in \mathbf{R}^n | \parallel x - y \parallel \leq \varepsilon \text{ for some } y \in A_N\}\,.$$

Now B is easily seen to be a closed set, so it must also contain each of the sets

$$\overline{\bigcup_{k \geq n} A_k}\,,$$

and therefore also the set A. Now A_N is a bounded set, and each point of B is within ε of some point of A_N, so B is also bounded. This proves that A is bounded.

Now we must show that $A_N \to A$ in the Hausdorff metric. That is, we must show that for every $\varepsilon > 0$ there is an integer K so that if $k \geq K$ then $d_H(A_k, A) \leq \varepsilon$. Put slightly differently, we must show that if $k \geq K$, then

$$\forall x \in A_k\, \exists z \in A \text{ such that } \parallel x - z \parallel \leq \varepsilon\,,$$

and

$$\forall z \in A\, \exists x \in A_k \text{ such that } \parallel x - z \parallel \leq \varepsilon\,,$$

So suppose $\varepsilon > 0$ given. Choose K_1 so that $k, \ell \geq K_1 \Rightarrow d_H(A_k, A_\ell) < \varepsilon$. Now suppose $x \in A_k$ for some $k \geq K_1$. Then for all $\ell \geq K_1, \exists x_\ell$ such that $\parallel x - x_\ell \parallel \leq \varepsilon$ Therefore the sequence (x_ℓ) has a convergent subsequence, say $(x_{\ell_i}) \to z$. Now for any given n, there is an index s so that $\ell_i \geq n$ for all $i \geq s$. Then

$$x_{\ell_i} \in \bigcup_{k \geq \ell_s} A_k \subset \bigcup_{k \geq n} A_k\,,$$

for $i \geq s$. Therefore

$$z \in \overline{\bigcup_{k \geq n} A_k}\,,$$

and thus $z \in A$.
Now suppose $z \in A$. Then $z \in \overline{\bigcup_{k \geq n} A_k}$ for each n. Therefore, for each n we can choose an element $x_n \in \bigcup_{k \geq n} A_k$ such that $\| x_n - z \| < \frac{1}{n}$. Note that then $x_n \to z$. Now choose K_2 so that on the one hand $1/K_2 < \varepsilon/2$, and on the other hand if $k, \ell \geq K_2$, then $d_H(A_k, A_\ell) < \varepsilon/2$. Then for $k, n \geq K_2$, we have $x_n \in A_m$ for some $m \geq n \geq K_2$. Therefore $\| x_n - z \| < \varepsilon/2$. Also $x \in A_k$, so that $\| x_n - x \| < \varepsilon/2$. Thus $\| z - x \| < \varepsilon$.
The proof is completed by letting $K = \max\{K_1, K_2\}$.

$\square$