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On the lack of monotonicity of Newton–Hewer updates for Riccati equations ☆

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ABSTRACT

We provide a set of counterexamples for the monotonicity of the Newton–Hewer method (Hewer, 1971) for solving the discrete-time algebraic Riccati equation in dynamic settings, drawing a contrast with the Riccati difference equation (Caines and Mayne, 1970).

1. Introduction

This note investigates the monotonicity properties of *iterative* methods for solving the discrete-time algebraic Riccati equation (DARE), which is given by

$$P = A^\top PA - A^\top PB(B^\top PB + R)^{-1}B^\top PA + Q. \tag{1}$$

As is well-known, given the discrete-time linear control system

$$x(t+1) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the system's state and controller at time $t \geq 0$, respectively, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and the cost function

$$J(u) = \sum_{t=0}^{\infty}\Big(x(t)^\top Qx(t) + u(t)^\top Ru(t)\Big),$$

where $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ are positive-definite matrices, and assuming the controllability of the system (A, B) and observability of $(A, Q^{1/2})$, the optimal controller which minimizes J is given by

$$u^*(t) = -(B^\top PB + R)^{-1}B^\top PAx(t),$$

where P satisfies (1), see, e.g., equation (3.3-21), p. 54 in Anderson and Moore (1990).

There are several classical iterative methods for solving the DARE in the literature, including the ones proposed in Caines and Mayne (1970), Hewer (1971), algebraic methods (Rodman & Lancaster, 1995), and semi-definite programming (Balakrishnan & Vandenberghe, 2003). In particular, these iterative methods generate a sequence of positive-definite matrices which converges to the solution of the DARE. Our main focus in this paper is on two commonly used methods, the so-called *Riccati difference equation* (Caines & Mayne, 1970), which provably converge to the fixed point solution of the DARE, and what we call the *Newton–Hewer method* which was introduced by Hewer in Hewer (1971), and uses a Newton-based update to generate a sequence of positive-definite matrices which monotonically converge to the solution of the DARE when initialized at a stable policy. Let us describe these in more detail:

The *Riccati difference equation* is given by

$$P_{t+1} = A^\top P_tA - A^\top P_tB(B^\top P_tB + R)^{-1}B^\top P_tA + Q.$$

It has been shown that the right hand side of this dynamics is monotone as a function of P_t, in the sense that $P_t \succeq \widehat{P}_t \succeq 0$ implies $P_{t+1} \succeq \widehat{P}_{t+1} \succeq 0$, see De Souza (1989, Lemma 3.1). Furthermore, this dynamics is monotone as a function of (A, B, Q, R) in the following sense: If $P_t \succeq \widehat{P}_t \succeq 0$ and

$$\begin{pmatrix} Q & A^\top \\ A & -BR^{-1}B^\top \end{pmatrix} \succeq \begin{pmatrix} \widehat{Q} & \widehat{A}^\top \\ \widehat{A} & -\widehat{B}\widehat{R}^{-1}\widehat{B}^\top \end{pmatrix}, \tag{2}$$

then $P_{t+1} \succeq \widehat{P}_{t+1} \succeq 0$, see Freiling, Jank, and Abou-Kandil (1996), Wimmer (1992). Notably and important to the discussion we will have in the next section, as long as (2) is satisfied, *this*

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monotonicity property holds even when the parameters A, B, Q, R are time-varying.

The *Newton-Hewer method* (Hewer, 1971) is given by

$$\begin{aligned} P_{t+1} &= A_t^\top P_{t+1} A_t + K_t^\top R K_t + Q, \\ A_t &= A - BK_t, \\ K_t &= (B^\top P_t B + R)^{-1} B^\top P_t A, \end{aligned} \tag{3}$$

and it has been shown that if the system is controllable, by initializing with a stable K_0; i.e., $\rho(A - BK_0) < 1$, where $\rho(\cdot)$ denotes the spectral radius, P_t converges monotonically, i.e., $P_1 \succeq P_2 \succeq \ldots \succeq P^*$, where P^* is the solution of (1).

The monotonicity property of the Riccati difference equation has been used to derive a robust stability condition for finite-horizon robust LQR problem (Zou & Gupta, 2000). In addition, a boundedness result for the solution of the Riccati difference equation has been derived using this property (De Nicolao, 1992; Freiling et al., 1996). These applications motivate us to ask the natural question whether the Newton–Hewer dynamics has the monotonicity property that Riccati difference equation enjoys. We will show in this note that this is not the case in general, i.e., $P_t \succeq \widehat{P}_t \succeq 0$ does not necessarily imply $P_{t+1} \succeq \widehat{P}_{t+1} \succeq 0$, by providing two counterexamples, each aimed to demonstrate a facet of this lack of monotonicity.

2. Counterexamples

The construction of our examples is done for the scalar case, and for this reason, we write the Newton-Hewer dynamics in this scenario. We assume that Q are R are positive real numbers, and do not change with time. By setting $n = m = 1$, the dynamics (3) can be written as:

$$P_{t+1} = \frac{A^2B^2P_t^2R + QB^4P_t^2 + 2QB^2P_tR + QR^2}{(P_tB^2 + R + AR)(P_tB^2 + R - AR)}.$$

By taking derivative, it can be shown that P_{t+1} as a function of P_t is increasing for $P_t > P^*$ and decreasing for $P_t < P^*$, where P^* is the solution to (1). For a stable policy K_t, P_t will be larger than P^* and monotonicity holds (Hewer, 1971). We have depicted P_{t+1} as a function of P_t in Fig. 1, and it can be observed that the Newton-Hewer dynamics is increasing for $P_t \geq P^*$, where P^* is at the intersection of the line $P_t = P_{t+1}$ and Newton-Hewer dynamics. We now show that if the system has time-varying Q and R, the stabilizability properties of the controller do not necessarily imply that the system is monotone, drawing a contrast with the Riccati difference equation.

To this end, note that this graph depends on A, B, R and Q, and if one of these parameters changes, P^* and the graph will change. To elaborate on this, we use Fig. 2 where we have depicted P_{t+1} as a function of P_t for two different Newton-Hewer dynamics with $(A, B, R, Q) = (1, 1, 1, 1)$ and $(A, B, R, Q) = (1, 1, 1, 2)$. In Fig. 2, the P_1^* and P_2^* refer to the solution to the DARE (1) for the systems $(A, B, R, Q) = (1, 1, 1, 1)$ and $(A, B, R, Q) = (1, 1, 1, 2)$, respectively. If $Q_t = 1, Q_{t+1} = 2$ and K_t are such that $P_1^* < P_t < P_2^*$, then the system for the next time step uses the orange graph to update P_{t+1}, and the reader can observe – we prove this with carefully chosen numerical values below – that this can lead to failure of monotonicity, i.e., $P_t \leq \widehat{P}_t$ does not necessarily imply $P_{t+1} \leq \widehat{P}_{t+1}$. Note that the system will remain monotone if $Q_{t+1} < Q_t$ for all t, in case the other system parameters A, B, and R remain fixed. Using this observation, we now explicitly construct the counterexample.

Example 2.1 (*Consider the Dynamics* (3)). Let the system be scalar, i.e., $n = m = 1$, and let $A = 1$, $B = 1$, $R = 1$ be fixed and Q_t be time-varying. Let P_t be the sequence generated by (3) at each time step. Given that A, B, R are fixed, P_t is a function of $\{Q_1, Q_2, \ldots, Q_t\}$ and K_0, where K_0 is a stable policy at time 0. Let $\widehat{P}_t$ be the sequence generated by (3) with $A = 1, B = 1, R = 1$ and $\widehat{Q}_t$ and $\widehat{K}_0$. We claim that $P_t \geq \widehat{P}_t$ does not necessarily imply that $P_{t+1} \geq \widehat{P}_{t+1}$.

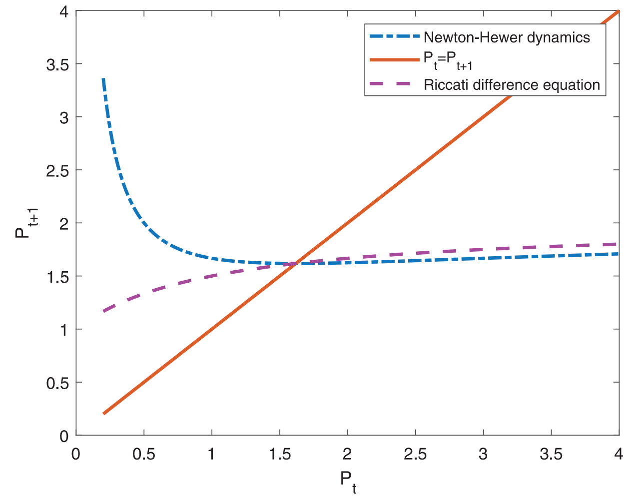

Fig. 1. P_{t+1} as a function of P_t is shown for $(A, B, R, Q) = (1, 1, 1, 1)$ of Newton-Hewer dynamics and Riccati difference equation.

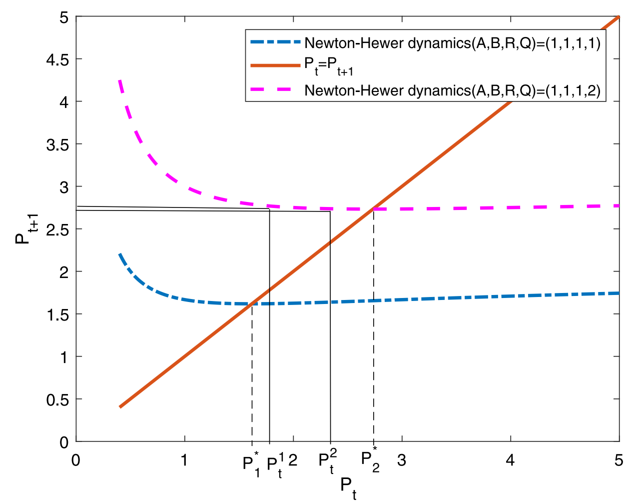

Fig. 2. P_{t+1} as a function of P_t is shown for $(A, B, R, Q) = (1, 1, 1, 1)$ and $(A, B, R, Q) = (1, 1, 1, 2)$ of Newton-Hewer dynamics.

To prove this claim, we need to chose K_0 properly. Let

$$K_0 = \sqrt{3} - 1,$$

which is a stabilizing policy. Hence, by (3) we have

$$P_1 = \frac{K_0^2 R_1 + Q_1}{1 - (A - BK_0)^2} = \frac{4 - 2\sqrt{3} + Q_1}{4\sqrt{3} - 6}.$$

Given this

$$\begin{aligned} K_1 &= \frac{BP_1A}{B^2P_1 + R_1} = \frac{4 - 2\sqrt{3} + Q_1}{2\sqrt{3} - 2 + Q_1}, \\ P_2 &= \frac{K_1^2 R_2 + Q_2}{1 - (A - BK_1)^2} \\ &= \frac{(8 - 4\sqrt{3})Q_1 + (16 - 8\sqrt{3})Q_2 + (Q_2 + 1)Q_1^2}{4(\sqrt{3} - 1)Q_1 + Q_1^2 - 68 + 40\sqrt{3}} \end{aligned}$$

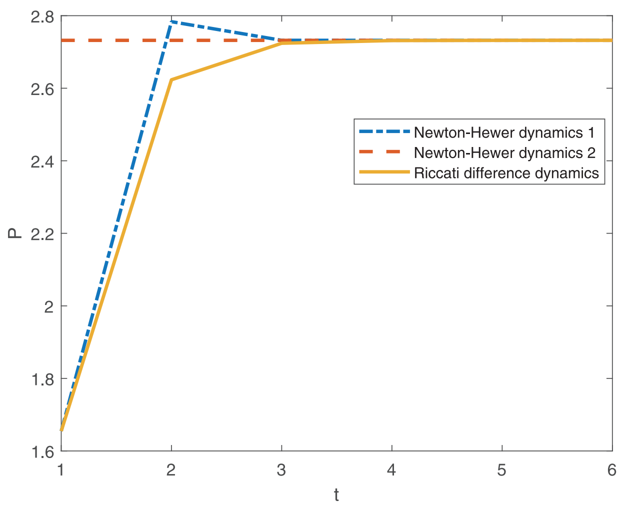

Fig. 3. This graph shows the Newton-Hewer dynamics for a system starts with $Q_1 = 1$ and $Q_t = 2$ for $t > 1$ (dotted line), Newton-Hewer dynamics for a system with $Q_t = 2$ for all t (dashed line) and Riccati difference dynamics with $Q_1 = 1$ and $Q_t = 2$ for $t > 1$ (solid line).

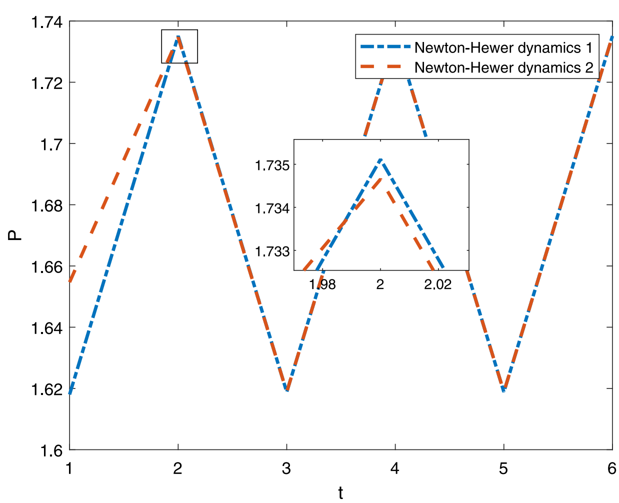

Fig. 4. Newton-Hewer dynamics for two systems with the same (A, B, R, Q_t) and different initial condition K_0.

$$+ \frac{4(\sqrt{3}-1)Q_1Q_2 + 28 - 16\sqrt{3}}{4(\sqrt{3}-1)Q_1 + Q_1^2 - 68 + 40\sqrt{3}}.$$

Now let $Q_1 = 1$ and $Q_2 = 2$ then $P_1 = 1.6547$, and $P_2 = 2.7835$. If we choose $\widehat{Q}_1 = 2$ and $\widehat{Q}_2 = 2$, then $\widehat{P}_1 = 2.7321$, and $\widehat{P}_2 = 2.7321$. This demonstrates that given $\widehat{P}_t \geq P_t$, it does not follow that $\widehat{P}_{t+1} \geq P_{t+1}$. Fig. 3 shows the sequence P_t (dotted line) and $\widehat{P}_t$ (dashed line) where $Q_1 = 1$ and $Q_t = 2$ for $t \geq 2$ and $\widehat{Q}_t = 2$. Furthermore, the sequence $\tilde{P}_t$ which is generated by the Riccati difference equation with initialization $\tilde{P}_1 = P_1$ and the same parameters $\tilde{A} = A, \tilde{B} = B, \tilde{Q}_t = Q_t, \tilde{R} = R$ is shown (solid line) for six time steps. •

We conclude with providing an example which demonstrates another aspect of lack of monotonicity of Newton-Hewer dynamics.

Example 2.2. We consider two dynamics with the same Q and R, albeit time-varying, but with different initial conditions K_0. Similar to the previous example, we assume $n = m = 1$, and $A = 1$, $B = 1$, $R = 1$ are fixed and Q_t is time-varying. We assume Q_t is 1 for odd time steps and 1.1 for even time steps. If we choose $K_0 = 0.7321$ for the first system and $\widehat{K}_0 = 0.6180$ for the second system, we will have $P_1 = 1.6180$ and $\widehat{P}_1 = 1.6547$, and for the next time, we have $P_2 = 1.7351$ and $\widehat{P}_2 = 1.7347$, which shows that the monotonicity does not hold. Fig. 4 illustrates the behaviour of two dynamics at the next time steps. •

References

Anderson, B. D., & Moore, J. B. (1990). *Optimal control: Linear quadratic methods*. Prentice-Hall.
Balakrishnan, V., & Vandenberghe, L. (2003). Semidefinite programming duality and linear time-invariant systems. *IEEE Transactions on Automatic Control*, *48*(1), 30–41.
Caines, P. E., & Mayne, D. Q. (1970). On the discrete time matrix Riccati equation of optimal control. *International Journal of Control*, *12*(5), 785–794.
De Nicolao, G. (1992). On the time-varying Riccati difference equation of optimal filtering. *SIAM Journal on Control and Optimization*, *30*(6), 1251–1269.
De Souza, C. E. (1989). On stabilizing properties of solutions of the Riccati difference equation. *IEEE Transactions on Automatic Control*, *34*(12), 1313–1316.
Freiling, G., Jank, G., & Abou-Kandil, H. (1996). Generalized Riccati difference and differential equations. *Linear Algebra and its Applications*, *241*, 291–303.
Hewer, G. (1971). An iterative technique for the computation of the steady state gains for the discrete optimal regulator. *IEEE Transactions on Automatic Control*, *16*(4), 382–384.
Rodman, L., & Lancaster, P. (1995). *Oxford mathematical monographs*, *Algebraic riccati equations*.
Wimmer, H. (1992). Monotonicity and maximality of solutions of discrete-time algebraic Riccati equations. *Mathematical Systems, Estimation, and Control*, *2*, 219–235.
Zou, J., & Gupta, Y. P. (2000). Robust stabilizing solution of the Riccati difference equation. *European Journal of Control*, *6*(4), 384–391.