# Zero-Delay Lossy Coding of Linear Vector Markov Sources: Optimality of Stationary Codes and Near Optimality of Finite Memory Codes

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Abstract—Optimal zero-delay coding (quantization) of  $\mathbb{R}^d$ -valued linearly generated Markov sources is studied under quadratic distortion. The structure and existence of deterministic and stationary coding policies that are optimal for the infinite horizon average cost (distortion) problem are established. Prior results studying the optimality of zero-delay codes for Markov sources for infinite horizons either considered finite alphabet sources or, for the  $\mathbb{R}^d$ -valued case, only showed the existence of deterministic and non-stationary Markov coding policies or those which are randomized. In addition to existence results, for finite blocklength (horizon) T the performance of an optimal coding policy is shown to approach the infinite time horizon optimum at a rate  $O(\frac{1}{T})$ . This gives an explicit rate of convergence that quantifies the near-optimality of finite window (finite-memory) codes among all optimal zero-delay codes.

Index Terms-Quantization, zero-delay coding, networked control systems.

## I. INTRODUCTION

**I** N TIME-SENSITIVE applications (such as networked control systems), causality in encoding and decoding is a natural limitation. With this motivation, in this paper we consider optimal zero-delay lossy coding for  $\mathbb{R}^d$ -valued Markov sources. In the zero-delay coding problem, the encoder encodes a source without delay and transmits it to a decoder which also operates without delay.

We assume that the source  $\{X_t\}_{t\geq 0}$  is a time-homogenous  $\mathbb{R}^d$ -valued discrete-time Markov process. For such a process, the distribution of  $\{X_t\}_{t\geq 0}$  is uniquely determined by the initial distribution  $\pi_0$  (i.e., the distribution of  $X_0$ ) and the transition kernel  $P(dx_{t+1}|x_t)$ .

The encoder encodes (quantizes) the source samples and transmits the encoded versions to a receiver over a discrete noiseless channel with finite input and output alphabet  $\mathcal{M} := \{1, 2, \dots, M\}$ . The encoder is defined by a coding

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policy  $\Pi$ , which is a sequence of Borel measurable functions  $\{\eta_t\}_{t\geq 0}$  with  $\eta_t : \mathcal{M}^t \times (\mathbb{R}^d)^{t+1} \to \mathcal{M}$ . At time *t*, the encoder transmits the  $\mathcal{M}$ -valued message

$$q_t = \eta_t(I_t)$$

where  $I_0 = X_0, I_t = (q_{[0,t-1]}, X_{[0,t]})$  for  $t \ge 1$ . Throughout the paper we use the notation  $q_{[0,t-1]} = (q_0, \ldots, q_{t-1})$  and  $X_{[0,t]} = (X_0, X_1, \ldots, X_t)$ . The set of admissible coding policies, denoted by  $\Pi_A$ , is the collection of all such zerodelay policies. Note that for fixed  $q_{[0,t-1]}$  and  $X_{[0,t-1]}$ , as a function of  $X_t$ , the encoder  $\eta_t(q_{[0,t-1]}, X_{[0,t-1]}, \cdot)$  is a Borel measurable mapping of  $\mathbb{R}^d$  into the finite set  $\mathcal{M}$ . Therefore, at each time  $t \ge 0$ , as noted in [2], the coding policy selects a quantizer  $Q_t : \mathbb{R}^d \to \mathcal{M}$  based on past information  $(q_{[0,t-1]}, X_{[0,t-1]})$ , and then quantizes  $X_t$  as  $q_t = Q_t(X_t)$ . Because of this, we refer to  $\Pi$  as a quantization policy.

The decoder without any delay generates the reconstruction  $U_t$  using decoder policy  $\gamma = {\gamma_t}_{t\geq 0}$ , where the  $\gamma_t : \mathcal{M}^{t+1} \to \mathcal{U}$ , are measurable functions for  $t \geq 0$ , with  $\mathcal{U} \subset \mathbb{R}^d$  being the reconstruction alphabet. Thus  $U_t$  is given by

$$U_t = \gamma_t(q_{[0,t]}).$$

In the finite horizon problem the goal is to minimize the average cumulative cost (distortion) for a time horizon  $T \in \mathbb{N}$  given by

$$J(\pi_0, \Pi, \gamma, T) \coloneqq \boldsymbol{E}_{\pi_0}^{\Pi, \gamma} \left[ \frac{1}{T} \sum_{t=0}^{T-1} c_0(X_t, U_t) \right], \qquad (1)$$

over the set of all admissible policies  $\Pi_A$ , where  $c_0 : \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$  is a nonnegative Borel measurable cost function (distortion measure) and  $E_{\pi_0}^{\Pi,\gamma}$  denotes expectation with initial distribution  $\pi_0$  for  $X_0$ , under the quantization policy  $\Pi$  and receiver policy  $\gamma$ .

In the infinite horizon problem, the goal is to minimize the long-term average cost (distortion) given by

$$J(\pi_0, \Pi, \gamma) \coloneqq \limsup_{T \to \infty} \boldsymbol{E}_{\pi_0}^{\Pi, \gamma} \left[ \frac{1}{T} \sum_{t=0}^{T-1} c_0(X_t, U_t) \right],$$

over all admissible policies.

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### A. Brief Literature Review and Contributions

Two important structural results for the finite horizon problem (1) have been developed by Witsenhausen [3], and Walrand and Varaiya [4]. These results are stated in the following two theorems. We adopt the presentation given in [2].

Theorem 1 ([3]): For the finite horizon problem, any zerodelay quantization policy  $\Pi = {\eta_t}$  can be replaced, without any loss in performance, by a policy  $\hat{\Pi} = {\hat{\eta}_t}$  which only uses  $q_{[0,t-1]}$  and  $X_t$  to generate  $q_t$ , i.e., such that  $q_t = \hat{\eta}_t(q_{[0,t-1]}, X_t)$  for all  $t = 1, \ldots, T - 1$ .

For a complete, separable, and metric (Polish) space  $\mathcal{X}$  and its Borel sets  $\mathcal{B}(\mathcal{X})$ , let  $\mathcal{P}(\mathcal{X})$  denote the space of probability measures on  $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$  equipped with the topology of weak convergence. Given a quantization policy  $\Pi$ , for all  $t \geq 1$  let  $\pi_t \in \mathcal{P}(\mathbb{R}^d)$  be the regular conditional probability defined by

$$\pi_t(A) \coloneqq P(X_t \in A | q_{[0,t-1]})$$

for any Borel set  $A \in \mathcal{B}(\mathbb{R}^d)$ .

The following result is by Walrand and Varaiya [4] where finite-alphabet sources were studied. In [6] this result was extended to the more general case of  $\mathbb{R}^d$ -valued sources.

Theorem 2 ([4], [6]): For the finite horizon problem, any zero-delay quantization policy can be replaced, without loss in performance, by a policy which at any time  $t = 1, \ldots, T - 1$  only uses the conditional probability measure  $\pi_t = P(dx_t|q_{[0,t-1]})$  and  $X_t$  to generate  $q_t$ . In other words, at time t such a policy  $\hat{\eta}_t$  uses  $\pi_t$  to select a quantizer  $Q_t = \hat{\eta}(\pi_t)$ , where  $Q_t : \mathbb{R}^d \to \mathcal{M}$ , and then  $q_t$  is generated as  $q_t = Q_t(X_t)$ .

We call a policy of the type in Theorem 2 a Walrand-Varaiya-type policy. Such a policy is also called a Markov coding policy. In the literature several results related to zero delay coding and causal coding are available. Notably, [7] and [8] consider causal lossy source coding where the reconstruction of the present source sample is restricted to be a function of the present and past source samples, while the code stream itself may be non-causal and have variable rate. In [7] it was shown that for memoryless sources, causal source coding cannot achieve any of the vector quantization advantages. In addition, [7] also showed that for stationary memoryless sources, an optimal causal coder can be replaced by one that time shares between at most two memoryless coders, without loss in performance. In [9], results on causal coding by Neuhoff and Gilbert are extended to (stationary) sources with memory, under high resolution conditions for mean squared error distortion.

Structural results for the finite horizon coding problem have been developed in a number of papers. As mentioned before, the classic works by Witsenhausen [3] and Walrand and Varaiya [4], which use two different approaches, are of particular relevance. An extension to the more general setting of non feedback communication was given by Teneketzis [5], and [6] also extended these results to more general state spaces; see also [2] and [30] for a more detailed overview. Optimal zero delay coding of Markov sources over noisy channels without feedback was considered in [5] and [31]. We refer to [32]–[34] for further results on zero-delay or causal coding in multi-user systems.

In this paper we also investigate how fast the optimum finite blocklength (time horizon) distortion converges to the optimum (infinite horizon) distortion. An analog of this problem in block coding is the speed of convergence of the finite block length encoding performance to Shannon's distortion rate function. For stationary and memoryless sources, this speed of convergence was shown to be of the type  $O\left(\frac{\log T}{T}\right)$  [10], [11]. See also [12] for a detailed literature review and further finite blocklength performance bounds.

A large body of work involves convex analytic or information theoretic relaxation of the operational problem presented above, where the constraint on the number of bits is replaced with entropy (which may replace the fixed-rate with variablerate constraints) or mutual information constraints (which has a more relaxed, Shannon theoretic infinite-dimensional, interpretation); see [30, Section 5.4] for a detailed discussion. In this case, the analysis often relies on deriving lower bounds and upper bounds on the optimal performance, or establishing asymptotic tightness conditions.

For lower bounds, primary methods build on Shannon lower bounding techniques (and the Gaussian measure's extremal properties), entropy-power inequality based bounds, or a *sequential-rate distortion theoretic* formulation where the minimization of directed mutual information is performed over causal kernels as in [13], and which has been investigated further in a series of recent publications including [14]–[19], [25]–[29].

Related to the above, when an actual channel is present, using channel-source coding separation based methods via the rate-distortion function and Shannon capacity dualities also leads to useful bounds. Perhaps the earliest papers giving such formulations are [35], [36] and [37]. Transmission over scalar Gaussian channels has been also studied in [35], [38] and [39], where the error exponents were shown to be unbounded (and the error probability was shown to decrease at least doubly exponentially in the block-length). Transmission of linear Gaussian sources over Gaussian channels (a matched pair, in the sense of rate-distortion achieving and capacity achieving properties of Gaussian models), in the scalar setup was considered in [40], [41], where the latter arrived at tightness of information theoretic inequalities; this result has been re-discovered later but also with some generalizations (e.g., [20] is a recent work considering linear systems and Gaussian channels in the presence of side information).

For upper bounds, methods based on high-rate quantization (and the corresponding uniform quantization and space filling analysis), dithering (allowing for uniformization), and entropypower inequalities (further refining Gaussian based bounds) have been studied; see e.g., [18], [21]–[24], [27])

In this paper we study linear Markovian systems driven by noise and consider the quadratic cost (mean squared distortion). Even though such systems are likely the most important and commonly adopted ones in applications (in systems and control theory, signal processing, and in estimation theory), their analysis in the context of zero-delay coding are quite challenging since the costs are not bounded. Accordingly, we will develop a number of results to address these technical challenges. To make the presentation accessible, many of the technical results will be presented in the appendix.

1) Contributions: We assume that the  $\mathbb{R}^d$ -valued source is a linearly generated stable Markov process and consider zerodelay quantization policies where the quantizers have convex codecells and the cost function is the squared distortion. Under these assumptions, our main result, Theorem 4, demonstrates the existence of globally optimal *deterministic* and *timeinvariant* (stationary) Markov policies. In addition, we also show that the (optimum) performance of such a policy for a finite time horizon T converges to the infinite-horizon optimal performance at a rate  $O(\frac{1}{T})$ .

The following papers studying the infinite horizon average cost optimality in the fixed-rate zero-delay quantization are most relevant to our work:

- In [42] a formulation for optimal average-cost zerodelay coding as an infinite horizon optimal stochastic control problem was introduced; this formulation has been an inspiration for our analysis. In particular, in [42] a stochastic control formulation of zero-delay quantization was given under more restrictive assumption than in this paper: the set of admissible quantizers in that paper was restricted to the set of nearest neighbor quantizers, and other conditions were placed on the dynamics of the system. In contrast, we impose the more relaxed assumption that the quantizers have convex codecells (this class of quantizers includes the set of nearest neighbor quantizers). Furthermore the proof technique used in [42] relies on the fact that the source is partially observed unlike in our case. As noted in [6], for the partially observed case, the structure of the encoder decoder pairs considered in [42] is suboptimal since the measurements are not Markovian.
- In [43], the source was assumed to have finite alphabet; however, in our case the source is taking values in R<sup>d</sup> and in this sense the present paper generalizes [43] to the technically more demanding continuous source case. On the other hand, [43] established a global optimality result with no restrictions on the structure of quantizers. Here, we impose codecell convexity for technical reasons.
- Finally, in [2] only the optimality of deterministic and *non-stationary* encoding policies, or of *randomized and stationary* policies were established, and here we prove the optimality and existence of *stationary and deterministic* quantization policies and also obtain convergence rates for finite-memory codes, thereby generalizing [2] in these two aspects.

The paper is organized as follows. In Section II, after reviewing some definitions we transform the problem into Markov decision process (MDP) framework, and we provide some preliminary results. The main result, Theorem 4, is presented in Section III; to prove it we consider the discounted infinite horizon problem followed by infinite horizon average cost problem and the proof of Theorem 4. Some background material on MDPs along with useful lemmas and theorems are presented in Appendix A. Finally, some proofs are relegated to Appendix B.

#### **II. PRELIMINARIES AND SOME SUPPORTING RESULTS**

In this section we present some properties of quantizers, from a different viewpoint than is usual in source coding, that will be important in the sequel.

A sequence of probability measures  $\{\mu_n\}$  on  $\mathbb{R}^d$  is said to converge *weakly* to  $\mu \in \mathcal{P}(\mathbb{R}^d)$  if for every continuous and bounded  $f : \mathbb{R}^d \to \mathbb{R}$ , we have  $\int_{\mathbb{R}^d} f(x)\mu_n(dx) \to \int_{\mathbb{R}^d} f(x)\mu(dx)$ .

For  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ , the *total variation* metric is defined as

$$\|\mu - \nu\|_{TV} = \sup_{g: \|g\|_{\infty} \le 1} \left| \int_{\mathbb{R}^d} g(x) \mu(dx) - \int_{\mathbb{R}^d} g(x) \nu(dx) \right|$$
(2)

where the supremum is over all measurable real g such that  $||g||_{\infty} = \sup_{x \in \mathbb{R}} |g(x)| \le 1.$ 

Definition 1 ([44]): The space of probability measures with finite second moment is

$$\mathcal{P}_2(\mathbb{R}^d) \coloneqq \{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int \|x\|^2 \mu(dx) < \infty \}$$

where  $\|\cdot\|$  denotes the Euclidean  $(l_2)$  norm.

Definition 2 ([44]): The order-2 Wasserstein distance for two probability distributions  $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$  is defined as

$$\rho_2(\mu,\nu) = \inf_{\lambda \in \mathcal{H}(\mu \times \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 \lambda(dx, dy) \right)^{\frac{1}{2}},$$

where  $\mathcal{H}(\mu \times \nu)$  denotes the set of probability measures on  $\mathbb{R}^d \times \mathbb{R}^d$  with first marginal  $\mu$  and second marginal  $\nu$ .

For compact subsets of  $\mathbb{R}^d$ , the Wasserstein distance of order 2 metrizes the weak topology on the set of probability measures on  $\mathbb{R}^d$  (see [44, Theorem 6.9]). For non-compact subsets, weak convergence combined with convergence of second moments (that is of  $\int ||x||^2 \mu_n(dx) \rightarrow \int ||x||^2 \mu(dx)$ ) is equivalent to convergence in order-2 Wasserstein distance.

Definition 3: An *M*-cell quantizer Q on  $\mathbb{R}^d$  is a (Borel) measurable mapping  $Q : \mathbb{R}^d \to \mathcal{M}$ . We let  $\mathcal{Q}$  denote the collection of all *M*-cell quantizers on  $\mathbb{R}^d$ .

Observe that each  $Q \in Q$  is uniquely characterized by its quantization cells (or bins)  $B_i = Q^{-1}(i) = \{x : Q(x) = i\}, i = 1, ..., M$  which form a measurable partition of  $\mathbb{R}^d$ .

Definition 4: An (admissible) quantization policy  $\Pi = \{\eta_t\}_{t\geq 0}$  belongs to  $\Pi_W$  (i.e., it is a Walrand-Varaiya type policy) if there exist a sequence of mappings  $\{\hat{\eta}_t\}$  of the type  $\hat{\eta}_t : \mathcal{P}(\mathbb{R}^d) \to \mathcal{Q}$  such that for  $Q_t = \hat{\eta}_t(\pi_t)$  we have  $q_t = Q_t(X_t) = \eta_t(I_t)$ .

Suppose we use a quantizer policy  $\Pi = {\hat{\eta}_t} \in \Pi_W$ . Then, using standard properties of conditional probability, building on [2] we can obtain the following filtering equation for the evolution of  $\pi_t$ :

$$\pi_{t+1}(dx_{t+1}) = \frac{P(dx_{t+1}, q_t | q_{[0,t-1]})}{P(q_t | q_{[0,t-1]})}$$

$$= \frac{\int_{\mathbb{R}^d} \pi_t(dx_t) P(q_t | \pi_t, x_t) P(dx_{t+1} | x_t)}{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \pi_t(dx_t) P(q_t | \pi_t, x_t) P(dx_{t+1} | x_t)}$$

$$= \frac{1}{\pi_t(Q_t^{-1}(q_t))} \int_{Q_t^{-1}(q_t)} \pi_t(dx_t) P(dx_{t+1} | x_t).$$
(3)

Thus  $\pi_{t+1}$  depends only on  $\pi_t, Q_t$ , and  $q_t$ , which implies that  $\pi_{t+1}$  is conditionally independent of  $(\pi_{[0,t-1]}, Q_{[0,t-1]})$ given  $\pi_t$  and  $Q_t$ . Thus,  $\{\pi_t\}$  can be viewed as  $\mathcal{P}(\mathbb{R}^d)$ -valued controlled Markov process [45] (see also Appendix A), with Q-valued control  $\{Q_t\}$  having transition kernel  $P(d\pi'|\pi, Q)$ determined by (3). The average cost up to time T-1 is given by (see also [2])

$$\boldsymbol{E}_{\pi_{0}}^{\Pi} \left[ \frac{1}{T} \sum_{t=0}^{T-1} c(\pi_{t}, Q_{t}) \right] = \inf_{\gamma} J(\pi_{0}, \Pi, \gamma, T), \qquad (4)$$

where

$$c(\pi_t, Q_t) \coloneqq \sum_{i=1}^{M} \inf_{u \in \mathcal{U}} \int_{Q_t^{-1}(i)} \pi_t(dx) c_0(x, u).$$
(5)

For the mean squared distortion  $c_0(x, u) = ||x-u||^2$  (which is our focus), the optimum receiver  $\gamma_t$  at time t is explicitly given by

$$\gamma_t(i) = \frac{1}{\pi(Q_t^{-1}(i))} \int_{Q_t^{-1}(i)} x \pi_t(dx), \quad i = 1, \dots, M.$$
(6)

Definition 5 ([2]): Let  $\mathcal{G}$  denote the set of all probability measures on  $\mathbb{R}^d$  admitting densities that are bounded by C and Lipschitz with constant  $C_1$ .

In [2, Lemma 3] it is shown that  $\mathcal{G}$  is closed in  $\mathcal{P}(\mathbb{R}^d)$ . Note that  $\mathcal{G}$  is also closed in  $\mathcal{P}_2(\mathbb{R}^d)$ , since the Wasserstein convergence is stronger than the weak convergence. Let  $\mathcal{Z} := \mathcal{G} \cap \mathcal{P}_2(\mathbb{R}^d)$ , be the intersection of  $\mathcal{G}$  and  $\mathcal{P}_2(\mathbb{R}^d)$ .

Remark 1: Due to our assumptions on the source  $\{X_t\}$  (see Section III), the distribution of  $X_{t+1}$  has (conditional) density function  $\phi(\cdot|x)$  given  $X_t = x$ , which is positive everywhere, bounded, and Lipschitz uniformly in x. Thus (with appropriate constants C and  $C_1$ ),  $P(dx_{t+1}|x_t) \in \mathcal{G}$  for all  $x_t \in \mathbb{R}^d$  and thus the filtering equation (3) implies that under any policy  $\Pi \in \Pi_W$ , we have  $\pi_t \in \mathcal{G}$  for all  $t \ge 0$  if  $\pi_0 \in \mathcal{G}$ . The assumptions on the source will also imply that  $\pi_t$  has finite second moment (with probability one) for all  $t \ge 0$  if  $\pi_0$  has finite second moment (see (18)), so we obtain  $\pi_t \in \mathcal{Z}$  for all  $t \ge 0$  if  $\pi_0 \in \mathcal{Z}$ . Thus we can make  $\mathcal{Z}$  the state space of our Markov decision process.

From now on, we restrict the set of quantizers to quantizers having convex cells [2]. Formally, this quantizer class  $Q_c$  is defined by

$$\mathcal{Q}_c = \{ Q \in \mathcal{Q} : Q^{-1}(i) \subset \mathbb{R}^d \text{ is convex for } i = 1, \dots, M \}.$$

Thus, we replace Q with  $Q_c$  in Definition 4 to obtain the new class of policies denoted by  $\Pi_W^C$ .

Definition 6: We denote by  $\Pi_W^C$  the set of all quantization policies  $\Pi = {\hat{\eta}_t} \in \Pi_W$  such that  $\hat{\eta}_t : \mathcal{P}(\mathbb{R}^d) \to \mathcal{Q}_c$ , i.e.,  $Q_t = \hat{\eta}_t(\pi_t) \in \mathcal{Q}_c$  for all  $t \ge 0$ . Furthermore,  $\Pi_{WS}^C$ denotes the set of all quantization policies in  $\Pi_W^C$  that are stationary, i.e., the policy  ${\hat{\eta}_t}$  does not depend on the time index t.

Remark 2:

(i) The set  $\Pi_W^C$  is called the set of Markov quantization policies and  $\Pi_{WS}^C$  is called the set of stationary Markov quantization policies.

(ii) The convex codecell restriction may lead to suboptimality in general; however it includes the class of nearest neighbor quantizers studied in [42]. For multiresolution scalar quantizers (MRSQ) and the squared error distortion measure, [46], [47] showed that for discrete and continuous sources (even with bounded continuous densities), optimal fixed rate multiresolution scalar quantizers cannot have only convex codecells, proving that the convex codecells assumption leads to a loss in performance. We introduce the convex codecell assumption for technical reasons; without this assumption the analysis of recursive policies seems very hard. Indeed, the parametric representation of convex codecell quantizers allowed [48] to establish compactness and desired convergence properties. In particular, in the absence of such a condition, it was shown in [48, p. 878] that the space of quantizers is not closed under weak convergence.

Following [2] and [48], in order to facilitate the stochastic control analysis of the quantization problem we will use an alternative representation of quantizers. A quantizer  $Q : \mathbb{R}^d \to \mathcal{M}$  with cells  $\{B_1, \ldots, B_M\}$ , can also be identified with the stochastic kernel (regular conditional probability) on  $\mathcal{M}$  given  $\mathbb{R}^d$ , also denoted by Q, defined by

$$Q(i|x) = 1_{\{x \in B_i\}}, \quad i = 1, \dots, M.$$

As in [2], [48], we say that a sequence of quantizers  $Q_n$ converges to Q at  $P \in \mathcal{P}(\mathbb{R}^d)$  if  $PQ_n \to PQ$ , where  $\mu \bar{Q}$ denotes the probability measure on  $\mathbb{R}^d \times \mathcal{M}$  induced by a  $\mu \in \mathcal{P}(\mathbb{R}^d)$  and a conditional probability  $\bar{Q}$  on  $\mathcal{M}$  given  $\mathbb{R}^d$ . Here we consider convergence in the order-2 Wasserstein distance. We note that by [2, Lemma 2], the convergence of quantizers with convex codecells holds simultaneously for all admissible input probability measures in  $\mathcal{Z}$  and accordingly we will not need to specify P explicitly.

The following lemma shows the compactness of  $Q_c$  in the order-2 Wasserstein topology. The proof is given in Appendix B.

Lemma 1:  $Q_c$  is compact in the order-2 Wasserstein topology at any input  $P \in \mathcal{Z}$ .

From now on we assume the mean squared distortion  $c_0(x, u) = ||x - u||^2$ . The following lemmas are proved in Appendix B.

Lemma 2: The cost function  $c(\pi, Q)$  is lower semicontinuous in  $(\pi, Q)$ , that is, when  $(\pi_n, Q_n) \to (\pi, Q)$  (in order-2 Wasserstein distance), then

$$\liminf_{n \to \infty} c(\pi_n, Q_n) \ge c(\pi, Q).$$

Also,  $c(\pi, Q)$  is continuous in Q for every fixed  $\pi \in \mathcal{Z}$ , i.e., if  $Q_n \to Q$ , then  $c(\pi, Q_n) \to c(\pi, Q)$ .

Recall the transition probability  $P(d\pi'|\pi, Q)$  of our MDP determined by the filtering equation (3).

*Lemma* 3: The function  $Pg(\pi, Q) \coloneqq \int_{\mathcal{P}_2(\mathbb{R}^d) \times \mathcal{Q}_c} g(\pi') P(d\pi' | \pi, Q)$  is continuous in  $(\pi, Q)$  (i.e. is continuous when  $(\pi_n, Q_n) \to (\pi, Q)$  in order-2 Wasserstein distance on  $\mathcal{Z} \times \mathcal{Q}_c$ ), for every continuous bounded function  $g : \mathcal{Z} \to \mathbb{R}$ . Moreover, for any fixed  $\pi \in \mathcal{Z}$ ,  $Pg(\pi, Q)$  is continuous in  $Q \in \mathcal{Q}_c$  for any continuous function g. In the theory of Markov decision processes (MDPs) (see also Appendix A), the so-called *measurable selection condition* ([45, Assumption 3.3.1]) guarantees the measurability of the value function and existence of a minimizer ([45, Theorem 3.3.5]). The following assumption, which is stated for the Markov control model of our zero-delay quantization setup, is referred to as the *measurable selection condition*.

Assumption 1: [45, Assumption 3.3.1] The Markov control model and a given measurable function  $u : \mathbb{Z} \to \mathbb{R}$  are such that  $u^* : \mathbb{Z} \to \mathbb{R}$  defined by

$$u^*(\pi) \coloneqq \inf_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \int_{\mathcal{Z}} u(\pi') P(d\pi' | \pi, Q) \right), \quad \pi \in \mathcal{Z}$$

is measurable and there exist a measurable  $\hat{\eta} : \mathbb{Z} \to \mathcal{Q}_c$  such that for any  $\pi \in \mathbb{Z}$ ,  $Q = \hat{\eta}(\pi)$  attains the minimum at  $\pi$ , i.e.,

$$u^{*}(\pi) = c(\pi, \hat{\eta}(\pi)) + \int_{\mathcal{Z}} u(\pi') P(d\pi' | \pi, \hat{\eta}(\pi)), \text{ for all } \pi \in \mathcal{Z}.$$
(7)

The following is a sufficient condition for the Assumption 1 to hold. Note that conditions (i)-(iii) hold in our setting by Lemmas 1, 2, and 3. Therefore Theorem 3 below holds for our model.

Condition 1:

- (i) The quantizer space (i.e. action space)  $Q_c$  is compact for every fixed  $\pi$ .
- (ii) The one-stage cost function  $c(\pi, Q)$  is lower semicontinuous in  $(\pi, Q)$ .
- (iii) The transition kernel P is such that

$$Pg(\pi, Q) \coloneqq \int_{\mathcal{P}_2(\mathbb{R}^d) \times \mathcal{Q}_c} g(\pi') P(d\pi' | \pi, Q)$$

is continuous in  $(\pi, Q)$  for every continuous and bounded g on  $\mathcal{Z}$ .

Theorem 3: [45, Theorem 3.3.5] Condition 1 implies Assumption 1 for any nonnegative measurable  $u : \mathbb{Z} \to \mathbb{R}$ . Moreover if, u is nonnegative and lower semi-continuous then the function  $u^*$  in (7) is lower semi-continuous.

## III. INFINITE HORIZON PROBLEM OF LINEAR SYSTEMS UNDER QUADRATIC COST

We consider the linear system given in the following assumption.

Assumption 2: The source  $\{X_t\}$  can be expressed in the linear stochastic realization form

$$X_{t+1} = AX_t + W_t, \tag{8}$$

where A is a  $d \times d$  real matrix and  $W_t$  is an independent and identically distributed (i.i.d.) vector noise sequence which is independent of  $X_0$ . Moreover, assume the following:

- (i) The maximum singular value of A, denoted by α, is less than 1 (i.e. maximum eigenvalue of the matrix A'A is less than 1, where A' is the transpose of the matrix A).
  (ii) U = R<sup>d</sup>.
- (iii) The cost for the pair (x, u) is given by  $c_0(x, u) = ||x u||^2$ .

- (iv) The  $W_t$  have a common probability density function  $\varphi$  that is positive, bounded, and Lipschitz continuous.
- (v)  $\sigma^2 \coloneqq \boldsymbol{E}[\|W_t\|^2] < \infty.$
- (vi) The initial distribution  $\pi_0$  for  $X_0$  admits a density such that  $E_{\pi_0}[||X||^2] < \infty$  or it is a point mass  $\pi_0 = \delta_{x_0}$ .

Note that assumption (iv) implies that for each fixed  $x \in \mathbb{R}^d$ , the distribution of  $X_{t+1}$ , (i.e.  $Ax + W_t$ ), has (conditional) density function  $\phi(\cdot|x)$  which is positive everywhere, bounded, and Lipschitz uniformly in x. Thus (with appropriate constants C and  $C_1$ ) we have  $\phi(dy|x) \in \mathcal{G}$  for all  $x \in \mathbb{R}^d$ , where  $\mathcal{G} \subset \mathcal{P}(\mathbb{R}^d)$  was defined in Definition 5. As we observed in Remark 1, this implies that  $\pi_t \in \mathcal{Z}$  for all  $t \ge 0$ .

For any initial distribution  $\pi_0 \in \mathbb{Z}$ , the long-term (infinitehorizon) minimum cost (distortion) of a quantization policy  $\Pi \in \mathbf{\Pi}_W^C$  is

$$J(\pi_0, \Pi) \coloneqq \limsup_{T \to \infty} \boldsymbol{E}_{\pi_0}^{\Pi} \left[ \frac{1}{T} \sum_{t=0}^{T-1} c(\pi_t, Q_t) \right]$$

and the optimal cost over all policies in  $\Pi_W^C$  is

$$J(\pi_0) \coloneqq \inf_{\Pi \in \mathbf{\Pi}_W^C} J(\pi_0, \Pi).$$

Our main result is the following theorem. *Theorem 4:* 

(i) Under Assumption 2, for any initial distribution  $\pi_0$ 

$$J(\pi_0, \Pi^*) = \inf_{\Pi \in \Pi_W^C} J(\pi_0, \Pi) = \min_{\Pi \in \Pi_{WS}^C} J(\pi_0, \Pi).$$
(9)

That is, there exists a deterministic and stationary policy  $\Pi^* \in \mathbf{\Pi}_{WS}^C$  that achieves the minimum above.

(ii) Furthermore, the finite horizon distortion of the optimal policy Π converges to its infinite horizon distortion at a rate O(<sup>1</sup>/<sub>T</sub>); in particular, for all π<sub>0</sub> and T ≥ 1,

$$\frac{1}{T} \boldsymbol{E}_{\pi_0}^{\Pi^*} \left[ \sum_{t=0}^{T-1} c(Q_t, \pi_t) \right] - J(\pi_0, \Pi^*) \right| \le \frac{K(\pi_0)}{T},$$

where  $K(\pi_0) < \infty$  only depends on  $\pi_0$ .

We will prove the theorem in Section III-B after obtaining auxiliary existence and optimality results for the easier-tohandle discounted cost problem in the next section.

### A. The Discounted Cost Problem

The discounted cost for some  $\beta \in (0, 1)$  and time horizon  $T \ge 1$  is defined as

$$J^{\beta}(\pi_0, \Pi, T) \coloneqq \boldsymbol{E}_{\pi_0}^{\Pi} \left[ \sum_{t=0}^{T-1} \beta^t c(\pi_t, Q_t) \right], \qquad (10)$$

and for the infinite horizon case,

$$J^{\beta}(\pi_0, \Pi) \coloneqq \boldsymbol{E}_{\pi_0}^{\Pi} \left[ \sum_{t=0}^{\infty} \beta^t c(\pi_t, Q_t) \right],$$

where  $c(\pi_t, Q_t)$  is defined in (5).

The goal is to find optimal policies that achieve

$$J^{\beta}(\pi_0) \coloneqq \inf_{\Pi \in \Pi^C_W} J^{\beta}(\pi_0, \Pi).$$
(11)

We call  $J^{\beta}$  the discounted *value function* of the MDP. Let us On the other hand, by the properties of the process we have, define  $[|| \mathbf{v} || 2]$ 

$$J^{\beta}(\pi_0, T) \coloneqq \inf_{\Pi \in \mathbf{\Pi}_W^C} J^{\beta}(\pi_0, \Pi, T),$$

so that we have

$$J^{\beta}(\pi_0) \ge \limsup_{T \to \infty} J^{\beta}(\pi_0, T).$$
(12)

Since  $J^{\beta}(\pi_0, T)$  is monotonically increasing in T, the limit superior becomes a limit and thus

$$J^{\beta}(\pi_0) \ge \lim_{T \to \infty} J^{\beta}(\pi_0, T).$$
(13)

Let  $v : \mathcal{Z} \to \mathbb{R}$  be lower semicontinuous and define the operator  $\mathbb{H}$  by

$$(\mathbb{H}v)(\pi) \coloneqq \min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \beta \int_{\mathcal{Z}} v(\pi_1) P(d\pi_1 | \pi, Q) \right).$$
(14)

Note that  $\mathbb{H}$  indeed maps lower semicontinuous functions into lower semicontinuous functions by Theorem 3. The discounted cost optimality equation (DCOE) is defined by

$$v(\pi) = (\mathbb{H}v)(\pi), \quad \pi \in \mathcal{Z}.$$
(15)

The following theorem is a version of a widely used result in the theory of Markov decision processes.

Theorem 5: Suppose Assumption 2 holds. Then, the value function  $J^{\beta}(\pi_0)$  is a fixed point of the operator  $\mathbb{H}$ , i.e.

$$J^{\beta} = \mathbb{H}J^{\beta}.$$
 (16)

Furthermore, there exists a deterministic stationary policy  $\Pi =$  $\{\hat{\eta}\} \in \mathbf{\Pi}_{WS}^C$  that is optimal, i.e.,  $J^{\beta}(\pi_0) = J^{\beta}(\pi_0, \Pi)$  for all  $\pi_0 \in \mathcal{Z}$  and this policy satisfies for all  $\pi_0 \in \mathcal{Z}$ ,

$$J^{\beta}(\pi_{0}) = c(\pi_{0}, \hat{\eta}(\pi_{0})) + \beta \int_{\mathcal{Z}} J^{\beta}(\pi') P(d\pi'|\pi_{0}, \hat{\eta}(\pi_{0})).$$
(17)

Since our setup is quite non-standard, we will have to give a separate proof of Theorem 5 after stating and proving some preliminary result. In what follows  $E_{\pi_0}[||X_t||^2]$  denotes the second moment of  $X_t$  when  $X_0 \sim \pi_0$  and  $E_{\pi_0}[||X||^2] =$  $\boldsymbol{E}_{\pi_0}[\|X_0\|^2] = \int_{\mathbb{R}^d} \|x\|^2 \pi_0(dx).$ 

*Lemma 4:* For every initial distribution  $\pi_0 \in \mathbb{Z}$ , the value function  $J^{\beta}(\pi_0)$ , and hence also  $J^{\beta}(\pi_0, T)$ , is uniformly bounded as

$$J^{\beta}(\pi_{0},T) \leq J^{\beta}(\pi_{0})$$
  
$$\leq \frac{1}{1-\beta} \left( \boldsymbol{E}_{\pi_{0}}[\|X\|^{2}] + \frac{1}{1-\alpha}\sigma^{2} \right).$$

Proof: Note that the receiver defined in (6) minimizes  $\int ||x - \gamma(Q(x))||^2 \pi(dx)$  over all  $\gamma$  and hence,

$$c(\pi, Q) = \min_{\gamma} \int_{\mathbb{R}^d} \|x - \gamma(Q(x))\|^2 \pi(dx) \le \int_{\mathbb{R}^d} \|x\|^2 \pi(dx)$$

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$$E_{\pi_{0}}[\|X_{t}\|^{-}] = E_{\pi_{0}}[\|AX_{t-1} + W_{t-1}\|^{2}] = E_{\pi_{0}}[X_{t-1}A'AX_{t-1} + W'_{t-1}W_{t-1}] \leq E_{\pi_{0}}[\alpha\|X_{t-1}\|^{2}] + \sigma^{2} \leq E_{\pi_{0}}[\alpha^{t}\|X_{0}\|^{2}] + (\alpha^{t-1} + \ldots + \alpha + 1)\sigma^{2} \leq E_{\pi_{0}}[\|X\|^{2}] + \frac{1}{1-\alpha}\sigma^{2} < \infty.$$
(18)

Therefore, we can bound  $J^{\beta}(\pi_0)$  as

$$J^{\beta}(\pi_{0}) \leq J^{\beta}(\pi_{0}, \Pi) \leq \boldsymbol{E}_{\pi_{0}}^{\Pi} \left[ \sum_{t=0}^{\infty} \beta^{t} c(\pi_{t}, Q_{t}) \right]$$
$$\leq \boldsymbol{E}_{\pi_{0}}^{\Pi} \left[ \sum_{t=0}^{\infty} \beta^{t} \|X_{t}\|^{2} \right]$$
$$\leq \frac{1}{1-\beta} \left( \boldsymbol{E}_{\pi_{0}}[\|X\|^{2}] + \frac{1}{1-\alpha} \sigma^{2} \right).$$
(19)

This together with (13) yields the lemma.

The following is a key equicontinuity lemma which is related to, but different from, Lemma 1 in [43]. The proof is also related to the approach of Borkar [49] (see also [50] and [42]), but our argument is different (and more direct) since the absolute continuity conditions in [49] are not applicable here due to quantization. As in [42], in the proof we will enlarge the space of admissible coding policies to allow for randomization at the encoder. Since for a discounted infinite horizon optimal encoding problem optimal policies are deterministic even among possibly randomized policies, allowing randomness does not change the optimal performance.

Lemma 5: Suppose the source is generated as in (8) and Assumption 2 holds. Then for any initial two distributions  $\mu_0, \nu_0 \in \mathbb{Z}$ , and any  $\beta \in (0, 1)$ , we have

$$\left|J^{\beta}(\nu_{0}) - J^{\beta}(\mu_{0})\right| \leq \left(\frac{\rho_{2}(\nu_{0},\mu_{0})}{1-\alpha} + \frac{2K_{1}}{1-\sqrt{\alpha}}\right)\rho_{2}(\nu_{0},\mu_{0}),$$
(20)

where  $K_1$  is a finite constant and  $\rho_2(\nu_0, \mu_0)$  is the order-2 Wasserstein distance of the two initial distributions.

 $\mathbb{R}^d \times \mathbb{R}^d$ -valued the *Proof:* Consider process  $\{(X_t,Y_t)\}_{t\geq 0}$  such that  $\{X_t\}_{t\geq 0} \sim (\nu_0,P), \{Y_t\}_{t\geq 0} \sim$  $(\mu_0, P), (X_0, Y_0) \sim \lambda$  where  $\lambda \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$  with marginals  $\nu_0$  and  $\mu_0$  respectively. We further assume identical noise realization  $W_t$  for these processes. Assume without loss of generality that  $J^{\beta}(\nu_0) - J^{\beta}(\mu_0) \ge 0$ . Then

$$|J^{\beta}(\nu_{0}) - J^{\beta}(\mu_{0})| = J^{\beta}(\nu_{0}) - J^{\beta}(\mu_{0}) = \boldsymbol{E}_{\nu_{0}}^{\Pi_{x}} \left[ \sum_{t=0}^{\infty} \beta^{t} c_{0}(X_{t}, U_{t}) \right] - \boldsymbol{E}_{\mu_{0}}^{\Pi_{y}} \left[ \sum_{t=0}^{\infty} \beta^{t} c_{0}(Y_{t}, \tilde{U}_{t}) \right],$$

where we assume that  $\Pi_x \in \Pi_W^C$  and  $\Pi_y \in \Pi_W^C$  achieve  $J^{\beta}(\nu_0)$  and  $J^{\beta}(\mu_0)$  respectively. (Note that we make this assumption only for convenience; at this point we do not know if such optimal policies exist. However, for any  $\delta > 0$  there exist  $\Pi_x, \Pi_y \in \mathbf{\Pi}_W^C$  such that  $J^{\beta}(\nu_0, \Pi_x) < J^{\beta}(\nu_0) + \delta$  and  $J^{\beta}(\mu_0, \Pi_u) < J^{\beta}(\mu_0) + \delta$  and using such  $\delta$ -optimal policies in the proof will lead to the same bound as in (20) since  $\delta > 0$  can be arbitrarily small.)

Consider the following suboptimal encoding and decoding policy for  $\{X_t\}$ : The encoder, in addition to observing the source  $\{X_t\}$ , has access to the noise process  $\{W_t\}$  which is independent of  $X_0$ . Then the encoder can generate the source  $Y_0$  through a simulation (which will be optimized later on with an optimal Wasserstein coupling), and then produce  $Y_t$ for  $t \ge 0$  according to the following equation

$$Y_t = A^t (Y_0 - X_0) + X_t. (21)$$

Then the encoder for  $\{X_t\}$  can use the quantizer policy  $\Pi_u$ and produce the same channel symbols  $\tilde{q}_t$  as  $\Pi_y$  and thus the same reproduction sequence  $U_t = \tilde{\gamma}_t(\tilde{q}_{[0,t]})$  as the encoder and decoder for  $\{Y_t\}$ . Denote this suboptimal policy by  $\Pi$ . Then we get the upper bound

$$\begin{aligned} \left| J^{\beta}(\nu_{0}) - J^{\beta}(\mu_{0}) \right| \\ \leq \boldsymbol{E}_{\nu_{0}}^{\hat{\Pi}} \left[ \sum_{t=0}^{\infty} \beta^{t} c_{0}(X_{t}, \tilde{U}_{t}) \right] - \boldsymbol{E}_{\mu_{0}}^{\Pi_{y}} \left[ \sum_{t=0}^{\infty} \beta^{t} c_{0}(Y_{t}, \tilde{U}_{t}) \right]. \end{aligned}$$

Since  $\beta \in (0, 1)$  and  $c_0(x, u) = ||x - u||^2$ , we have

$$\begin{aligned} \left| J^{\beta}(\nu_{0}) - J^{\beta}(\mu_{0}) \right| \\ &\leq \sum_{t=0}^{\infty} \boldsymbol{E}_{\lambda} \left[ \left| X_{t}'X_{t} + 2\tilde{U}_{t}'(Y_{t} - X_{t}) - Y_{t}'Y_{t} \right| \right] \\ &= \sum_{t=0}^{\infty} \boldsymbol{E}_{\lambda} \left[ \left| X_{t}'X_{t} + 2(\tilde{U}_{t} - Y_{t} + Y_{t})'(Y_{t} - X_{t}) - Y_{t}'Y_{t} \right| \right] \\ &= \sum_{t=0}^{\infty} \boldsymbol{E}_{\lambda} \left[ \left| \|X_{t} - Y_{t}\|^{2} + 2(\tilde{U}_{t} - Y_{t})'(Y_{t} - X_{t}) \right| \right] \\ &\leq \sum_{t=0}^{\infty} \left( \boldsymbol{E}_{\lambda} \left[ \|X_{t} - Y_{t}\|^{2} \right] \\ &+ 2 \left( \boldsymbol{E}_{\lambda} \left[ \|X_{t} - Y_{t}\|^{2} \right] \boldsymbol{E}_{\lambda} \left[ \|\tilde{U}_{t} - Y_{t}\|^{2} \right] \right)^{\frac{1}{2}} \right), \quad (22) \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality.

Since  $\tilde{U}_t$  is produced by the optimal decoder for the source  $Y_t$ , if we use suboptimal reconstruction  $U_t = 0$  for all  $t \ge 0$ , we get an upper bound

$$\boldsymbol{E}_{\lambda}\left[\|\tilde{U}_{t}-Y_{t}\|^{2}
ight]\leq \boldsymbol{E}_{\lambda}\left[\|Y_{t}\|^{2}
ight],$$

and moreover, (18) implies

$$\boldsymbol{E}_{\lambda}\left[\left\|\boldsymbol{Y}_{t}\right\|^{2}\right] \leq \boldsymbol{E}_{\lambda}\left[\left\|\boldsymbol{Y}_{0}\right\|^{2}\right] + \frac{1}{1-\alpha}\sigma^{2}.$$
 (23)

By (21) we can write

$$||X_t - Y_t||^2 = (X_0 - Y_0)'(A^t)'A^t(X_0 - Y_0),$$

so by the Assumption 2(i), we get that

$$||X_t - Y_t||^2 \le \alpha^t ||X_0 - Y_0||^2$$

Since  $\alpha < 1$ , (22) gives, with  $K_2 \coloneqq \boldsymbol{E}_{\lambda} \left[ \|Y_0\|^2 \right] + \frac{1}{1-\alpha} \sigma^2$ ,

$$\begin{aligned} \left| J^{\beta}(\nu_{0}) - J^{\beta}(\mu_{0}) \right| \\ &\leq \boldsymbol{E}_{\lambda} \left[ \sum_{t=0}^{\infty} \alpha^{t} \| X_{0} - Y_{0} \|^{2} \right] \\ &+ 2 \sum_{t=0}^{\infty} (\sqrt{\alpha})^{t} \sqrt{\boldsymbol{E}_{\lambda} \left[ \| X_{0} - Y_{0} \|^{2} \right]} \sqrt{\boldsymbol{E}_{\lambda} \left[ \| Y_{t} \|^{2} \right]} \\ &\leq \left( \sum_{t=0}^{\infty} \alpha^{t} \right) \boldsymbol{E}_{\lambda} \left[ \| X_{0} - Y_{0} \|^{2} \right] \\ &+ 2 \sqrt{\boldsymbol{E}_{\lambda} \left[ \| X_{0} - Y_{0} \|^{2} \right]} \sum_{t=0}^{\infty} (\sqrt{\alpha})^{t} \sqrt{K_{2}} \\ &= \left( \frac{1}{1 - \alpha} \right) \boldsymbol{E}_{\lambda} \left[ \| X_{0} - Y_{0} \|^{2} \right] \\ &+ \left( \frac{2 \sqrt{K_{2}}}{1 - \sqrt{\alpha}} \right) \sqrt{\boldsymbol{E}_{\lambda} \left[ \| X_{0} - Y_{0} \|^{2} \right]}. \end{aligned}$$

By the definition of the Wasserstein distance, for any  $\epsilon > 0$ , by suitably choosing the joint law  $\lambda$  of  $(X_0, Y_0)$ , we have

$$E_{\lambda} \left[ \|X_0 - Y_0\|^2 \right] \le \rho_2^2(\nu_0, \mu_0) + \epsilon$$

Since  $\epsilon$  was arbitrary, we get

$$\left| J^{\beta}(\nu_{0}) - J^{\beta}(\mu_{0}) \right| \leq \left( \frac{\rho_{2}(\nu_{0}, \mu_{0})}{1 - \alpha} + \frac{2K_{1}}{1 - \sqrt{\alpha}} \right) \rho_{2}(\nu_{0}, \mu_{0}),$$
  
where  $K_{1} = \sqrt{K_{2}}.$ 

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Given the equicontinuity result of Lemma 5, we have that  $J^{\beta}(\pi)$  is (uniformly) continuous. Note that the proof of the lemma also applies almost verbatim to the finite horizon case so that the bound in Lemma 5 also holds for the discounted finite horizon optimal cost for all  $T \ge 1$ , which implies that the family of functions  $\{J^{\beta}(\pi,T): T \geq 1\}$  is (uniformly) equicontinuous on  $\mathcal{Z}$ .

Corollary 1: For any  $T \ge 1$ ,  $\mu_0, \nu_0 \in \mathbb{Z}$ , and  $\beta \in (0, 1)$ ,

$$\begin{aligned} \left| J^{\beta}(\nu_{0},T) - J^{\beta}(\mu_{0},T) \right| \\ &\leq \left( \frac{\rho_{2}(\nu_{0},\mu_{0})}{1-\alpha} + \frac{2K_{1}}{1-\sqrt{\alpha}} \right) \rho_{2}(\nu_{0},\mu_{0}). \end{aligned}$$
(24)

*Proof of Theorem 5:* With Lemmas 1–3 and Theorem 3, Condition 1 (the measurable selection condition) is satisfied and the function  $v^*$  is lower semi-continuous, so we can now define the so-called value iteration (VI) updates recursively (see, e.g., [45, (4.2.2)]): For any  $\pi \in \mathbb{Z}$ , let

$$v_n(\pi) = \min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \beta \int_{\mathcal{Z}} v_{n-1}(\pi') P(d\pi' | \pi, Q) \right),$$
(25)

for  $n \ge 1$  with  $v_0(\pi) = 0$  for all  $\pi$ . Since  $v_0 \equiv 0$  is continuous for n = 1, we get that  $v_1 = \min_Q c(\pi, Q)$ , and since  $c(\pi, Q)$ is lower semi-continuous and  $\mathcal{Q}_c$  is compact, we obtain that  $v_1$  is also lower semi-continuous. For  $n \ge 2$ , by Theorem 3, the iterations are well defined and  $v_n$  is lower semi-continuous for all n.

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It is known that  $v_n$  is the value function of the *n*-stage discounted cost  $J^{\beta}(\pi, \Pi, n)$  in (10) with zero terminal cost (see [45, Chapter 4, p.45]), i.e.,

$$v_n(\pi) = \inf_{\Pi \in \Pi_W^C} J^\beta(\pi, \Pi, n) = J^\beta(\pi, n) \quad \text{for all } \pi \in \mathcal{Z}.$$
(26)

Note that, using the operator  $\mathbb{H}$  defined in (14), we may rewrite the DCOE (15) and the VI functions in (25) as

$$v = \mathbb{H}v, \text{ and } v_n = \mathbb{H}v_{n-1} \text{ for } n \ge 1,$$
 (27)

respectively.

In addition, note that since  $c(\pi, Q)$  is non-negative,  $\mathbb{H}$  is monotone, i.e., for u and u' if  $u \ge u'$  then  $\mathbb{H}u \ge \mathbb{H}u'$ . Therefore, since we start from  $v_0 = 0$ , then  $v_n$  is a non-decreasing sequence of lower semi-continuous functions. By (13) and (26) we know that for all  $n \ge 0$ ,

$$J^{\beta}(\pi) \ge v_n(\pi)$$

Thus  $v_n$  is a non-decreasing and bounded sequence and hence it converges pointwise to some function v. Since  $v_n(\pi) = J^{\beta}(\pi, n)$ , by Corollary 1  $v_n$  is continuous and the sequence  $\{v_n : n \ge 1\}$  is a (uniformly) equicontinuous which converges pointwise (on the metric space  $\mathbb{Z}$ ). Therefore, the limit function v is continuous.

Now since both  $v_n$  and v are continuous we have that, by Lemma 2 and Lemma 3, the functions

$$V_n^{\beta}(\pi, Q) \coloneqq \left( c(\pi, Q) + \beta \int_{\mathcal{Z}} v_n(\pi') P(d\pi' | \pi, Q) \right)$$

and

$$V^{\beta}(\pi, Q) \coloneqq \left( c(\pi, Q) + \beta \int_{\mathcal{Z}} v(\pi') P(d\pi' | \pi, Q) \right)$$

are continuous in Q, for each fixed  $\pi$  for all  $n \geq 1$ . Also, as  $v_n \uparrow v$ , by the dominated convergence theorem  $V_n^\beta(\pi, Q) \uparrow V^\beta(\pi, Q)$  for all  $(\pi, Q) \in \mathbb{Z} \times \mathcal{Q}_c$ . Thus by [45, Lemma 4.2.4], we can change the order of limit and minimum as

$$\lim_{n \to \infty} \min_{Q \in \mathcal{Q}_c} V_n^\beta(\pi, Q) = \min_{Q \in \mathcal{Q}_c} V_n^\beta(\pi, Q).$$

Since the left hand side is  $\lim_{n\to\infty} v_{n+1} = v$  and the right hand side is  $\mathbb{H}v$ , we obtain the DCOE  $v = \mathbb{H}v$ , i.e., for all  $\pi \in \mathcal{Z}$ ,

$$v(\pi) = \min_{Q \in \mathcal{Q}_c} V^{\beta}(\pi, Q)$$
$$= \min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \beta \int_{\mathcal{Z}} v(\pi') P(d\pi' | \pi, Q) \right).$$
(28)

According to Theorem 3, the measurable selection condition Assumption 1 holds in (28) (with  $u^* = u$ ) and therefore there exists a (measurable)  $\hat{\eta} : \mathbb{Z} \to Q_c$  such that for all  $\pi \in \mathbb{Z}$ ,

$$v(\pi) = \left(c(\pi, \hat{\eta}(\pi)) + \beta \int_{\mathcal{Z}} v(\pi') P(d\pi' | \pi, \hat{\eta}(\pi))\right).$$
(29)

Thus to finish the proof of the theorem we need only show that  $v = J^{\beta}$  and that the stationary and deterministic policy

 $\Pi = {\hat{\eta}}$  is optimal. This is done with the aid of the following lemma which has a simple proof (see, e.g., [58, Lemma 5.4.4]). *Lemma 6:* Assume  $v(\pi_0) = \lim_{n \to \infty} J^{\beta}(\pi_0, n)$  satisfies

the DCOE  $v = \mathbb{H}v$  and the stationary and deterministic policy  $\Pi = \{\hat{\eta}\}$  is such that it satisfies (29). Assume furthermore that

$$\lim_{t \to \infty} \beta^t \boldsymbol{E}_{\pi_0}^{\Pi}[\boldsymbol{v}(\pi_t)] = 0, \tag{30}$$

for all  $\pi_0 \in \mathcal{Z}$ , where  $\{\pi_t\}$  is the state process of our MDP with initial distribution  $\pi_0$  and policy  $\Pi$ . Then  $v(\pi_0) = J^{\beta}(\pi_0)$ and  $J^{\beta}(\pi_0, \Pi) = J^{\beta}(\pi_0)$  for all  $\pi_0 \in \mathcal{Z}$ , i.e.,  $\Pi \in \Pi_{WS}^C$  is an optimal policy.

Note that we have already shown that the first two conditions of the lemma hold, so we have only to check that (30) holds in our case. By the bound (19) in the proof of Lemma 4, for any initial condition  $\pi_0$  and policy  $\Pi$ , we have

$$v(\pi_t) \leq J^{\beta}(\pi_t) \leq \frac{1}{1-\beta} \Big( \boldsymbol{E}_{\pi_t}[\|X\|^2] + \frac{1}{1-\alpha} \sigma^2 \Big) \\ = \frac{1}{1-\beta} \Big( \boldsymbol{E}_{\pi_0}^{\Pi} \big[ \|X_t\|^2 |q_{[0,...,t-1]}] + \frac{1}{1-\alpha} \sigma^2 \Big).$$

Thus

$$egin{split} m{E}_{\pi_0}^{\Pi}[v(\pi_t)] &\leq rac{1}{1-eta} \Big( m{E}_{\pi_0}^{\Pi}ig[\|X_t\|^2ig] + rac{1}{1-lpha} \sigma^2 \Big) \ &\leq rac{1}{1-eta} \Big( m{E}_{\pi_0}ig[\|X_0\|^2ig] + rac{2}{1-lpha} \sigma^2 \Big), \end{split}$$

where the last inequality holds by (18). Therefore, since  $\beta \in (0, 1)$ ,

$$\lim_{t\to\infty}\beta^t \boldsymbol{E}^{\Pi}_{\pi_0}[v(\pi_t)] = 0.$$

In summary, we have shown that  $J^{\beta}$  satisfies the DCOE and there exists a stationary deterministic policy  $\Pi$  that is optimal. This finishes the proof of Theorem 5.

### B. Proof of Theorem 4

This section is devoted to proving our main result. The proof is done by showing the existence of a so-called canonical triplet for our MDP (see Definition 8 in Appendix A) which in term, after checking that the conditions of Theorem 6 in Appendix A, proves the existence of optimal stationary and deterministic quantization policies and the stated convergence rate. Due to the nature of the controlled Markov process  $\{\pi_t, Q_t\}$  in our problem, verifying these sufficient conditions is technically challenging.

In the following we present the proof of Theorem 4, our main result.

Proof of Theorem 4: We will prove Theorem 4 via the approach of vanishing discounted cost (see [45, Chapter 5.3]). Recall that our state space is  $\mathcal{Z} := \mathcal{G} \cap \mathcal{P}_2(\mathbb{R}^d)$ . In [2], it was shown that,  $\mathcal{G}$  is closed in  $\mathcal{P}(\mathbb{R}^d)$ . Note that  $\mathcal{P}_2(\mathbb{R}^d) = \bigcup_{m \in \mathbb{N}} Z_m$ , where

$$Z_m = \left\{ \mu \in \mathcal{P}_2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \|x\|^2 \mu(dx) \le m \right\},\$$

and this implies that  $\mathcal{Z}$  is  $\sigma$ -compact.

Next note that by Lemma 5, the family of functions

$$h_{\beta}(\pi) \coloneqq J^{\beta}(\pi) - J^{\beta}(\mu), \quad \pi \in \mathcal{Z},$$
(31)

with fixed  $\mu \in \mathcal{Z}$  is equicontinuous on  $\mathcal{Z}$ . Theorem 5 in the previous section proved that  $J^{\beta}$  solves the equation

$$J^{\beta}(\pi) = \min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \beta \int_{\mathcal{Z}} J^{\beta}(\pi') P(d\pi' | \pi, Q) \right).$$
(32)

With the definition of  $h_{\beta}$  and an elementary calculation we can rewrite (32) as

$$(1-\beta)J^{\beta}(\mu) + h_{\beta}(\pi)$$
  
=  $\min_{Q \in \mathcal{Q}_{c}} \left( c(\pi, Q) + \beta \int_{\mathcal{Z}} h_{\beta}(\pi') P(d\pi'|\pi, Q) \right).$  (33)

Recall that by Lemma 4 for all  $\beta \in (0,1)$ , we have the upper bound

$$(1-\beta)J^{\beta}(\mu) \le \mathbf{E}_{\mu}[||X||^{2}] + \frac{1}{1-\alpha}\sigma^{2},$$

which is independent of  $\beta$ .

Since the range of  $(1 - \beta)J^{\beta}(\mu)$ ,  $\beta \in (0, 1)$  is bounded, there exists a limit point  $\rho^*$  as  $\beta \uparrow 1$ . Let  $\beta(l)$  be a sequence such that

$$\lim_{l \to \infty} (1 - \beta(l)) J^{\beta(l)}(\mu) = \rho^*$$

(Note that  $\rho^*$  depends on the fixed  $\mu \in \mathbb{Z}$ , but not on  $\pi$ .) By the conditions on the state space  $\mathcal{Z}$ , the equicontinuity of  $h_{\beta}$ , and the Arzela-Ascoli theorem (see Theorem 7 in Appendix A), there exists a subsequence  $\{h_{\beta(k)}\}$  of  $\{h_{\beta(l)}\}$ which converges pointwise to a continuous function h

$$h(\pi) \coloneqq \lim_{k \to \infty} h_{\beta(k)}(\pi), \quad \pi \in \mathcal{Z}.$$
 (34)

Then, (33) along the subsequence  $\beta(k)$  becomes

$$(1 - \beta(k))J^{\beta(k)}(\mu) + h_{\beta(k)}(\pi)$$
  
= 
$$\min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \beta(k) \int_{\mathcal{Z}} h_{\beta(k)}(\pi') P(d\pi'|\pi, Q) \right).$$
(35)

If we take the limit of (35) as  $k \to \infty$  we get

$$\rho^* + h(\pi)$$

$$= \lim_{k \to \infty} \min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \beta(k) \int_{\mathcal{Z}} h_{\beta(k)}(\pi') P(d\pi' | \pi, Q) \right).$$
(36)

Define

$$V_k(\pi, Q) \coloneqq c(\pi, Q) + \beta(k) \int_{\mathcal{Z}} h_{\beta(k)}(\pi') P(d\pi'|\pi, Q),$$
$$V(\pi, Q) \coloneqq c(\pi, Q) + \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, Q).$$

In the following we show that average cost optimality equation (ACOE) in Definition 8 in Appendix A holds, i.e.,

$$\rho^* + h(\pi) = \min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \int_{\mathcal{Z}} h(\pi') P(d\pi' | \pi, Q) \right),$$

for all  $\pi \in \mathcal{Z}$ .

Lemma 7: Consider  $V_k(\pi, Q)$  and  $V(\pi, Q)$  defined above. Then,

$$\lim_{k \to \infty} \min_{Q \in \mathcal{Q}_c} V_k(\pi, Q) = \min_{Q \in \mathcal{Q}_c} V(\pi, Q).$$

*Proof:* Let  $Q_k^*$  and  $Q^*$  minimize  $V_k(\pi, Q)$  and  $V(\pi, Q)$ respectively. Then by the definition of  $V_k$  and V, it suffices to show that the upper bound in the following inequality converges to zero as  $k \to \infty$ :

$$\left| \min_{Q \in \mathcal{Q}_{c}} \left( c(\pi, Q) + \int_{\mathcal{Z}} \beta(k) h_{\beta(k)}(\pi') P(d\pi'|\pi, Q) \right) - \min_{Q \in \mathcal{Q}_{c}} \left( c(\pi, Q) + \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, Q) \right) \right| \\
\leq \max \left( \left| \int_{\mathcal{Z}} \beta(k) h_{\beta(k)}(\pi') P(d\pi'|\pi, Q_{k}^{*}) - \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, Q_{k}^{*}) \right| \right) - \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, Q_{k}^{*}) \\
= \int_{\mathcal{Z}} \beta(k) h_{\beta(k)}(\pi') P(d\pi'|\pi, Q^{*}) \\
- \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, Q^{*}) \Big| \right), \quad (38)$$

Since  $\beta(k) \uparrow 1$ , it is enough to show that (37) and (38) go to zero as  $k \to \infty$  with the  $\beta(k)$  multiplicative terms replaced by 1.

Let

$$g(\pi) \coloneqq \left(\frac{\rho_2(\pi,\mu)}{1-\alpha} + \frac{2K_1}{1-\sqrt{\alpha}}\right)\rho_2(\pi,\mu), \qquad (39)$$

where  $\mu \in \mathcal{Z}$  is given from the definition of  $h_{\beta}$  in (31). Observe that by Lemma 5,

$$h_{\beta(k)}(\pi)| \le g(\pi) < \infty, \tag{40}$$

for all  $\pi \in \mathcal{Z}$ . Note that by choosing the joint measure so that the marginals are independent, the Wasserstein distance  $\rho_2$  can be upper bounded as

$$\rho_2^2(\pi,\mu) \le 2\boldsymbol{E}_{\pi}[\|X\|^2] + 2\boldsymbol{E}_{\mu}[\|X\|^2] < \infty.$$
 (41)

Now we show that the term in (37) converges to zero (the convergence of (37) will follow from this proof too). Suppose otherwise that for some  $\epsilon > 0$  there exists a subsequence  $Q_{k_{\ell}}^*$ such that

$$\left|\int_{\mathcal{Z}} h_{\beta(k_l)}(\pi') P(d\pi'|\pi, Q_{k_l}^*) - \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, Q_{k_l}^*)\right| \ge \epsilon.$$
(42)

By the compactness of  $Q_c$  there exists further subsequence  $Q_{k_{l_{a}}}^{*}$  that converges to a quantizer  $\bar{Q} \in Q_{c}$ . In the following we prove that along the subsequence  $Q_{k_{l_m}}^*$  the term on the left hand side of (42) goes to zero and reach a contradiction. To do so, we use Lemma 10 in the Appendix. Note that by (40) and (41), for all  $n \ge 1$  we have

$$|h_{\beta(k_{l_n})}(\pi)| \le g(\pi) \le g_1(\pi), \tag{43}$$

where

$$g_{1}(\pi) \coloneqq \frac{2}{1-\alpha} \left( \boldsymbol{E}_{\pi}[\|X\|^{2}] + \boldsymbol{E}_{\mu}[\|X\|^{2}] \right) \\ + \frac{2K_{1}}{1-\sqrt{\alpha}} \sqrt{2\boldsymbol{E}_{\pi}[\|X\|^{2}] + 2\boldsymbol{E}_{\mu}[\|X\|^{2}]}.$$
(44)

Furthermore, for any sequence  $\{\pi_n\} \in \mathbb{Z}$ , with  $\rho_2(\pi_n, \pi) \to 0$ , we have that  $h_{\beta(k_{l_n})}$  continuously converges to h (i.e.  $\lim_{n\to\infty} h_{\beta(k_{l_n})}(\pi_n) = h(\pi)$ ) since

$$\begin{split} |h_{\beta(k_{l_n})}(\pi_n) - h(\pi)| \\ &\leq |h_{\beta(k_{l_n})}(\pi_n) - h_{\beta(k_{l_n})}(\pi)| + |h_{\beta(k_{l_n})}(\pi) - h(\pi)| \\ &= |J^{\beta(k_{l_n})}(\pi_n) - J^{\beta(k_{l_n})}(\pi)| + |h_{\beta(k_{l_n})}(\pi) - h(\pi)| \\ &\leq \left(\frac{\rho_2(\pi_n, \pi)}{1 - \alpha} + \frac{2K_1}{1 - \sqrt{\alpha}}\right) \rho_2(\pi_n, \pi) \\ &+ |h_{\beta(k_{l_n})}(\pi) - h(\pi)| \to 0, \quad \text{as } n \to \infty, \end{split}$$

where the last inequality follows from Lemma 5. Since  $Q_{k_{l_n}}^* \to \overline{Q}$  in order-2 Wasserstein distance and since  $g_1$  is continuous for the order-2 Wasserstein convergence of its argument, we have by Lemma 3,

$$\lim_{n \to \infty} \int_{\mathcal{Z}} g_1(\pi') P(d\pi' | \pi, Q_{k_{l_n}}^*) = \int_{\mathcal{Z}} g_1(\pi') P(d\pi' | \pi, \bar{Q}) < \infty.$$

Hence the conditions of the generalized dominated convergence theorem in Lemma 10 in Appendix A are satisfied, which gives

$$\lim_{n \to \infty} \int_{\mathcal{Z}} h_{\beta(k_{l_n})}(\pi') P(d\pi'|\pi, Q_{k_{l_n}}^*) = \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, \bar{Q})$$

Since we also clearly have

$$\lim_{n \to \infty} \int_{\mathcal{Z}} h(\pi') P(d\pi' | \pi, Q_{k_{l_n}}^*) = \int_{\mathcal{Z}} h(\pi') P(d\pi' | \pi, \bar{Q}),$$

we obtain

$$\int_{\mathcal{Z}} h_{\beta(k_{l_n})}(\pi') P(d\pi'|\pi, Q_{k_{l_n}}^*) - \int_{\mathcal{Z}} h(\pi') P(d\pi'|\pi, Q_{k_{l_n}}^*) \to 0$$

which contradicts (42). Hence the term in (37) also goes to zero and this concludes the proof.  $\Box$ 

Thus, by Lemma 7 we can change the order of limit and minimum in (36), then we get

$$\begin{split} \rho^* + h(\pi) \\ &= \min_{Q \in \mathcal{Q}_c} \lim_{k \to \infty} \left( c(\pi, Q) + \beta(k) \int_{\mathcal{Z}} h_{\beta(k)}(\pi') P(d\pi' | \pi, Q) \right) \\ &= \min_{Q \in \mathcal{Q}_c} \left( c(\pi, Q) + \int_{\mathcal{Z}} h(\pi') P(d\pi' | \pi, Q) \right) \\ &= c(\pi, Q^*) + \int_{\mathcal{Z}} h(\pi') P(d\pi' | \pi, Q^*). \end{split}$$

Noting that  $Q^* = Q^*_{\pi}$  is a function of  $\pi$  in the last equation and defining  $\hat{\eta} : \mathbb{Z} \to Q_c$  by  $\hat{\eta}(\pi) = Q^*_{\pi}$ , we obtain that  $(\rho^*, h, \hat{\eta})$  is a canonical triplet for which the ACOE holds (see Definition 8 in Appendix A).

Now we are ready to apply Theorem 6 in Appendix A to complete the proof of Theorem 4. For this recall that for all  $\pi \in \mathbb{Z}$ , by (43) and (44) we have

$$\begin{aligned} |h_{\beta}(\pi)| &= |J^{\beta}(\pi) - J^{\beta}(\mu)| \\ &\leq \frac{2}{1 - \alpha} \left( \boldsymbol{E}_{\pi}[\|X\|^{2}] + \boldsymbol{E}_{\mu}[\|X\|^{2}] \right) \\ &+ \frac{2K_{1}}{1 - \sqrt{\alpha}} \sqrt{2\boldsymbol{E}_{\pi}[\|X\|^{2}] + 2\boldsymbol{E}_{\mu}[\|X\|^{2}]}. \end{aligned}$$
(45)

Fix the initial distribution  $\pi_0$ , let  $\Pi \in \Pi_W^C$  be arbitrary, and let  $\{\pi_t\}$  be the states generated by this policy. Since the inequality in (45) holds for all  $\pi \in \mathcal{Z}$ , in particular it holds for  $\pi_T \in \mathcal{Z}$ . Thus, from (34) and (45) we get

$$\begin{aligned} |h(\pi_T)| &= \lim_{k \to \infty} |h_{\beta(k)}(\pi_T)| \\ &\leq \frac{2}{1 - \alpha} \left( \boldsymbol{E}_{\pi_T}[\|X\|^2] \\ &+ \boldsymbol{E}_{\mu}[\|X\|^2] \right) + \frac{2K_1}{1 - \sqrt{\alpha}} \sqrt{2\boldsymbol{E}_{\pi_T}[\|X\|^2] + 2\boldsymbol{E}_{\mu}[\|X\|^2]}. \end{aligned}$$

$$(46)$$

Note that

where the inequality follows from (18). Now choose  $\mu$  in the definition of  $h_{\beta}$  as  $\mu = \pi_0$ . Then (47), (46), and Jensen's inequality give

$$E_{\pi_{0}}^{\Pi}[|h(\pi_{T})|] \leq \frac{2}{1-\alpha} \left( 2E_{\pi_{0}}[||X||^{2}] + \frac{1}{1-\alpha}\sigma^{2} \right) + \frac{2K_{1}}{1-\sqrt{\alpha}}\sqrt{4E_{\pi_{0}}[||X||^{2}] + \frac{2}{1-\alpha}\sigma^{2}}.$$
(48)

Hence, we have

$$\limsup_{T \to \infty} \frac{1}{T} \boldsymbol{E}_{\pi_0}^{\Pi} \left[ h(\pi_T) \right] = 0,$$

for all  $\pi_0$  and under every policy  $\Pi$ . Therefore by Theorem 6 there exists a deterministic stationary policy  $\Pi^* \in \Pi_{WS}^C$ that achieves the minimum in (9) simultaneously for all  $\pi_0$ . Furthermore, by Theorem 6

$$|J(\pi_0, \Pi^*, T) - J(\pi_0, \Pi^*)| \\ \leq \frac{1}{T} \left( \left| \boldsymbol{E}_{\pi_0}^{\Pi^*}[h(\pi_T)] - h(\pi_0) \right| \right) \\ = \frac{1}{T} \left( \left| \boldsymbol{E}_{\pi_0}^{\Pi^*}[h(\pi_T)] \right| \right) \leq \frac{K(\pi_0)}{T}$$

where the equality follows by the definition of h,

$$h(\pi_0) = \lim_{k \to \infty} h_{\beta(k)}(\pi_0) = \lim_{k \to \infty} (J^{\beta(k)}(\pi_0) - J^{\beta(k)}(\pi_0)) = 0,$$

and  $K(\pi_0)$  is the upper bound in (48). This concludes the proof of the second part of Theorem 4.

## **IV. CONCLUSION**

In this paper we have considered the problem of zerodelay coding of  $\mathbb{R}^d$ -valued linearly generated Markov sources. We have proved structural, existence, and converge rate results for optimal zero-delay coding under the assumption that the allowable quantizers have convex codecells. Applications to closed-loop control systems, especially to optimal quadratic control under information constraints for infinite horizons, see e.g., [51]–[56], are currently under study.

## APPENDIX A

## A. Average Cost Optimality in Markov Decision Processes

Let  $\mathcal{Z}$  be a Borel space (i.e., a Borel subset of a complete and separable metric space) and let  $\mathcal{P}(\mathcal{Z})$  denote the set of all probability measures on  $\mathcal{Z}$ . A discrete time Markov control model (Markov decision process) is a system characterized by the 4-tuple  $(\mathcal{Z}, \mathcal{A}, \mathcal{K}, c)$ , where (i)  $\mathcal{Z}$  is the state space, the set of all possible states of the system; (ii)  $\mathcal{A}$  (a Borel space) is the control space (or action space), the set of all controls (actions)  $a \in \mathcal{A}$  that can act on the system; (iii)  $\mathcal{K} = \mathcal{K}(\cdot | z, a)$  is the transition probability of the system, a stochastic kernel on Zgiven  $\mathcal{Z} \times \mathcal{A}$ , i.e.,  $\mathcal{K}(\cdot | z, a)$  is a probability measure on  $\mathcal{Z}$ for all state-action pairs (z, a), and  $\mathcal{K}(B|\cdot, \cdot)$  is a measurable function from  $\mathcal{Z} \times \mathcal{A}$  to [0, 1] for each Borel set  $B \subset \mathcal{Z}$ ; (iv)  $c: \mathcal{Z} \times \mathcal{A} \to [0, \infty)$  is the cost per time stage function of the system, a Borel measurable function c(z, a) of the state and the control.

Define the *history* spaces  $\mathcal{H}_t$  at time  $t \ge 0$  of the Markov control model by  $\mathcal{H}_0 \coloneqq \mathcal{Z}$  and  $\mathcal{H}_t \coloneqq (\mathcal{Z} \times \mathcal{A})^t \times \mathcal{Z}$ . Thus a specific history  $h_t \in \mathcal{H}_t$  has the form  $h_t =$  $(z_0, a_0, \ldots, z_{t-1}, a_{t-1}, z_t).$ 

Definition 7 (Admissible Control Policy [45]): An admissible control policy  $\Pi = \{\alpha_t\}_{t>0}$ , also called a randomized control policy (more simply a control policy or a policy) is a sequence of stochastic kernels on the action space  $\mathcal{A}$  given the history  $\mathcal{H}_t$ . The set of all randomized control policies is denoted by  $\Pi_A$ . A *deterministic policy*  $\Pi$  is a sequence of functions  $\{\alpha_t\}_{t>0}, \alpha_t : \mathcal{H}_t \to \mathcal{A}$ , that determine the control used at each time stage deterministically, i.e.,  $a_t = \alpha_t(h_t)$ . The set of all deterministic policies is denoted  $\Pi_D$ . Note that  $\Pi_D \subset \Pi_A$ . A Markov policy is a policy  $\Pi$  such that for each time stage the choice of control only depends on the current state  $z_t$ , i.e.,  $\Pi = \{\alpha_t\}_{t>0}$  with  $\alpha_t : \mathcal{Z} \to \mathcal{P}(\mathcal{A})$ . The set of all Markov policies is denoted by  $\Pi_M$ . The set of deterministic Markov policies is denoted by  $\Pi_{MD}$ . A stationary policy is a Markov policy  $\Pi = \{\alpha_t\}_{t>0}$  such that  $\alpha_t = \alpha$  for all t > 0 for some  $\alpha : \mathcal{Z} \to \mathcal{P}(\mathcal{A})$ . The set of all stationary policies is denoted by  $\Pi_S$  and the set of deterministic stationary policies is denoted by  $\Pi_{SD}$ .

The transition kernel  $\mathcal{K}$ , an initial probability distribution  $\pi_0$  on  $\mathcal{Z}$ , and a policy  $\Pi$  define a unique probability measure  $P_{\pi_0}^{\Pi}$  on  $\mathcal{H}_{\infty} = (\mathcal{Z} \times \mathcal{A})^{\infty}$ , the distribution of the state-action process  $\{(Z_t, A_t)\}_{t\geq 0}$ . The resulting state process  $\{Z_t\}_{t\geq 0}$ is called a controlled Markov process. The expectation with respect to  $P_{\pi_0}^{\Pi}$  is denoted by  $E_{\pi_0}^{\Pi}$ . If  $\pi_0 = \delta_z$ , the point mass at  $z \in \mathcal{Z}$ , we write  $P_z^{\Pi}$  and  $E_z^{\Pi}$  instead of  $P_{\delta_z}^{\Pi}$  and  $E_{\delta_z}^{\Pi}$ . In an optimal control problem, a performance objective J of the system is given and the goal is to find the controls that minimize (or maximize) that objective. Some common optimal control problems for Markov control models are the following:

1) Finite Horizon Average Cost Problem: Here the goal is to find policies that minimize the average cost

$$J(\pi_0, \Pi, T) \coloneqq \boldsymbol{E}_{\pi_0}^{\Pi} \left[ \frac{1}{T} \sum_{t=0}^{T-1} c(Z_t, A_t) \right],$$

for some  $T \geq 1$ .

2) Infinite Horizon Discounted Cost Problem: Here the goal is to find policies that minimize

$$J^{\beta}(\pi_0,\Pi) \coloneqq \lim_{T \to \infty} \boldsymbol{E}_{\pi_0}^{\Pi} \left[ \sum_{t=0}^{T-1} \beta^t c(Z_t, A_t) \right],$$

for some  $\beta \in (0, 1)$ .

3) Infinite Horizon Average Cost Problem: In the more challenging infinite horizon control problem the goal is to find policies that minimize the average cost

$$J(\pi_0, \Pi) \coloneqq \limsup_{T \to \infty} \boldsymbol{E}_{\pi_0}^{\Pi} \left[ \frac{1}{T} \sum_{t=0}^{T-1} c(Z_t, A_t) \right].$$

The Markov control model together with the performance objective is called a Markov decision process (MDP).

Definition 8 ([57]): Let h and g be measurable real functions on  $\mathcal{Z}$  and let  $f: \mathcal{Z} \to \mathcal{A}$  be measurable. Then (g, h, f) is said to be a canonical triplet if for all  $z \in \mathbb{Z}$ ,

$$g(z) = \inf_{a \in \mathcal{A}} \int_{\mathcal{Z}} g(z') \mathcal{K}(dz'|z, a)$$
(49)

$$g(z) + h(z) = \inf_{a \in \mathcal{A}} \left( c(z, a) + \int_{\mathcal{Z}} h(z') \mathcal{K}(dz'|z, a) \right)$$
(50)

and

$$g(z) = \int_{\mathcal{Z}} g(z') \mathcal{K}(dz'|z, f(z))$$
(51)

$$g(z) + h(z) = c(z, f(z)) + \int_{\mathcal{Z}} h(z') \mathcal{K}(dz'|z, f(z)).$$
 (52)

Equations (49)-(50) and (51)-(52) are called the canonical equations. In case g is a constant,  $g \equiv g^*$ , these equations reduce to

$$g^* + h(z) = \inf_{a \in \mathcal{A}} \left( c(z, a) + \int_{\mathcal{Z}} g(z') \mathcal{K}(dz'|z, a) \right)$$
(53)

$$g^* + h(z) = c(z, f(z)) + \int_{\mathcal{Z}} h(z') \mathcal{K}(dz'|z, f(z)), \quad (54)$$

and (53)-(54) is called the average cost optimality equation (ACOE).

Theorem 6: [58, Theorem 7.1.1] Let (g, h, f) be a canonical triplet. If  $g \equiv g^*$  is a constant and

$$\limsup_{T \to \infty} \frac{1}{T} \boldsymbol{E}_{z}^{\Pi}[h(z_{T})] = 0,$$

for all z and under every policy  $\Pi \in \Pi_A$ , then the stationary deterministic policy  $\Pi^* = \{f\} \in \Pi_{SD}$  is optimal so that

$$g^* = J(z, \Pi^*) = \inf_{\Pi \in \Pi_A} J(z, \Pi).$$

where

$$J(z,\Pi) = \limsup_{T \to \infty} \frac{1}{T} \boldsymbol{E}_z^{\Pi} \left[ \sum_{t=0}^{T-1} c(z_t, a_t) \right].$$

Furthermore,

$$\left|\frac{1}{T}\boldsymbol{E}_{z}^{\Pi^{*}}\sum_{t=0}^{T-1}c(z_{t},a_{t})-g^{*}\right| \leq \frac{1}{T}\left(\left|\boldsymbol{E}_{z}^{\Pi^{*}}[h(z_{T})]-h(z)\right|\right),$$
 i.e.

$$J(z,\Pi^*,T) - g^*| = |J(z,\Pi^*,T) - J(z,\Pi^*)| \\ \leq \frac{1}{T} \left( \left| \boldsymbol{E}_z^{\Pi^*}[h(z_T)] - h(z) \right| \right).$$
(55)

## B. Auxiliary Results

Theorem 7: [59, Theorem 2.4.7] Let F be an equicontinuous family of real functions on a compact space  $\mathcal{X}$  and let  $f_n$  be a sequence in F such that the range of  $f_n$  is compact. Then, there exists a subsequence  $f_{n_k}$  which converges uniformly to a continuous function. If  $\mathcal{X}$  is  $\sigma$ -compact,  $f_{n_k}$  converges pointwise to a continuous function, and the convergence is uniform on compact subsets of  $\mathcal{X}$ .

Lemma 8: Let  $\mathcal{A}$  be compact, and let V(z, a) be continuous on  $\mathcal{Z} \times \mathcal{A}$ . Then,  $\min_{a \in \mathcal{A}} V(z, a)$  is continuous on  $\mathcal{Z}$ .

*Lemma 9:* [2, Lemma 2]

- (a) Let  $\{\pi_n\}$  be a sequence of probability density functions on  $\mathbb{R}^d$  which are uniformly equicontinuous and uniformly bounded and assume  $\pi_n \to \pi$  weakly. Then  $\pi_n \to \pi$  in total variation.
- (b) Let {Q<sub>n</sub>} be a sequence in Q<sub>c</sub> such that Q<sub>n</sub> → Q weakly at P for some Q ∈ Q<sub>c</sub>. If P admits a density, then Q<sub>n</sub> → Q in total variation at P. If the density of P is positive, then Q<sub>n</sub> → Q in total variation at any P' admitting a density.

Lemma 10: [60, Theorem 3.5] Suppose  $f_n, b_n, f$ , and b are measurable real functions on a standard Borel space  $\mathcal{X}$ . Let  $\{\mu_n\}$  be a sequence of probability measures in  $\mathcal{P}(\mathcal{X})$ , converging weakly to some  $\mu \in \mathcal{P}(\mathcal{X})$ . Assume that

$$|f_n| \le b_n, \quad n \ge 1,$$

and that

$$f_n \xrightarrow{c} f, \quad b_n \xrightarrow{c} b,$$
$$\lim_{n \to \infty} \int_{\mathcal{X}} b_n(x) \mu_n(dx) = \int_{\mathcal{X}} b(x) \mu(dx) < \infty$$

where  $f_n \xrightarrow{c} f$  means that for any  $x \in \mathcal{X}$  and any sequence  $x_n \to x$ , we have  $f_n(x_n) \to f(x)$  (i.e.,  $f_n$  continuously converges to f). Then,

$$\lim_{n \to \infty} \int_{\mathcal{X}} f_n(x) \mu_n(dx) = \int_{\mathcal{X}} f(x) \mu(dx).$$

#### APPENDIX B

Proof of Lemma 1: [2, Lemma 3] shows that  $Q_c$  is compact in the weak topology, so by [61, Theorem 7.12] we only need to prove the convergence of second moment, i.e., we have to show that for any  $P \in \mathbb{Z}$ ,

$$\int_{\mathbb{R}^d \times \mathcal{M}} \|z\|^2 dPQ_n \to \int_{\mathbb{R}^d \times \mathcal{M}} \|z\|^2 dPQ, \qquad (56)$$

whenever  $PQ_n \rightarrow PQ$  weakly. Note that

$$\int_{\mathbb{R}^d \times \mathcal{M}} \|z\|^2 dPQ_n = \sum_{i=1}^M \int_{B_i^n} (\|x\|^2 + i^2) P(dx),$$

where the  $\{B_i^n\}$  are the bins of  $Q_n$ .

For  $\epsilon > 0$ , let L > 0 be such that  $\int_{\{\|x\| \ge L\}} \|x\|^2 P(dx) < \epsilon$ . Letting  $\{B_i\}$  be the bins of Q, we have

$$\left\| \int \|z\|^2 dPQ_n - \int \|z\|^2 dPQ \right\|$$
  
$$\leq \sum_{i=1}^M \int_{\mathbb{R}^d} (\|x\|^2 + i^2) (|1_{B_i^n} - 1_{B_i}|) P(dx)$$

$$\begin{split} &\leq \sum_{i=1}^{M} \left( \int_{\mathbb{R}^{d}} \|x\|^{2} (|1_{B_{i}^{n}} - 1_{B_{i}}|) P(dx) \right) \\ &+ \int_{\mathbb{R}^{d}} M^{2} (|1_{B_{i}^{n}} - 1_{B_{i}}|) P(dx) \right) \\ &= \sum_{i=1}^{M} \left( \int_{\{\|x\| \ge L\}} \|x\|^{2} (|1_{B_{i}^{n}} - 1_{B_{i}}|) P(dx) \right) \\ &+ \int_{\{\|x\| < L\}} \|x\|^{2} (|1_{B_{i}^{n}} - 1_{B_{i}}|) P(dx) \right) \\ &+ M^{2} \sum_{i=1}^{M} P(B_{i}^{n} \Delta B_{i}) \\ &\leq \sum_{i=1}^{M} \left( \epsilon + L^{2} P(B_{i}^{n} \Delta B_{i}) \right) + M^{2} \sum_{i=1}^{M} P(B_{i}^{n} \Delta B_{i}) \\ &\leq \epsilon M + (L^{2} + M^{2}) \sum_{i=1}^{M} P(B_{i}^{n} \Delta B_{i}) \to \epsilon M, \end{split}$$

as  $n \to \infty$ . It was shown in the proof of [2, Lemma 2] that if  $PQ_n \to PQ$  weakly then by the assumption that P admits a density we have  $P(B_i^n \Delta B_i) \to 0$  for all i = 1, ..., M. Since  $\epsilon$  was arbitrary we obtain (56), which completes the proof.  $\Box$ 

The next lemma is needed in the proof of Lemma 2.

Lemma 11: Let  $\{B_1^n, \ldots, B_M^n\}$  and  $\{B_1, \ldots, B_M\}$  denote the cells of quantizers  $Q_n$  and Q respectively. If  $(\pi_n, Q_n) \rightarrow (\pi, Q)$  in  $\mathcal{Z} \times \mathcal{Q}_c$ , the optimal receiver  $\gamma_n$  for  $Q_n$  converges to optimal receiver  $\gamma$  of Q in the sense that

$$\gamma_n(i) = \frac{1}{\pi_n(B_i^n)} \int_{B_i^n} x \, \pi_n(dx) \to \frac{1}{\pi(B_i)} \int_{B_i} x \, \pi(dx) = \gamma(i),$$

for every  $i \in \{1, ..., M\}$  such that  $\pi(B_i) > 0$ . *Proof:* We have

$$\left\| \int_{B_{i}^{n}} x \, \pi_{n}(dx) - \int_{B_{i}} x \, \pi(dx) \right\|$$

$$\leq \left\| \int_{B_{i}^{n}} x \, \pi_{n}(dx) - \int_{B_{i}^{n}} x \, \pi(dx) \right\|$$

$$+ \left\| \int_{B_{i}^{n}} x \, \pi(dx) - \int_{B_{i}} x \, \pi(dx) \right\|$$

$$\leq \left\| \int_{B_{i}^{n}} x \, \pi_{n}(dx) - \int_{B_{i}^{n}} x \, \pi(dx) \right\|$$

$$+ \int_{B_{i}^{n} \bigtriangleup B_{i}} \|x\| \, \pi(dx).$$
(57)

Since  $(\pi_n, Q_n) \to (\pi, Q)$ , we have  $\pi(B_i^n \Delta B_i) \to 0$ (see [48]). Since  $E_{\pi}[||X||] \leq \sqrt{E_{\pi}[||X||^2]} < \infty$ , ||x|| is integrable with respect to  $\pi$  and so the absolute continuity of the integral implies that

$$\lim_{n \to \infty} \int_{B_i^n \triangle B_i} \|x\| \, \pi(dx) = 0. \tag{58}$$

To bound the first term in (57), have for any L > 0

$$\left\| \int_{B_i^n} x \, \pi_n(dx) - \int_{B_i^n} x \, \pi(dx) \right\|$$
  
$$\leq \left\| \int_{\mathbb{R}^d} x \mathbf{1}_{B_i^n \cap \{ \|x\| \le L \}} \, \pi_n(dx) \right\|$$

$$-\int_{\mathbb{R}^{d}} x \mathbf{1}_{B_{i}^{n} \cap \{\|x\| \leq L\}} \pi(dx) \|$$
  
+ 
$$\int_{\mathbb{R}^{d}} \|x\| \mathbf{1}_{B_{i}^{n} \cap \{\|x\| > L\}} \pi_{n}(dx)$$
  
+ 
$$\int_{\mathbb{R}^{d}} \|x\| \mathbf{1}_{B_{i}^{n} \cap \{\|x\| > L\}} \pi(dx).$$
(59)

Letting  $Y_n = ||U_n||$  and Y = ||U||, where the  $\mathbb{R}^d$ -valued random variables  $U_n$  and U are distributed according to  $\pi_n$ and  $\pi$ , respectively, the order-2 Wasserstein convergence of  $\pi_n$  to  $\pi$  in  $\mathcal{Z}$  implies that  $E[Y_n^2] \to E[Y] < \infty$  so that  $\{Y_n\}$  is an  $L_2$ -bounded sequence and therefore it is uniformly integrable [59]. Therefore

$$\lim_{L \to \infty} \sup_{n \ge 1} \left( \int_{\mathbb{R}^d} \|x\| 1_{B_i^n \cap \{\|x\| > L\}} \pi_n(dx) + \int_{\mathbb{R}^d} \|x\| 1_{B_i^n \cap \{\|x\| > L\}} \pi(dx) \right) = 0.$$

Moreover, since each component of  $x \mathbb{1}_{B_i^n \cap \{\|x\| \le L\}} \in \mathbb{R}^d$  is bounded by L and since  $\|\pi_n - \pi\|_{TV} \to 0$ , the definition (2) of the total variation distance implies that for any fixed L, the first term on the right hand side of equation (59) converges to zero as  $n \to \infty$ . In summary, for any  $\epsilon > 0$  there is an L > 0 such that the right hand side of (59) is less than  $\epsilon$  for all n large enough. This and (58) then give that the leftmost term in (57) converges to zero, i.e.,

$$\lim_{n \to \infty} \int_{B_i^n} x \, \pi_n(dx) = \int_{B_i} x \, \pi(dx).$$

Since  $\pi_n \to \pi$  implies  $\pi_n(B_i^n) \to \pi(B_i)$ , we obtain the lemma statement for all *i* such that  $\pi(B_i) > 0$ .

Proof of Lemma 2: To prove the first statement, assume that  $(\pi_n, Q_n) \to (\pi, Q)$  in  $\mathcal{Z} \times \mathcal{Q}_c$ . Then for any L > 0

$$\liminf_{n \to \infty} c(\pi_n, Q_n) = \liminf_{n \to \infty} \int \|x - \gamma_n(Q_n(x))\|^2 \pi_n(dx) 
\geq \sum_{i=1}^M \liminf_{n \to \infty} \int_{B_i^n} \|x - \gamma_n(i)\|^2 \mathbf{1}_{\{\|x\|^2 \le L\}} \pi_n(dx) 
= \sum_{i=1}^M \int_{B_i} \|x - \gamma(i)\|^2 \mathbf{1}_{\{\|x\|^2 \le L\}} \pi(dx) 
= \int_{\{\|x\|^2 \le L\}} \|x - \gamma(Q(x))\|^2 \pi(dx),$$
(60)

since  $\gamma_n(i) \to \gamma(i)$  for any i with  $\pi(B_i) > 0$  by Lemma 11 and so  $\gamma_n$  is bounded and so  $||x - \gamma_n(i)||^2 \mathbb{1}_{\{||x||^2 \leq L\}}$  is uniformly bounded. This together with the fact  $\pi_n \to \pi$  in total variation,  $\pi_n(B_i^n) \to \pi(B_i)$  and  $\pi_n(B_i^n \Delta B_i) \to 0$ , implies the equality in (60). Since L is arbitrary, taking the limit as  $L \to \infty$  then we get

$$\liminf_{n \to \infty} c(\pi_n, Q_n) \ge c(\pi, Q)$$

The proof of the continuity of  $c(\pi, Q)$  in Q is similar to [2, Lemma 7]. Assume  $(Q_n, \pi) \to (Q, \pi)$ . First observe that by Lemma 11, we have  $\gamma_n(i) \to \gamma(i)$  for any  $i \in \{1, \ldots, M\}$  such that  $\pi(B_i) > 0$ . Let  $I = \{i \in \{1, \ldots, M\} : \pi(B_i) > 0\}$ . Then we have for all  $i \in I$ ,

$$D_i \coloneqq \sup_{n \ge 1} \|\gamma_n(i)\| < \infty$$

Letting  $D = \max_{i \in I} D_i$ , by the parallelogram law we have for all  $i \in I$ ,

$$\|x - \gamma_n(Q_n(x))\|^2 \le 2\|x\|^2 + 2\|\gamma_n(Q_n(x))\|^2 \le 2\|x\|^2 + 2D.$$

Since  $\pi \in \mathcal{Z}$  has finite second moment, we obtain

$$\lim_{L \to \infty} \sup_{n \ge 1} \int_{B_i^n} \|x - \gamma_n(Q_n(x))\|^2 \mathbf{1}_{\{\|x\|^2 > L\}} \pi(dx)$$
  
$$\leq \lim_{L \to \infty} \int_{\mathbb{R}^d} \left(2\|x\|^2 + 2D\right) \mathbf{1}_{\{\|x\|^2 > L\}} \pi(dx) = 0.$$
(61)

When  $||x||^2 \leq L$ ,  $||x - \gamma_n(Q_n(x))||^2$  is uniformly bounded and since  $\pi(B_i^n \Delta B_i) \to 0$ , we have

$$\int_{B_i^n} \|x - \gamma_n(Q_n(x))\|^2 \mathbf{1}_{\{\|x\|^2 \le L\}} \pi(dx)$$
(62)

$$\to \int_{B_i} \|x - \gamma(Q(x))\|^2 \mathbf{1}_{\{\|x\|^2 \le L\}} \pi(dx).$$
(63)

Then, using truncation by L together with (61) and (62) we obtain

$$\int_{B_i^n} \|x - \gamma_n(Q_n(x))\|^2 \pi(dx) \to \int_{B_i} \|x - \gamma(Q(x))\|^2 \pi(dx)$$

and therefore  $c(Q_n, \pi) \to c(Q, \pi)$  as  $n \to \infty$ .

*Proof of Lemma 3:* Consider the conditional probability distribution given by [2]

$$\hat{\pi}(i,\pi,Q)(A) \coloneqq P(x_{t+1} \in A | \pi_t = \pi, Q_t = Q, q = i)$$
$$= \frac{1}{\pi(B_i)} \int_A \left( \int_{B_i} \pi(dx) \phi(z|x) \right) dz,$$

for  $i \in I$  (see also [2]). We have

$$\begin{split} \left| \int_{\mathcal{Z}} g(\pi') P(d\pi' | \pi_n, Q_n) - \int_{\mathcal{Z}} g(\pi') P(d\pi' | \pi, Q) \right| \\ &= \left| \sum_{i=1}^M \left( g(\hat{\pi}(i, \pi_n, Q_n)) P(\hat{\pi}(i, \pi_n, Q_n) | \pi_n, Q_n) - g(\hat{\pi}(i, \pi, Q)) P(\hat{\pi}(i, \pi, Q) | \pi, Q) \right) \right| \\ &= \left| \sum_{i=1}^M \left( g(\hat{\pi}(i, \pi_n, Q_n)) \pi_n(Q_n^{-1}(i)) - g(\hat{\pi}(i, \pi, Q)) \pi(Q^{-1}(i)) \right) \right| \\ &= \left| \sum_{i=1}^M \left( g(\hat{\pi}(i, \pi_n, Q_n)) \pi_n(B_i^n) - g(\hat{\pi}(i, \pi, Q)) \pi(B_i) \right) \right|. \end{split}$$

Thus in view of the fact that  $\pi_n(B_i^n) \to \pi(B_i)$  and that  $g(\cdot)$  is a continuous function, it is enough to prove that for all  $i \in I$ ,  $\hat{\pi}(i, \pi_n, Q_n) \to \hat{\pi}(i, \pi, Q)$ . In turn, this is implied by

$$\begin{split} & \left| \int_{B_i^n} \pi_n(dx)\phi(z|x) - \int_{B_i} \pi(dx)\phi(z|x) \right| \\ & \leq \left| \int_{\mathbb{R}^d} (\mathbf{1}_{B_i^n} - \mathbf{1}_{B_i})\pi_n(dx)\phi(z|x) \right| \\ & + \left| \int_{\mathbb{R}^d} \mathbf{1}_{B_i}(\pi_n(x) - \pi(x))\phi(z|x)dx \right| \\ & \leq C(\pi_n(B_i^n\Delta B_i) + \|\pi_n - \pi\|_{TV}) \to 0, \end{split}$$

where C is the uniform upper bound on  $\phi$  and by Lemma 9 we have  $\|\pi_n - \pi\|_{TV} \to 0$ .

The proof that  $Pg(\pi, Q)$  is continuous in Q for every fixed  $\pi$  follows from the proof above by setting  $\pi_n = \pi$  for all n and noting that in this case that the argument only requires the continuity of q but not its boundedness.

#### REFERENCES

- M. Ghomi, T. Linder, and S. Yüksel, "Optimality of stationary and deterministic policies for zero-delay coding of R<sup>d</sup>-valued linear Markov sources with quadratic cost," in *Proc. 59th IEEE Conf. Decis. Control* (*CDC*), Dec. 2020, pp. 4859–4864.
- [2] T. Linder and S. Yüksel, "On optimal zero-delay coding of vector Markov sources," *IEEE Trans. Inf. Theory*, vol. 60, no. 10, pp. 5975–5991, Oct. 2014.
- [3] H. S. Witsenhausen, "On the structure of real-time source coders," *Bell Syst. Tech. J.*, vol. 58, no. 6, pp. 1437–1451, Jul. 1979.
- [4] J. Walrand and P. Varaiya, "Optimal causal coding-decoding problems," *IEEE Trans. Inf. Theory*, vol. IT-29, no. 6, pp. 814–820, Nov. 1983.
- [5] D. Teneketzis, "On the structure of optimal real-time encoders and decoders in noisy communication," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 4017–4035, Sep. 2006.
- [6] S. Yüksel, "On optimal causal coding of partially observed Markov sources in single and multiterminal settings," *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 424–437, Sep. 2013.
- [7] D. Neuhoff and R. Gilbert, "Causal source codes," *IEEE Trans. Inf. Theory*, vol. IT-28, no. 5, pp. 701–713, Sep. 1982.
- [8] H. Asnani and T. Weissman, "Real-time coding with limited lookahead," *IEEE Trans. Inf. Theory*, vol. 59, no. 6, pp. 3582–3606, Jun. 2013.
- [9] T. Linder and R. Zamir, "Causal coding of stationary sources and individual sequences with high resolution," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 662–680, Feb. 2006.
- [10] R. Pilc, "Coding theorems for discrete source-channel pairs," Ph.D. dissertation, Dept. Elect. Eng., Massachusetts Inst. Technol., Cambridge, MA, USA, 1967.
- [11] Z. Zhang, E.-H. Yang, and V. K. Wei, "The redundancy of source coding with a fidelity criterion. 1. Known statistics," *IEEE Trans. Inf. Theory*, vol. 43, no. 1, pp. 71–91, Jan. 1997.
- [12] V. Kostina and S. Verdú, "Fixed-length lossy compression in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 58, no. 6, pp. 3309–3338, Jun. 2012.
- [13] A. K. Gorbunov and M. S. Pinsker, "Nonanticipatory and prognostic epsilon entropies and message generation rates," *Problemy Peredachi Informatsii*, vol. 9, no. 3, pp. 12–21, 1973.
- [14] M. S. Derpich and J. Østergaard, "Improved upper bounds to the causal quadratic rate-distortion function for Gaussian stationary sources," *IEEE Trans. Inf. Theory*, vol. 58, no. 5, pp. 3131–3152, May 2012.
- [15] T. Tanaka, "Semidefinite representation of sequential rate-distortion function for stationary gauss-Markov processes," in *Proc. IEEE Conf. Control Appl. (CCA)*, Sep. 2015, pp. 1217–1222.
- [16] T. Tanaka, K.-K.-K. Kim, P. A. Parrilo, and S. K. Mitter, "Semidefinite programming approach to Gaussian sequential rate-distortion trade-offs," *IEEE Trans. Autom. Control*, vol. 62, no. 4, pp. 1896–1910, Apr. 2017.
- [17] P. A. Stavrou, T. Charalambous, C. D. Charalambous, and S. Loyka, "Optimal estimation via nonanticipative rate distortion function and applications to time-varying gauss-Markov processes," *SIAM J. Control Optim.*, vol. 56, no. 5, pp. 3731–3765, Jan. 2018.
- [18] P. A. Stavrou, J. Østergaard, and C. D. Charalambous, "Zero-delay rate distortion via filtering for vector-valued Gaussian sources," *IEEE J. Sel. Topics Signal Process.*, vol. 12, no. 5, pp. 841–856, Oct. 2018.
- [19] P. A. Stavrou and M. Skoglund, "Asymptotic reverse waterfilling algorithm of NRDF for certain classes of vector Gauss–Markov processes," *IEEE Trans. Autom. Control*, early access, Jul. 26, 2022, doi: 10.1109/TAC.2021.3099444.
- [20] P. A. Stavrou and M. Skoglund, "LQG control and linear policies for noisy communication links with synchronized side information at the decoder," *Automatica*, vol. 123, Jan. 2021, Art. no. 109306.
- [21] R. Zamir and M. Feder, "On universal quantization by randomized uniform/lattice quantizers," *IEEE Trans. Inf. Theory*, vol. 38, no. 2, pp. 428–436, Mar. 1992.
- [22] E. I. Silva, M. S. Derpich, and J. Østergaard, "A framework for control system design subject to average data-rate constraints," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1886–1899, Aug. 2011.

- [23] J. Østergaard, "Stabilizing error correction codes for controlling LTI systems over erasure channels," in *Proc. IEEE Conf. Decis. Control* (CDC), 2021.
- [24] E. I. Silva, M. S. Derpich, J. Østergaard, and M. A. Encina, "A characterization of the minimal average data rate that guarantees a given closed-loop performance level," *IEEE Trans. Autom. Control*, vol. 61, no. 8, pp. 2171–2186, Aug. 2016.
- [25] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1549–1561, Sep. 2004.
- [26] C. D. Charalambous, P. A. Stavrou, and N. U. Ahmed, "Nonanticipative rate distortion function and relations to filtering theory," *IEEE Trans. Autom. Control*, vol. 59, no. 4, pp. 937–952, Apr. 2014.
- [27] V. Kostina and B. Hassibi, "Rate-cost tradeoffs in control," *IEEE Trans. Autom. Control*, vol. 64, no. 11, pp. 4525–4540, Nov. 2019.
- [28] P. A. Stavrou and J. Østergaard, "Fixed-rate zero-delay source coding for stationary vector-valued gauss-Markov sources," in *Proc. Data Compress. Conf.*, Mar. 2018, pp. 257–266.
- [29] V. Kostina, "Data compression with low distortion and finite blocklength," *IEEE Trans. Inf. Theory*, vol. 63, no. 7, pp. 4268–4285, Jul. 2017.
- [30] S. Yüksel and T. Başar, Stochastic Networked Control Systems: Stabilization and Optimization Under Information Constraints. New York, NY, USA: Springer, 2013.
- [31] A. Mahajan and D. Teneketzis, "Optimal design of sequential realtime communication systems," *IEEE Trans. Inf. Theory*, vol. 55, no. 11, pp. 5317–5338, Nov. 2009.
- [32] Y. Kaspi and N. Merhav, "Structure theorems for real-time variable rate coding with and without side information," *IEEE Trans. Inf. Theory*, vol. 58, no. 12, pp. 7135–7153, Dec. 2012.
- [33] Y. Kaspi and N. Merhav, "Zero-delay and causal single-user and multiuser lossy source coding with decoder side information," *IEEE Trans. Inf. Theory*, vol. 60, no. 11, pp. 6931–6942, Nov. 2014.
- [34] A. Nayyar and D. Teneketzis, "On the structure of real-time encoding and decoding functions in a multiterminal communication system," *IEEE Trans. Inf. Theory*, vol. 57, no. 9, pp. 6196–6214, Sep. 2011.
- [35] P. Elias, "Channel capacity without coding," *Inst. Radio Eng.*, vol. 45, no. 3, 1957.
- [36] T. J. Goblick, Jr., "Theoretical limitations on the transmission of data from analog sources," *IEEE Trans. Inf. Theory*, vol. IT-11, no. 4, pp. 558–567, Oct. 1965.
- [37] J. Ziv, "The behavior of analog communication systems," *IEEE Trans. Inf. Theory*, vol. IT-16, no. 5, pp. 587–594, Sep. 1970.
- [38] J. Schalkwijk and T. Kailath, "A coding scheme for additive noise channels with feedback—Part I: No bandwidth constraint," *IEEE Trans. Inf. Theory*, vol. IT-12, no. 2, pp. 172–182, Apr. 1966.
- [39] R. G. Gallager and B. Nakiboğlu, "Variations on a theme by Schalkwijk and Kailath," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 6–17, Jan. 2010.
- [40] R. Bansal and T. Başar, "Solutions to a class of linear-quadratic-Gaussian (LQG) stochastic team problems with nonclassical information," *Syst. Control Lett.*, vol. 9, no. 2, pp. 125–130, Aug. 1987.
- [41] R. Bansal and T. Başar, "Simultaneous design of measurement and control strategies for stochastic systems with feedback," *Automatica*, vol. 25, no. 5, pp. 679–694, Sep. 1989.
- [42] V. S. Borkar, S. K. Mitter, and S. Tatikonda, "Optimal sequential vector quantization of Markov sources," *SIAM J. Control Optim.*, vol. 40, no. 1, pp. 135–148, 2001.
- [43] R. G. Wood, T. Linder, and S. Yüksel, "Optimal zero delay coding of Markov sources: Stationary and finite memory codes," *IEEE Trans. Inf. Theory*, vol. 63, no. 9, pp. 5968–5980, Sep. 2017.
- [44] C. Villani, Optimal Transport: Old and New. Berlin, Germany: Springer-Verlag, 2009.
- [45] O. Hernández-Lerma and J. Lasserre, Discrete-Time Markov Control Processes: Basic Optimality Criteria. Berlin, Germany: Springer-Verlag, 1996.
- [46] A. Antos, "On codecell convexity of optimal multiresolution scalar quantizers for continuous sources," *IEEE Trans. Inf. Theory*, vol. 58, no. 2, pp. 1147–1157, Feb. 2012.
- [47] D. Muresan and M. Effros, "Quantization as histogram segmentation: Optimal scalar quantizer design in network systems," *IEEE Trans. Inf. Theory*, vol. 54, no. 1, pp. 344–366, Jan. 2008.
- [48] S. Yüksel and T. Linder, "Optimization and convergence of observation channels in stochastic control," *SIAM J. Control Optim.*, vol. 50, no. 2, pp. 864–887, Jan. 2012.

- [49] V. S. Borkar, "Average cost dynamic programming equations for controlled Markov chains with partial observations," *SIAM J. Control Optim.*, vol. 39, no. 3, pp. 673–681, 2000.
- [50] V. S. Borkar, "Dynamic programming for ergodic control of Markov chains under partial observations: A correction," *SIAM J. Control Optim.*, vol. 45, no. 6, pp. 2299–2304, 2007.
- [51] L. Bao, M. Skoglund, and K. H. Johansson, "Iterative encodercontroller design for feedback control over noisy channels," *IEEE Trans. Autom. Control*, vol. 56, no. 2, pp. 265–278, Feb. 2011.
- [52] V. S. Borkar and S. K. Mitter, "LQG control with communication constraints," in *Kailath Festschrift*. Boston, MA, USA: Kluwer, 1997.
- [53] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," *Proc. IEEE*, vol. 95, no. 1, pp. 108–137, Jan. 2007.
- [54] M. Fu, "Lack of separation principle for quantized linear quadratic Gaussian control," *IEEE Trans. Autom. Control*, vol. 57, no. 9, pp. 2385–2390, Sep. 2012.
- [55] S. Yuksel, "Jointly optimal LQG quantization and control policies for multi-dimensional systems," *IEEE Trans. Autom. Control*, vol. 59, no. 6, pp. 1612–1617, Jun. 2014.
- [56] S. Yüksel, "A note on the separation of optimal quantization and control policies in networked control," *SIAM J. Control Optim.*, vol. 57, no. 1, pp. 773–782, Jan. 2019.
- [57] A. Arapostathis, V. S. Borkar, E. Fernández-Gaucherand, M. K. Ghosh, and S. I. Marcus, "Discrete-time controlled Markov processes with average cost criterion: A survey," *SIAM J. Control Optim.*, vol. 31, no. 2, pp. 282–344, Mar. 1993.
- [58] S. Yüksel, "Control of stochastic systems," Queen's Univ., Kingston, ON, USA, Lecture Notes, 2021.
- [59] R. M. Dudley, *Real Analysis and Probability*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 2002.
- [60] H.-J. Langen, "Convergence of dynamic programming models," Math. Oper. Res., vol. 6, no. 4, pp. 493–512, Nov. 1981.
- [61] C. Villani, *Topics in Optimal Transportation*, no. 58. Providence, RI, USA: American Mathematical Society, 2003.

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