# An Asymptotically Optimal Two-Part Fixed-Rate Coding Scheme for Networked Control With Unbounded Noise

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***Abstract*— It is known that under fixed-rate information constraints, adaptive quantizers can be used to stabilize an open-loop-unstable linear system on $\mathbb{R}^n$ driven by unbounded noise. These adaptive schemes can be designed so that they have near-optimal rate, and the resulting system will be stable in the sense of having an invariant probability measure, or ergodicity, as well as boundedness of the state second moment. Although structural results and information theoretic bounds of encoders have been studied, the performance of such adaptive fixed-rate quantizers beyond stabilization has not been addressed. In this paper, we propose a two-part adaptive (fixed-rate) coding scheme that achieves state second moment convergence to the classical optimum (i.e., for the fully observed setting) under mild moment conditions on the noise process. The first part, as in prior work, leads to ergodicity (via positive Harris recurrence) and the second part ensures that the state second moment converges to the classical optimum at high rates. These results are established using an intricate analysis which uses random-time state-dependent Lyapunov stochastic drift criteria as a core tool.**

***Index Terms*— Networked control, stochastic stability, ergodicity, source coding, quantization, stochastic optimal control.**

## I. Introduction

NETWORKED *control* or *information-constrained control* refers to control systems in which the controllers, sensors, and systems (actuators/plants) are connected through communication channels or a data-rate constrained network. Thus, there may be a data link between the sensors (which collect information), the controllers (which make decisions), and the actuators (which execute the controller commands). Moreover, the sensors, controllers and the plant themselves could be geographically separated. For such information-constrained control (or networked control) systems, one needs to jointly design encoders and controllers for satisfactory performance, which may have stability or optimality as a design objective.

In cases where stability is the primary design objective, one is typically concerned with the minimum capacity above which stabilization is possible, and there are many results of this flavour in the existing literature (these are discussed in detail in Section I-B).

This paper is primarily concerned with optimality as the primary design objective. In this context, the notion of optimality is the minimization of some cost function over a specified time horizon (which here is infinite), and for the kinds of systems we consider here, one seeks asymptotic bounds on the cost function as the data rate becomes large.

### A. Problem Statement

Let us first introduce the system to be controlled under no information constraints. The optimal cost in this "classical" case will yield a lower bound over all information-constrained policies. Consider the linear (but not necessarily Gaussian) discrete-time multi-dimensional control system,

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{1}$$

where the state process $x_t$, the control $u_t$, and the noise process $w_t$ live in $\mathbb{R}^n$, $A$ and $B$ are $n \times n$ real matrices, and we assume $B$ to be invertible (this can be relaxed to controllability by a sampling argument for stability [2], though in this case the optimality results we present are not maintained).

The initial state $x_0$ may be distributed according to some probability measure $\nu$ on $\mathbb{R}^n$, as long as $x_0 \sim \nu$ admits at least the same finite moments as the noise process $w_t$. That is, whenever $E\left[\|w_t\|_\infty^\beta\right] < \infty$ for some $\beta > 0$, we have $E\left[\|x_0\|_\infty^\beta\right] < \infty$ (this is trivially satisfied if $\nu = \delta_x$ for some $x \in \mathbb{R}^n$). In addition, we suppose that the initial state $x_0$ is independent of the noise process $w_t$.

In the classical, fully observed setting, at each time stage $t \geq 0$, the controller has access to the history $I_t = x_{[0,t]} = (x_0, \ldots, x_t)$. An admissible control policy $\gamma$ is a sequence of Borel measurable mappings $\{\gamma_t; t \geq 0\}$ where $\gamma_t : I_t \to \mathbb{R}^n$ is such that it produces the control $u_t = \gamma_t(I_t)$ at each time stage.

The noise process $\{w_t\}_{t=0}^\infty$ is assumed to be i.i.d., zero-mean with covariance matrix $\Sigma = E\left[w_0 w_0^\top\right]$. Furthermore, we suppose that the noise process admits a pdf $\eta$ with respect

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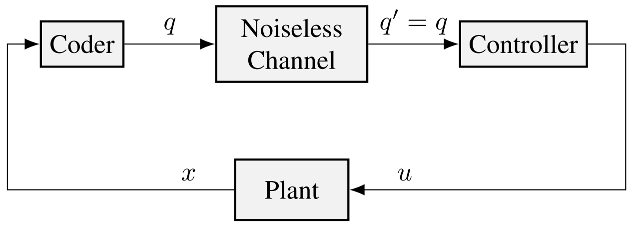


Fig. 1. Block diagram of the communication and control loop.

to the Lebesgue measure on $\mathbb{R}^n$ which is positive everywhere (and thus has unbounded support).

For an $n \times n$ positive definite matrix $Q$, the optimal control problem is to choose a policy $\gamma$ which minimizes the infinite-horizon average quadratic form,

$$\limsup_{T\to\infty} \frac{1}{T} E^{\gamma} \left[ \sum_{t=0}^{T-1} x_t^\top Q x_t \right], \tag{2}$$

where the expectation above is with respect to the policy $\gamma$ and $x^\top$ denotes the transpose of the column vector $x \in \mathbb{R}^n$.

The following statement is a special case of a well-known result for a more general setup (where, e.g., $B$ may not be invertible) and can be obtained by solving the algebraic Riccati equation (see, e.g., [3, Chap. 4.1]. However, we provide a short and direct proof in the Appendix.

*Proposition 1:* For the fully observed setup described above, the optimal control policy is $u_t = -B^{-1}Ax_t$, achieving an optimal cost of $\operatorname{tr}(Q\Sigma)$ where $\operatorname{tr}(\cdot)$ is the trace operator.

In contrast to this idealized fully observed setup, in this paper we assume that the controller only has access to $x_t$ through a discrete noiseless channel of capacity $C$ bits. We assume that the encoder is causal. In particular, letting $\mathcal{M}$ be a finite alphabet of cardinality $|\mathcal{M}| = \lfloor 2^C \rfloor$, the encoder is specified by a quantization policy $\Pi$, which, with $\mathbb{X} = \mathbb{R}^n$ as the state space, is a sequence of functions $\{\eta_t\}_{t=0}^{\infty}$ of the type $\eta_t : \mathcal{M}^t \times \mathbb{X}^{t+1} \to \mathcal{M}$. At time $t$, the encoder transmits the $\mathcal{M}$-valued message

$$q_t = \eta_t(I_t^e),$$

where $I_0^e = x_0$ and $I_t^e = (q_{[0,t-1]}, x_{[0,t]})$ for $t \geq 1$. The collection of all such zero-delay policies is called the set of admissible quantization policies and is denoted by $\mathbf{\Pi}_A$.

Upon receiving $q_t$, the receiver generates the control $u_t$, also without delay. A zero-delay controller policy is a sequence of functions $\gamma = \{\gamma_t\}_{t=0}^{\infty}$ such that $\gamma_t : \mathcal{M}^{t+1} \to \mathbb{U}$, where $\mathbb{U} = \mathbb{R}^n$ is the control action space, so that $u_t = \gamma_t(I_t^d)$, where $I_t^d = q_{[0,t]}$.

Thus, the data rate is fixed, and we assume zero coding delay. In this setup, it becomes necessary to describe not just a control policy, but also a coding scheme with which to communicate information about the current state vector.

### B. Literature Review

For systems of this nature, various authors have obtained the minimum channel capacity above which stabilization is possible, under various assumptions on the system and the admissible coders and controllers. Here, "stabilization" can be in several senses, for example positive Harris recurrence, asymptotic mean stationarity or more generally, limiting moment stability of the state.

Such a result is usually referred to as a *data-rate theorem* and takes the following form, with $\{\lambda_i\}$ being the eigenvalues of the system matrix $A$:

$$C > R_{\min} := \sum_{|\lambda_i|\geq 1} \log_2 |\lambda_i| \tag{3}$$

That is, the capacity must exceed the sum of the unstable eigenvalue logarithms.

Some of the earliest works in this context are [4] and [5]. More general versions of the data-rate theorem have been proven in [6] and [7]. For noisy systems and mean-square stabilization, or more generally, moment-stabilization, analogous data-rate theorems have been proven in [8] and [9], see also [10], [11].

In [12] and [13], a joint fixed-rate coding and control scheme is given which, in the scalar case $n = 1$ with unstable eigenvalue $|\lambda| \geq 1$ and where $w_t$ is Gaussian, stabilizes the system (1) while being nearly rate-optimal, in that the rate used satisfies only $C > \log(|\lambda| + 1)$. This is achieved using an adaptive uniform quantization scheme, where the quantizer bin sizes "zoom" in and out exponentially to track the state $x_t$. Here, the notion of stability is ergodicity and finiteness of all limiting system moments. By increasing a sampling period $T$, the achievable rate $\frac{1}{T}\log(|\lambda|^T + 1)$ gets arbitrarily close to $C > R_{\min} = |\lambda|$ [2, Theorem 2.3]. This scheme can be generalized to one which stabilizes the multi-dimensional system (1) (where the noise is more general than Gaussian) using a similar approach [2]. Furthermore, this leads to a closed loop system which is positive Harris recurrent (and hence, ergodic) and admits finite limiting system second moment [2, Theorem 2.2]. For related recent fixed-rate constructions which also utilize modest delay, we refer the reader to [14] and [15].

Despite being near rate-optimal for achieving stability (i.e., finite asymptotic system moments), the schemes in [2] and [12] have not been shown to yield second moment convergence to the classical optimum $\operatorname{tr}(Q\Sigma)$ as the data rate $C$ grows large.

In the literature, information theoretic relaxations of the problem noted have been studied, for obtaining both lower and upper bounds. Lower bounding methods typically build on replacing the number of bins with entropy of the quantization symbols, or the latter with mutual information bounds, and the use of Shannon lower bounding techniques. Upper bounding methods include entropy coding and dithering methods; dithering [16] "uniformizes" the noise even for low rates (though which critically requires the presence of common randomness at the encoder and the decoder); see [17] and [21] [22], [23]. Using ergodicity and invariance properties, [24] has established time-invariant (though still variable-rate) coding schemes using dithering. Making use of lattice quantization performance bounds developed in [19], without the use of dithering [20] investigated distributed control with lattice quantization followed by entropy coding (and thus also with variable rate coding) and established near optimality of such schemes for high-rates.

In this paper, we will only use fixed-rate codes, which may be more suitable for a large class of zero-delay systems.

A further common approach in the literature has been to minimize the directed information [25] subject to the distortion rate:

$$\limsup_{N\to\infty} \min\Big\{\frac{1}{N} I(X^N \to \hat{X}^N) : \frac{1}{N} E\Big[\sum_{k=0}^{N-1} (X_k - \hat{X}_k)^2\Big] \le D\Big\}.$$

This leads to an information theoretic lower bound to the optimal estimation error subject to an information rate constraint. There has been a surge of research activity on this problem since [26], where explicit solutions, bounds, as well as convex analytic numerical solutions (including via semi-definite programming) have been presented; see, e.g., [18], [23], [27], [28], [29], [30], [31], [32], [33], and [34]. For noisy channels, using channel-source coding separation based methods via the rate-distortion function and Shannon capacity duality can be utilized to establish tightness results, especially for the Gaussian case. When an additive Gaussian channel is present, sequential rate-distortion theoretic ideas presented above in (also via generalizing the scalar Gaussian analysis [27]) lead to explicit optimality conditions studied in [30] and [35] (see also [20])

We refer to [36] and [37] for a detailed review on structural results for optimal coding of controlled linear systems. Infinite horizon zero-delay coding for linear systems is studied in [38] and [39].

### C. Contributions

We study the class of stochastic linear systems having transition dynamics given in (1) and driven by unbounded noise that are to be controlled across a discrete noiseless channel of finite capacity. For such systems, we present a novel two-stage coding scheme, in which the first stage is time-adaptive and stabilizing (in the sense of positive Harris recurrence and finite limiting system moments), while the second stage is fixed in time. The first coding stage is a variation on the schemes in [2], [12], and [13].

Crucially utilizing the ergodicity results of the first coding stage, we show that this two-part coding scheme attains convergence of the limiting system second moment to the classical optimum as the data rate $C$ grows large, with explicit rate of convergence. While multi-stage quantization schemes have been studied before in the source coding literature [40], our implementation is novel in that one stage of the code is time-adaptive (though still with fixed-rate) and stabilizing, and the second stage is designed for near-optimal performance at high-rates. An inspiration for this approach also comes from Berger [41] and Sahai [42]. Our analysis complements recent studies in the literature [20], [24] which have utilized variable rate codes (without and with dithers, respectively), but more importantly, it presents a two-part architecture in code design, separately targeting stability and optimality.

To our knowledge, this is the first scheme proven to have this convergence property for systems of this type (in particular, with unbounded noise) in networked control. The algorithm to be presented is relatively simple, and being fixed-rate, highly practical, but its performance analysis requires quite involved technical steps which necessitate a technically involved analysis using, among other tools, random-time state-dependent stochastic Lyapunov drift conditions [12] and a careful analysis of the performance of finite-level uniform quantizers for unbounded sources that satisfy certain moment constraints.

### D. Organization

The paper is organized as follows. Section II contains relevant preliminaries, some background on stochastic stability for general state-space Markov chains, and statements of some useful random-time Lyapunov drift theorems that are central to our analysis.

Section III presents the construction of our two-part scheme in the scalar case for simplicity, and provides a somewhat detailed proof program to guide the reader. Section IV provides the construction of our two-part scheme in the vector case, in full generality. A detailed proof program in the vector case is provided in Section IV-D.

Finally, as many of the proofs are rather mechanical and quite involved, the Appendix contains complete proofs of many key results stated in the main body of the paper.

## II. Preliminaries

### A. Definitions and Conventions

We denote the nonnegative and strictly positive reals by $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively. We let $\mathbb{N}$ denote the nonnegative integers.

For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ and $p \in [1, \infty]$, $\|x\|_p$ denotes the $p$-norm on $\mathbb{R}^n$, defined by $\|x\|_p := \left(\sum_{i=1}^n |x^i|^p\right)^{\frac{1}{p}}$ for $p \in [1, \infty)$ and $\|x\|_\infty := \max_{1\le i\le n} |x^i|$ for $p = \infty$.

The following pair of inequalities is well-known (e.g., see [43, Exercise 3.5(a)]):

*Proposition 2:* Suppose $1 \le p \le q \le \infty$. Then for any $x \in \mathbb{R}^n$,

$$\|x\|_q \le \|x\|_p \le n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q, \tag{4}$$

where $\frac{1}{\infty}$ is taken to be 0 by convention.
The first inequality follows by a simple renormalization argument, while the second follows from Hölder's inequality.

For a matrix $V \in \mathbb{R}^{m\times n}$, we will refer to the $(i, j)$-th component either as $V_{ij}$ or as $[V]_{ij}$. The latter notation will be used when $V$ takes on an expression involving square brackets (e.g., an expectation) so as to avoid ambiguity.

For a matrix $A \in \mathbb{R}^{m\times n}$ we define its $\infty$-norm as $\|A\|_\infty := \max_{1\le i\le m} \sum_{j=1}^n |A_{ij}|$. This norm is consistent with the vector $\infty$-norm in the following sense. Let $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$. Then,

$$\|Av\|_\infty \le \|A\|_\infty \|v\|_\infty. \tag{5}$$

We will find it useful to use the Landau notation for comparing function asymptotics. For two functions $f, g : [a, \infty) \to \mathbb{R}_+$, where $a \in \mathbb{R}$, we say that $f = \mathcal{O}_u(g)$ if for all $u$ sufficiently large one has $f(u) \le cg(u)$ for some constant $c > 0$. This is equivalent to the condition that $\limsup_{u\to\infty} \frac{f(u)}{g(u)} < \infty$.

Finally, for a vector-valued random variable $X$ we denote its "tail function" as

$$T_X(u) := P(\|X\|_\infty > u), \quad u \geq 0.$$

### B. Stochastic Stability

In this section we provide some brief background on stochastic stability, particularly that which will be relevant to the stabilizing properties of the two-stage scheme we present (namely, positive Harris recurrence).

Suppose $\{\phi_t\}_{t=0}^\infty$ is a time-homogeneous Markov chain with state space $\mathbb{X}$, where $\mathbb{X}$ is a complete separable metric space that is locally compact; its Borel sigma algebra is denoted $\mathcal{B}(\mathbb{X})$. The transition probability is denoted by $P$, so that for any $\phi \in \mathbb{X}$ and $A \in \mathcal{B}(\mathbb{X})$, the probability of moving in one step from state $\phi$ to the set $A$ is given by $P(\phi_{t+1} \in A \mid \phi_t = \phi) =: P(\phi, A)$. For any $n \geq 2$, the $n$-step transitions $P(\phi_{t+n} \in A \mid \phi_t = \phi) =: P^n(\phi, A)$ are obtained recursively in the usual way:

$$P^n(\phi, A) = \int_{\mathbb{X}} P(y, A) P^{n-1}(\phi, dy).$$

The transition law acts on measurable functions $f : \mathbb{X} \to \mathbb{R}$ and measures $\mu$ on $\mathcal{B}(\mathbb{X})$ via

$$Pf(\phi) := \int_{\mathbb{X}} P(\phi, dy) f(y) \; = E\left[f(\phi_{t+1}) \mid \phi_t = \phi\right],$$

for all $\phi \in \mathbb{X}$, and

$$\mu P(A) := \int_{\mathbb{X}} \mu(d\phi) P(\phi, A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

A probability measure $\pi$ on $\mathcal{B}(\mathbb{X})$ is called invariant if $\pi P = \pi$, that is,

$$\int_{\mathbb{X}} \pi(d\phi) P(\phi, A) = \pi(A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

For any initial probability measure $\nu$ on $\mathcal{B}(\mathbb{X})$ we can construct a stochastic process with transition law $P$ and $\phi_0 \sim \nu$. We let $P_\nu$ denote the resulting probability measure on the sample space, with the usual convention that $\nu = \delta_\phi$ (i.e., $\nu(\{\phi\}) = 1$) when the initial state is $\phi \in \mathbb{X}$. When $\nu = \pi$ is invariant, the resulting process is stationary.

There is at most one stationary solution under the following irreducibility assumption. For a set $A \in \mathcal{B}(\mathbb{X})$ we denote,

$$\tau_A := \min\{t \geq 1 : \phi_t \in A\}. \tag{6}$$

*Definition 3:* Let $\varphi$ denote a $\sigma$-finite measure on $\mathcal{B}(\mathbb{X})$.

(i) The Markov chain is called *$\varphi$-irreducible* if for any $\phi \in \mathbb{X}$ and $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$ we have

$$P_\phi(\tau_B < \infty) > 0.$$

(ii) A $\varphi$-irreducible Markov chain is *aperiodic* if for any $\phi \in \mathbb{X}$ and any $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$, there exists $n_0 = n_0(\phi, B)$ such that for all $n \geq n_0$,

$$P^n(\phi, B) > 0.$$

(iii) A $\varphi$-irreducible Markov chain is *Harris recurrent* if $P_\phi(\tau_B < \infty) = 1$ for any $\phi \in \mathbb{X}$ and any $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$. It is *positive Harris recurrent* if in addition there is an invariant probability measure $\pi$.

The notion of full and absorbing sets will be useful to us.

*Definition 4:* For a $\varphi$-irreducible Markov chain $\{\phi_t\}_{t=0}^\infty$, a set $A \in \mathcal{B}(\mathbb{X})$ is called full if $\varphi(A^C) = 0$.

*Definition 5:* A set $A \in \mathcal{B}(\mathbb{X})$ is called absorbing if $P(x, A) = 1$ for all $x \in A$.

Finally, we define the notion of small sets for a Markov chain.

*Definition 6:* A set $C \in \mathcal{B}(\mathbb{X})$ is $(m, \delta, \nu)$-small on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ (for integer $m \geq 1$, $\delta \in (0, 1]$ and a probability measure $\nu$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$) if for all $x \in C$ and $B \in \mathcal{B}(\mathbb{X})$,

$$P^m(x, B) \geq \delta \nu(B).$$

A set is called small (or sometimes $m$-small) if it is $(m, \delta, \nu)$-small for some $(m, \delta, \nu)$.

Briefly we provide an intuition for small sets. Suppose $C \in \mathcal{B}(\mathbb{X})$ is $(m, \delta, \nu)$-small. Whenever the state $\phi_t$ happens to visit the set $C$, it forgets its entire past with probability at least $\delta > 0$ and transitions according to the probability measure $\nu$ over the next $m$ time stages. In this way, the small set $C$ acts as a "regenerative set" from which the process can forget its history. This line of investigation leads one to Nummelin's splitting technique [44], [45], which is a key tool in many stability results for irreducible Markov chains.

### C. Random-Time State-Dependent Stochastic Lyapunov Drift Conditions for Stability

In this section, we present a drift condition [13] and use it to show two stability results which will be crucial to our analysis. As in the previous section, we consider a general Markov chain $\{\phi_t\}_{t=0}^\infty$. We consider a sequence of stopping times $\{\mathcal{T}_z\}_{z=0}^\infty$, measurable with respect to the natural filtration of $\{\phi_t\}_{t=0}^\infty$, such that $\{\mathcal{T}_z\}_{z=0}^\infty$ is strictly increasing and $\mathcal{T}_0 = 0$. Finally, we will make use of the filtration $\{\mathcal{F}_{\mathcal{T}_z}\}_{z=0}^\infty$ which is informally the filtration of "information generated by $\{\phi_t\}_{t=0}^\infty$ up to time $\mathcal{T}_z$" (for full details, see [46]).

The following is a condition on the general Markov chain $\{\phi_t\}_{t=0}^\infty$ introduced in [13].

*Condition 7 (Random-Time Lyapunov Drift):* For a measurable function $V : \mathbb{X} \to (0, \infty)$, measurable functions $f, d : \mathbb{X} \to [0, \infty)$, a constant $b \in \mathbb{R}$ and a set $C \in \mathcal{B}(\mathbb{X})$, we say that $\{\phi_t\}_{t=0}^\infty$ satisfies the random-time Lyapunov drift condition at $\phi \in \mathbb{X}$ if for all $z = 0, 1, 2 \ldots$

$$E\left[V(\phi_{\mathcal{T}_{z+1}}) \mid \mathcal{F}_{\mathcal{T}_z}\right] \leq V(\phi_{\mathcal{T}_z}) - d(\phi_{\mathcal{T}_z}) + b \mathbf{1}_{\{\phi_{\mathcal{T}_z} \in C\}},$$

and

$$E\left[\sum_{t=\mathcal{T}_z}^{\mathcal{T}_{z+1}-1} f(\phi_t) \;\middle|\; \mathcal{F}_{\mathcal{T}_z}\right] \leq d(\phi_{\mathcal{T}_z}). \tag{7}$$

when $\phi_0 = \phi$.

*Remark 8:* Suppose that the stopping times $\mathcal{T}_z$ are the sequential return times to some set $\Lambda \in \mathcal{B}(\mathbb{X})$, that is $\mathcal{T}_0 = 0$ and

$$\mathcal{T}_{z+1} = \min\{t > \mathcal{T}_z : \phi_t \in \Lambda\}.$$

In this case, if one is able to verify for all $\phi \in \Lambda$ that

$$E_\phi \left[V(\phi_{\tau_\Lambda})\right] \leq V(\phi) - d(\phi) + b\mathbf{1}_{\{\phi \in C\}},$$

and

$$E_\phi \left[\sum_{t=0}^{\tau_\Lambda - 1} f(\phi_t)\right] \leq d(\phi), \tag{8}$$

then it follows automatically that Condition 7 holds at every $\phi \in \Lambda$. Notably, if $C \subseteq \Lambda$ then Condition 7 holds at every $\phi \in C$.

This drift condition, in combination with different assumptions on the functions and sets involved, can lead to many useful results on stability (e.g., [13, Theorem 2.1]). We present two such results here.

*Remark 9:* The results presented here are variations of [13, Theorem 2.1], presented in the form most useful for our application. The proofs of these results draw heavily from the proof program in [13] and rely on supermartingale arguments. For brevity we have omitted the proofs here; for details, see [13] and [46].

*Lemma 10:* Suppose $\{\phi_t\}_{t=0}^\infty$ is $\varphi$-irreducible and satisfies Condition 7 at all $\phi \in C$, with the restrictions that $C$ is a small set, $\sup_{\phi \in C} V(\phi) < \infty$, $f \equiv 1$ and $d(\phi) \geq 1$. Then the set

$$\mathcal{X} := \{\phi \in \mathbb{X} : P_\phi \left(\tau_C < \infty\right) = 1\}$$

is full and absorbing, and the restriction of $\{\phi_t\}_{t=0}^\infty$ to $\mathcal{X}$ is positive Harris recurrent.

If in addition one can show that $P_\phi \left(\tau_C < \infty\right) = 1$ for all $\phi \in \mathbb{X}$ (i.e., $\mathcal{X} = \mathbb{X}$), then $\{\phi_t\}_{t=0}^\infty$ is positive Harris recurrent.

*Lemma 11:* Suppose that $\{\phi_t\}_{t=0}^\infty$ is positive Harris recurrent with invariant measure $\pi$ and satisfies Condition 7 at some $\phi \in \mathbb{X}$. Then,

$$E_\pi \left[f(\phi_t)\right] \leq b$$

and for any function $g : \mathbb{X} \to [0, \infty)$ which is bounded by $f$ in that $g(\cdot) \leq cf(\cdot)$ for some constant $c > 0$, we have the following ergodic theorem for $g$,

$$\lim_{n \to \infty} \frac{1}{n} E_{\phi_0} \left[\sum_{t=0}^{n-1} g(\phi_t)\right] = E_\pi \left[g(\phi_t)\right] \tag{9}$$

for every $\phi_0 \in \mathbb{X}$.

## III. Scalar Linear Systems

We begin with considering the scalar case for simplicity. Here we consider control of the scalar system,

$$x_{t+1} = ax_t + bu_t + w_t, \tag{10}$$

where $b \neq 0$ and the noise process $w_t$ is assumed to be i.i.d., zero-mean with finite second moment $\sigma^2 = E\left[w_0^2\right]$. Furthermore, we suppose the noise process admits a pdf $\eta$ with respect to Lebesgue measure on $\mathbb{R}$ which is positive everywhere. Here we will also suppose that $|a| \geq 1$ so that the system is open-loop-unstable. That is, if we set $u_t = 0$ for all $t \geq 1$, then the system is transient and $x_t$ tends to infinity in magnitude (almost surely and in mean-square).

As in Section I-A, $x_0$ may be distributed with some probability measure $\nu$ on $\mathbb{R}$, so long as $x_0 \sim \nu$ admits the same finite absolute moments as the noise process $w_t$. In addition, we suppose that the initial state $x_0$ is independent of the noise process $w_t$.

Then the optimal control problem we study is to choose a policy $\gamma$ which minimizes the infinite-horizon quadratic cost,

$$\limsup_{T \to \infty} \frac{1}{T} E^\gamma \left[\sum_{t=0}^{T-1} x_t^2\right]. \tag{11}$$

By Proposition 1, in the fully observed setup the optimal control policy is $u_t = -\frac{a}{b}x_t$ which achieves an optimal cost of $\sigma^2$. We suppose now that the controller only has access to $x_t$ through a discrete noiseless channel of capacity $C$ bits.

Our goal here is to minimize $\limsup_{T \to \infty} \frac{1}{T} E^\gamma \left[\sum_{t=0}^{T-1} x_t^2\right]$ which we know is bounded below by $\sigma^2$. As the state will only be partially observed by the controller, we seek to arrive at asymptotic bounds on the "optimality gap" $\limsup_{T \to \infty} E^\gamma \left[\sum_{t=0}^{T-1} x_t^2\right] - \sigma^2$ in terms of the channel capacity $C$.

We say that $\phi_C$ is a joint coding and control scheme for $C > 0$ if $\phi_C$ specifies both a coding scheme for communicating over the channel of capacity $C$ as well as a corresponding control scheme at the channel receiver.

The following result demonstrates a limit on achievable rates of convergence.

*Lemma 12:* Under any joint coding and control scheme $\phi_C$ we have

$$\limsup_{T \to \infty} \frac{1}{T} E \left[\sum_{t=0}^{T-1} x_t^2\right] - \sigma^2 \geq \frac{a^2 \sigma^2}{2^{2C} - a^2}. \tag{12}$$

The proof follows the core arguments of the proof of [47, Theorem 11.3.2] (with several critical changes, as the assumed limit of distortions utilized in [47, Theorem 11.3.2] is to be replaced with a limit of average distortions, requiring additional steps) and is provided in the Appendix.

Intuitively, the above lemma implies that the fastest rate of convergence (of the optimality gap to zero) one can hope for is $2^{-2C}$. Motivated by this, we normalize potential rate functions by $2^{2C}$ and make the following definition.

*Definition 13:* A joint coding and control scheme $\phi_C$ achieves second moment convergence with rate function $r : \mathbb{R}_+ \to \mathbb{R}_+$ if,

$$\limsup_{T \to \infty} \frac{1}{T} E \left[\sum_{t=0}^{T-1} x_t^2\right] - \sigma^2 = \mathcal{O}_C \left(\frac{r(C)}{2^{2C}}\right).$$

For brevity we may say simply that $\phi_C$ achieves the rate function $r(C)$.

Intuitively, one seeks to achieve a rate function $r(C)$ which grows slowly in order to maximize the rate of convergence. Lemma 12 implies that the best achievable rate function is the constant function, so any achievable rate function satisfies $\limsup_{C \to \infty} r(C) > 0$.

We will impose the following mild condition on the noise process.

*Condition 14:* For some $\beta > 2$, $E\left[|w_0|^\beta\right] < \infty$. That is, the noise process has finite $\beta$th moment.

In the scalar case, our main result is the following.

*Theorem 15:* Supposing Condition 14 holds with $\beta > 2$, for any $\varepsilon \in (0, \beta - 2)$ there exists a joint coding and control scheme, denoted Scheme $\mathrm{P}(\beta, \varepsilon)$, which achieves the exponential rate function $r(C) = 2^{\frac{4}{\beta-\varepsilon}C}$.

Intuitively, with only the condition that the noise admits finite $\beta$th moment, we are able to construct schemes that nearly (as $\varepsilon \to 0$) achieve convergence of the optimality gap like $2^{(-2+\frac{4}{\beta})C}$. If $\beta$ is quite large, then the convergence becomes much closer to $2^{-2C}$ and, in the extreme case where $w_t$ admits finite moments of all orders (such as in the case of a Gaussian), one can construct schemes achieving convergence of the optimality gap at a speed $\mathcal{O}_C\left(2^{(-2+\delta)C}\right)$ for any $\delta > 0$.

The rest of Section III is dedicated to the construction of Scheme $\mathrm{P}(\beta, \varepsilon)$ and a high-level proof program of Theorem 15. All proofs are relegated to the Appendix.

### *A. Two-Part Code With Uniform Quantization*

In this section we describe the joint coding and control scheme used in proving Theorem 15.

The coding scheme is in two parts where the first part is adaptive in time and the second is fixed. The adaptive part will yield stability, and the fixed part will yield an optimal rate of convergence via simple iterated expectation arguments.

To communicate over a finite capacity channel, will employ uniform quantizers. Let $M \geq 2$ be an even integer and $\Delta > 0$ be a scalar "bin size". We define the *scalar modified uniform quantizer* $Q^{\Delta}_M$ by,

$$Q^{\Delta}_M(x) = \begin{cases} \Delta\left\lfloor \frac{x}{\Delta}\right\rfloor + \frac{\Delta}{2}, & \text{if } x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right) \\ \frac{M}{2}\Delta - \frac{\Delta}{2}, & \text{if } x = \frac{M}{2}\Delta \\ 0, & \text{if } |x| > \frac{M}{2}\Delta. \end{cases} \quad (13)$$

This quantizer uniformly quantizes $x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right]$ into $M$ bins of size $\Delta$ and maps all larger $x$ to zero. This requires $M+1$ output levels.

We will use this quantizer for two different purposes.

(i) The first is to use adaptive bin sizes which vary with time to achieve stability (in the sense of positive Harris recurrence and finite system moments). Let $K \geq 2$ be an even integer, and suppose that $\{\Delta_t\}_{t=0}^{\infty}$ is a sequence of strictly positive "bin sizes" varying with time. We will make use of the quantizer $Q^{\Delta_t}_K$.

(ii) Secondly, we will use this quantizer to achieve optimal convergence. For a given even number of bins $N \geq 2$, let $\Delta_{(N)}$ be a bin size which is a function of $N$. We will make use of the quantizer $U^{\Delta_{(N)}}_N$. For brevity, we denote $U_N := U^{\Delta_{(N)}}_N$ and where necessary, specify the dependence of $\Delta_{(N)}$ on $N$. Note that this quantizer is fixed in time, in contrast to $Q^{\Delta_t}_K$.

Suppose that the sequence of bin sizes $\{\Delta_t\}_{t=0}^{\infty}$ is such that $\Delta_{t+1}$ is a function of only $\Delta_t$ and the indicator random variable $\mathbf{1}_{\{|x_t| \leq \frac{K}{2}\Delta_t\}}$. Also assume that both the encoder and decoder (controller) know $\Delta_0$. If $Q^{\Delta_t}_K(x_t)$ is sent over the channel, it is possible to synchronize knowledge of $\Delta_t$ between the quantizer and the controller since $|x_t| \leq \frac{K}{2}\Delta_t$ if and only if $Q^{\Delta_t}_K(x_t) \neq 0$.

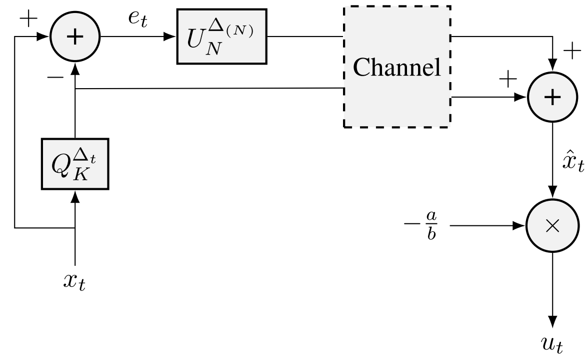


Fig. 2. Block diagram of the two-stage coding and control scheme in the scalar case.

The coding scheme is constructed as follows. For $\{\Delta_t\}_{t=0}^{\infty}$ as above, we calculate the adaptive quantizer output $Q^{\Delta_t}_K(x_t)$ and the adaptive system error $e_t := x_t - Q^{\Delta_t}_K(x_t)$. Then for integer $N \geq 2$ we use a fixed quantizer $U_N$ with bin size $\Delta_{(N)}$ as mentioned above to calculate the fixed quantizer output $U_N(e_t)$. We then send $Q^{\Delta_t}_K(x_t)$ and $U_N(e_t)$ across the noiseless channel where the channel capacity is at least,

$$C = \log_2(K+1) + \log_2(N+1). \quad (14)$$

We estimate the state $x_t$ as $\hat{x}_t := Q^{\Delta_t}_K(x_t) + U_N(e_t)$. To mirror the fully observed case, the controller applies the control

$$u_t = -\frac{a}{b}(Q^{\Delta_t}_K(x_t) + U_N(e_t)) = -\frac{a}{b}\hat{x}_t.$$

The scheme is illustrated in Figure 2.

The controlled system dynamics resulting from this scheme are

$$x_{t+1} = a(x_t - \hat{x}_t) + w_t = a(e_t - U_N(e_t)) + w_t. \quad (15)$$

Finally, we describe the adaptive bin size update dynamics where, as in prior work in this context [12], [13], a simple *zooming* scheme is employed. We assume that $K \geq 2$ is even and large enough that $K > |a|$ and choose scalars $\alpha, \rho$ and $L$ such that $\frac{|a|}{K} < \alpha < 1$, $\rho > |a|$ and $L > 0$. We also assume that $\rho \geq K\alpha$. Choose $\Delta_0 \geq L$ arbitrarily, then for $t \geq 1$ the bin update is

$$\Delta_{t+1} = \begin{cases} \rho\Delta_t, & \text{if } |x_t| > \frac{K}{2}\Delta_t \\ \alpha\Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t \geq L \\ \Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t < L. \end{cases} \quad (16)$$

The following result is proved, e.g,. in [13].

*Proposition 16:* With dynamics (15) and (16), the process $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is a time-homogeneous Markov chain.

The motivation for this scheme is that the adaptive part leads to stability in the sense of positive Harris recurrence, while the fixed quantizer $U_N$ leads to order-optimal convergence of the ergodic second moment (11) as the fixed quantization rate $N$ grows large.

The state space for the process $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ highly depends on the following "countability condition".

*Condition 17:* There exist relatively prime integers $j$, $k \geq 1$ such that $\alpha^j \rho^k = 1$. Equivalently, $\log_\alpha \rho$ is rational.

If this condition holds, then starting from an arbitrary $\Delta_0 > 0$ there exists $\kappa, b \in \mathbb{R}$ such that $\log \Delta_t$ always belongs to a subset of $\mathbb{Z}\kappa + b = \{n\kappa + b : n \in \mathbb{Z}\}$ (see e.g. [13, Theorem 3.1]). If the condition fails, then starting from any fixed $\Delta_0$ the set of reachable bin sizes is a dense but countable subset of $\mathbb{R}_{++}$.

We restrict our analysis to the case where Condition 17 holds. This is not restrictive; it can be shown that for any arbitrary $(\alpha, \rho)$ there exists $(\alpha', \rho')$ arbitrarily close that satisfy Condition 17. We let the state space for $\Delta_t$ be

$$\Omega_\Delta := \left\{ \alpha^j \rho^k \Delta_0 : j, k \in \mathbb{Z}_{\geq 0} \right\}.$$

The state space for the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is then $\mathbb{R} \times \Omega_\Delta$. What remains is to specify additional constraints which complete our proposed scheme.

We assume that Condition 14 holds for some $\beta > 2$. For any $\varepsilon \in (0, \beta - 2)$ we finish our construction of Scheme $\mathrm{P}(\beta, \varepsilon)$ by requiring that $\rho > |a|^{\frac{\beta}{\varepsilon}}$ and specifying the dependence of $\Delta_{(N)}$ on $N$ as $\Delta_{(N)} = 2N^{-1+\frac{2}{\beta-\varepsilon}}$.

*Remark 18:* In this scheme, the constant multiplying $\Delta_{(N)}$ is arbitrary for convergence purposes.

We also impose the following condition in our construction.

*Condition 19:* The minimum adaptive bin size is at least $\alpha L > \frac{|a|}{K\alpha - |a|}\Delta_{(N)}$.

This may place an implicit dependence of $L$ on the number of fixed quantization bins $N \geq 2$, though if all other parameters remain fixed as $N$ increases then one can ensure Condition 19 holds by ensuring that it holds for $N = 2$ (since $\Delta_{(N)}$ as specified above is monotone decreasing in $N$).

Finally, as $C \to \infty$ we fix $K$ and let $N \to \infty$ to take advantage of fixed quantization results at high rates, to be presented shortly.

Since the proof of Theorem 15 is rather tedious, we present a somewhat detailed proof program to guide the reader for the scalar setup.

We note that while the proof method for stabilization of our scheme builds on the random-time Lyapunov drift approach introduced in [12] and [13], the coupling between the two parts of the coding scheme significantly complicates the analysis. Furthermore, we consider performance bounds as the data rate grows without bound. Altogether, this requires a cautious analysis between moments and high-rate quantization coupled with random-time drift criteria.

### *B. High-Level Proof Program*

In this section we outline the high-level proof program for Theorem 15, i.e., the proof that Scheme $\mathrm{P}(\beta, \varepsilon)$ achieves the exponential rate function $r(C) = 2^{\frac{4}{\beta-\varepsilon}C}$. Results without complete proofs admit more general sister results in the vector case, which is discussed in detail in Section IV-D.

*Theorem 20:* Under Scheme $\mathrm{P}(\beta, \varepsilon)$ with $K > |a|$, $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is positive Harris recurrent for every even $N \geq 2$ (i.e., as $C$ grows without bound). Therefore, for every even $N \geq 2$, Scheme $\mathrm{P}(\beta, \varepsilon)$ yields a unique invariant measure $\pi_N$ for the process $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$.

*Sketch of Proof:* We establish $\varphi$-irreducibility and aperiodicity for $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ where $\varphi$ is the product of the Lebesgue and discrete measures on $\mathbb{R} \times \Omega_\Delta$. The logarithmic function $V(x, \Delta) = c \log_\alpha \Delta$ is shown to satisfy Condition 7 with $d(x, \Delta)$ constant and $f \equiv 1$ (i.e., in the form required by Lemma 10), leading to positive Harris recurrence. For a complete proof, see the proof of Theorem 36 (vector case). □

*Remark 21:* The analysis here is highly similar to that of the proof of [13, Theorem 3.1]. The only major change is that the upper bound of [13, Lemma 5.2] for the tail probabilities $P_{x_0,\Delta_0}(\tau_\Lambda \geq k)$ must be re-derived in some form, as the out-of-view state dynamics change significantly due to the fixed quantization stage. We address this by providing a similar bound in Lemma 45 (compare to equation (26) in [13]), which decays suitably fast for summability in the proof program under Condition 14.

We denote $(x_{*,N}, \Delta_{*,N}) \sim \pi_N$ as the state under invariant measure. This will also induce an invariant measure for the system adaptive error $e_t$, which we denote by $e_{*,N} \sim \pi_N^{\mathrm{err}}$.

We have the following ergodicity result.

*Proposition 22:* The infinite-horizon second moment and the invariant second moment agree, that is

$$\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] = E\left[(x_{*,N})^2\right].$$

*Sketch of Proof:* We are able to show using the drift conditions of Section II-C that functions $g(x, \Delta)$ that are bounded asymptotically by $|x|^{\beta-\varepsilon}$ satisfy the above ergodicity condition. Since $\varepsilon < \beta - 2$, $g(x, \Delta) = x^2$ is bounded by $|x|^{\beta-\varepsilon}$ and the result follows. For a complete proof, see the proof of Proposition 37. □

Finally, we use a simple iterated expectation argument. Suppose that $(x_0, \Delta_0) \sim \pi_N$. Let $e_0 = x_0 - Q_K^{\Delta_0}(x_0)$ and $x_1 = a(e_0 - U_N(e_0)) + Z$, where $Z \sim \eta$ (recall $\eta$ is the distribution of $w_t$). Since we have applied the one-step transition kernel and $\pi_N$ is the invariant measure, the marginal distributions of $x_0$ and $x_1$ are identical.

For brevity, we denote $s_0 := e_0 - U_N(e_0)$ so that $x_1 = as_0 + Z$. Supposing that $E\left[(x_{*,N})^2\right] < \infty$ (this will be shown as part of the full proof program), we then have by invariance and iterated expectations that

$$\begin{aligned} E\left[x_0^2\right] &= E\left[x_1^2\right] = E\left[E\left[x_1^2 \mid s_0\right]\right] \\ &= E\left[E\left[(as_0 + Z)^2 \mid s_0\right]\right] \\ &= E\left[a^2 s_0^2 + 2as_0 E[Z] + E\left[Z^2\right]\right] = a^2 E\left[s_0^2\right] + \sigma^2 \\ &= a^2 E\left[(e_0 - U_N(e_0))^2\right] + \sigma^2. \end{aligned} \tag{17}$$

Let $e_{*,N}$ denote the system adaptive error under invariant measure, i.e., $e_{*,N} = x_{*,N} - Q_K^{\Delta_{*,N}}(x_{*,N})$. Rearranging (17), we find that the optimality gap is given as

$$E\left[(x_{*,N})^2\right] - \sigma^2 = a^2 E\left[(e_{*,N} - U_N(e_{*,N}))^2\right], \tag{18}$$

which is (up to a constant) the distortion of the fixed quantizer $U_N$ applied to the random variable $e_{*,N}$.

With this in mind, we state the following result for high-rate distortion of $U_N$ on sequences of suitably well-behaved random variables. The proof builds on balancing the trade-off between distortion due to the high-rate granular region and the overflow region.

*Lemma 23:* Let $\{x_N\}_{N=2}^{\infty}$ be a sequence of random variables that satisfy,

$$\sup_{N\geq 2} E\left[|x_N|^m\right] =: B_m < \infty \tag{19}$$

for some $m > 2$ (not necessarily integer). Set the bin size for the quantizer $U_N$ as $\Delta_{(N)} = 2N^{-1+\frac{2}{m}}$. Then we have,

$$E\left[(x_N - U_N(x_N))^2\right] = \mathcal{O}_N\left(N^{-2+\frac{4}{m}}\right).$$

The proof is mostly mechanical; for a proof, see the proof of Lemma 35 (vector case) in the Appendix.

Briefly, we note that if the sequence $\{x_N\}_{N=2}^{\infty}$ is for instance uniformly sub-Gaussian (in the sense that $\sup_{N\geq 2} E\left[e^{s(x_N)^2}\right] < \infty$ for some $s > 0$, then (19) holds for every $m > 2$ and so it is possible to achieve distortion asymptotic to $\mathcal{O}_N(N^{-2+\delta})$ for any $\delta > 0$ by taking $m$ sufficiently large.

However, this sub-Gaussian condition is much stronger than (19), and if one follows the proof of Lemma 23 using this stronger tail condition carefully, setting the bin size $\Delta_{(N)}$ slightly differently, it is possible to achieve convergence that is faster than $\mathcal{O}_N\left(N^{-2+\delta}\right)$ (for instance, $\mathcal{O}_N(N^{-2}\ln N)$).

*Remark 24:* One can justify that Lemma 23 cannot be improved without strengthening the imposed condition or using more complex (i.e., non-uniform) quantizers. Consider the special case where $x_N = X$ for all $N \geq 2$ and where $X$ admits the "Bucklew-Gallagher" pdf,

$$p(x) = \frac{1+\delta/2}{(1+|x|)^{3+\delta}} \quad \text{for all } x \in \mathbb{R}, \tag{20}$$

for arbitrary $\delta > 0$. This distribution has finite moments only of order $m < 2+\delta$ and since $X$ is independent of $N$, the problem here is essentially optimal quantization of $X \sim p$. Let $D_N$ denote the infimum (MSE) distortion achievable over all uniform quantizers with $N > 0$ output levels. In [48], it is shown that for the source above, $D_N$ satisfies asymptotically (see the first equation in [48, p. 963]),

$$\lim_{N\to\infty} N^{\frac{2\delta}{2+\delta}} D_N = c_\delta$$

for some constant $c_\delta > 0$, so that asymptotically the best distortion achievable by uniform quantizers for the above source $X$ is $D_N = \mathcal{O}_N(N^{-\frac{2\delta}{2+\delta}})$.

Now we compare this to Lemma 23. Here, since $X$ admits finite moments $m$ for any $m < 2+\delta$, by employing our uniform quantizer with step size $\Delta_{(N)} = 2N^{-1+\frac{2}{m}}$ we achieve asymptotic distortion $\mathcal{O}_N(N^{-2+\frac{4}{m}})$. As we let $m \to 2+\delta$ from below, $-2+\frac{4}{m}$ becomes arbitrarily close to $-\frac{2\delta}{2+\delta}$ and so the result we have stated here achieves asymptotic distortion which can be arbitrarily close to the best achievable asymptotic distortion one can get using uniform quantization. It is in this sense that Lemma 23 cannot be improved without strengthening the conditions imposed on the sequence $\{x_N\}$, or without employing more complex quantization schemes.

In light of (18), we would like to apply Lemma 23 to the sequence of random variables $\{e_{*,N}\}_{N=2}^{\infty}$. To do this, we need to establish (19). This is ensured by the following result.

*Lemma 25:* Under Scheme P$(\beta,\varepsilon)$, the invariant system error has finite $(\beta-\varepsilon)$th moment uniformly in $N \geq 2$. That is,

$$\sup_{N\geq 2} E\left[|e_{*,N}|^{\beta-\varepsilon}\right] < \infty. \tag{21}$$

*Sketch of Proof:* With Lyapunov functions $V(x,\Delta)$ and $d(x,\Delta)$ proportional to $\Delta^{\beta-\varepsilon}$ and appropriate set $C$ and constant $b$, we show that these functions satisfy the drift condition (8). In particular, we do this for $f$ proportional to $|x|^{\beta-\varepsilon}$ and $f$ proportional to $\Delta^{\beta-\varepsilon}$ which, with the Lyapunov parameters independent of $N$ leads to results of the form (21) by Lemma 11 for the invariant state and adaptive bin size. A simple invariance argument finishes the result. Condition 14 is crucial to the proof of this result. For a complete proof, see the proof of Lemma 40 in the Appendix. □

Therefore, in light of (18) and the above lemma, we find that the optimality gap decays asymptotically at the rate $\mathcal{O}_N(N^{-2+\frac{4}{\beta-\varepsilon}})$. Since $K$ is fixed and $2^C$ is linearly proportional to $N$, we find that the optimality gap is asymptotically in C,

$$E\left[(x_{*,N})^2\right] - \sigma^2 = \mathcal{O}_C\left(\frac{2^{\frac{4}{\beta-\varepsilon}C}}{2^{2C}}\right),$$

which establishes Theorem 15.

## IV. The Multi-Dimensional Case

We now present the more general vector case. Recall that we consider control of the multi-dimensional system (1) over a discrete noiseless channel of capacity $C$ bits and the aim is to minimize the infinite-horizon average quadratic form (2).

As stated earlier, in the fully observed setup the optimal control policy is $u_t = -B^{-1}Ax_t$ which achieves an optimal cost of $\mathrm{tr}\,(Q\Sigma)$. In light of this, we seek to arrive at asymptotic bounds on the optimality gap,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^\top Q x_t\right] - \mathrm{tr}\,(Q\Sigma)$$

in terms of the channel capacity $C$.

We first generalize Definition 13 to the vector case.

*Definition 26:* A joint coding and control scheme $\phi_C$ achieves second moment convergence with rate function $r : \mathbb{R}_+ \to \mathbb{R}_+$ if,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^\top Q x_t\right] - \mathrm{tr}\,(Q\Sigma) = \mathcal{O}_C\left(\frac{r(C)}{2^{\frac{2}{n}C}}\right).$$

For brevity we may say simply that $\phi_C$ achieves the rate function $r(C)$.

We refer the reader to [49, Theorem 4.2] for a fundamental lower bound which leads the vector analog to the expression presented in Lemma 12.

*Remark 27:* In the vector case, the factor of $\frac{1}{n}$ in the above definition comes intuitively from the fact that to tile a hypercube in $\mathbb{R}^n$ of width $L$ it takes $(L/\Delta)^n$ bins of width $\Delta$.

As in the scalar case, one seeks to achieve a rate function $r(C)$ which grows slowly to maximize the rate of convergence.

We will impose the following condition on the noise process, analogous to Condition 14.

*Condition 28:* For some $\beta > 2$, $E\left[\|w_0\|_\infty^\beta\right] < \infty$, i.e., the noise process has finite $\beta$th moment.

The following generalization of Theorem 15 is our main result.

*Theorem 29:* Supposing Condition 28 holds with $\beta > 2$, for any $\varepsilon \in (0, \beta - 2)$ there exists a joint coding and control scheme Scheme P$(\beta, \varepsilon)$ which achieves the exponential rate function $r(C) = 2^{\left(\frac{4}{\beta-\varepsilon}\right)\frac{1}{n}C}$.

The rest of Section IV is dedicated to proving Theorem 29.

### A. Vector Quantization

In the multi-dimensional case, we will make use of vector quantization. We will use scalar uniform quantizers to define two types of cubic lattice vector quantizers.

Let $M \geq 2$ be an even integer and $\Delta > 0$ be a scalar "bin size". With $\lfloor\cdot\rfloor$ the usual floor function, we define the partial uniform quantizer $u_M^\Delta : [-\frac{M}{2}\Delta, \frac{M}{2}\Delta] \to \mathbb{R}$ as

$$u_M^\Delta(x) = \begin{cases} \Delta\left\lfloor\frac{x}{\Delta}\right\rfloor + \frac{\Delta}{2}, & \text{if } x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right) \\ \frac{M}{2}\Delta - \frac{\Delta}{2}, & \text{if } x = \frac{M}{2}\Delta. \end{cases}$$

Note that $0 \notin \text{range}(u_M^\Delta)$. Then, we define the type I vector quantizer $Q_M^\Delta : \mathbb{R}^n \to \mathbb{R}^n$ as

$$Q_M^\Delta(x) = \begin{cases} \left(u_M^\Delta(x^i)\right)_{i=1}^n, & \text{if } \|x\|_\infty \leq \frac{M}{2}\Delta \\ 0, & \text{if } |x^i| > \frac{M}{2}\Delta^i \text{ for some } 1 \leq i \leq n, \end{cases}$$

and the type II vector quantizer $U_M^\Delta : \mathbb{R}^n \to \mathbb{R}^n$ component-wise as

$$\left(U_M^\Delta(x)\right)^i = \begin{cases} u_M^\Delta(x^i), & \text{if } |x^i| \leq \frac{M}{2}\Delta \\ 0, & \text{otherwise.} \end{cases}$$

*Remark 30:* If $n = 1$ then the above definitions both correspond to the scalar quantizer (13). For this reason, the scheme we present for the vector case is a direct generalization of the scalar scheme.

As in the scalar case, we will use these vector quantizers for two different purposes. The type I quantizer will be used with adaptive bin sizes to achieve stability. Let $K \geq 2$ be an even integer, and suppose that $\{\Delta_t\}_{t=0}^\infty$ is some sequence of strictly positive bin sizes varying with time. We will make use of the quantizer $Q_K^{\Delta_t}$.

The type II quantizer will be used to achieve convergence to the optimum. For a given even number of bins $N \geq 2$, let $\Delta_{(N)}$ be a bin size which is a function of $N$. We will make use of the quantizer $U_N^{\Delta_{(N)}}$. For brevity we denote $U_N := U_N^{\Delta_{(N)}}$ and where necessary, specify the dependence of $\Delta_{(N)}$ on $N$.

### B. System in a Jordan Form

Briefly, we discuss a reduction of the system matrix $A$ into its separate modes. This will have the useful side effect of making $\|A\|_\infty$ very close to the absolute eigenvalue of each mode.

Let $\{\lambda_i\}_{i=1}^n$ be the (possibly repeated) eigenvalues of the system matrix $A$. Without loss of generality, we assume that $A$ is in real Jordan normal form, as any matrix $A$ there exists an invertible matrix $P$ such that $\tilde{A} := P^{-1}AP$ is the real Jordan normal form of $A$ [50, Theorem 3.4.1.5]. Let $P\tilde{x}_t = x_t$ and left-multiply the system (1) by $P^{-1}$. Defining $\tilde{B} = P^{-1}B$ and $\tilde{w}_t = P^{-1}w_t$, the system dynamics become

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{B}u_t + \tilde{w}_t,$$

which is in the same form as (1) but with the system matrix in real Jordan normal form.

For the purposes of controlling this system, it suffices to consider each of the Jordan blocks of $A$ individually. Since the matrix $B$ is invertible, the control of each of these blocks will be identical to the control problem for the full system (1). Therefore, by a slight abuse of notation, we will consider the control of a single mode or Jordan block $A \in \mathbb{R}^{n\times n}$ (which may now be part of a larger system) with the single repeated eigenvalue $\lambda \in \mathbb{C}$. By the real Jordan normal form [50, Theorem 3.4.1.5], we know then that $A$ takes the form

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} D & I & & \\ & D & \ddots & \\ & & \ddots & I \\ & & & D \end{bmatrix} \tag{22}$$

where $\lambda \in \mathbb{R}$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ respectively. For $\lambda = a+bi \in \mathbb{C}\backslash\mathbb{R}$ above, $I$ is the $2\times 2$ identity matrix and $D$ is the $2\times 2$ matrix

$$D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

We note that for $A$ in either of the forms above, we can describe $\|A\|_\infty$ quite easily:

*Proposition 31:* There are four cases to consider, which are:

- If $\lambda \in \mathbb{R}$ and $n = 1$ then $\|A\|_\infty = |\lambda|$.
- If $\lambda \in \mathbb{R}$ and $n > 1$ then $\|A\|_\infty = |\lambda| + 1$.
- If $\lambda = a+bi \in \mathbb{C}\backslash\mathbb{R}$ and $n = 2$ then $\|A\|_\infty = |a|+|b| \leq \sqrt{2}\,|\lambda|$.
- If $\lambda = a + bi \in \mathbb{C} \setminus \mathbb{R}$ and $n > 2$ then $\|A\|_\infty = |a| + |b| + 1 \leq \sqrt{2}\,|\lambda| + 1$.

The equalities follow just by observing the rows of $A$ under each assumption. In the complex case, the upper bound follows by Cauchy-Schwarz inequality in $\mathbb{R}^2$. The largest upper bound is $\sqrt{2}\,|\lambda| + 1$, so in view of (5) we may always write that

$$\|Ax\|_\infty \leq (\sqrt{2}\,|\lambda| + 1)\,\|x\|_\infty\,. \tag{23}$$

Briefly, we remark that the above bound can be improved in the case $n > 1$ by applying an invertible transform $S$. That is, as before we let $\tilde{A} = S^{-1}AS$ and apply the same "change-of-view" transformations $\tilde{x}_t = Sx_t$, $\tilde{B} = S^{-1}B$ and $\tilde{w}_t = S^{-1}w_t$. By left-multiplying the system (1) by $S^{-1}$ we arrive at an identical control problem now with the system matrix $\tilde{A} = S^{-1}AS$.

From here it suffices to determine how small $\left\|S^{-1}AS\right\|_\infty$ can be made over all invertible transformations $S$. At least in the case $\lambda \in \mathbb{R}$ we demonstrate that the infimum is at most $|\lambda|$ using the following construction.

First, suppose $A$ is a real Jordan block, i.e. that of (22) for $\lambda \in \mathbb{R}$. For any $\varepsilon > 0$ we let $S_\varepsilon := \text{diag}(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1})$

with inverse $S_\varepsilon^{-1} = \mathrm{diag}(1, \varepsilon^{-1}, \varepsilon^{-2}, \ldots, \varepsilon^{-(n-1)})$. Then,

$$S_\varepsilon^{-1} A S_\varepsilon = \begin{bmatrix} \lambda & \varepsilon & & \\ & \lambda & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda \end{bmatrix}.$$

Therefore, $\left\|S_\varepsilon^{-1} A S_\varepsilon\right\|_\infty = |\lambda| + \varepsilon$. By taking $\varepsilon \to 0$ this value is arbitrarily close to $|\lambda|$. If one allows for complex vector and matrix entries, an identical argument can be made in the case $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

The minimization of $\|A\|_\infty$ will be relevant to our scheme in terms of the minimum capacity required for stabilization.

*Remark 32:* It is important to distinguish between the cases $|\lambda| < 1$ and $|\lambda| \geq 1$. In the former case, the system is open-loop-stable so stability is easy to achieve. Here, if one employs only the fixed quantization stage of the scheme we present in the next section, then stability of the process is nearly automatic and the asymptotic optimality will follow by simple iterated expectation arguments that are almost identical to what we will present shortly. Therefore, we will ignore the stable case and suppose moving forward that $|\lambda| \geq 1$.

### C. Scheme $P(\beta, \varepsilon)$ in the Vector Case

In this section we describe the joint coding and control scheme of Theorem 29. The scheme presented is very similar to that of the scalar case and is in fact a direct generalization of the joint scheme of Theorem 15.

Let $K \geq 2$ be an even integer and suppose $\{\Delta_t\}_{t=0}^\infty$ is a sequence of positive "bin sizes" such that $\Delta_{t+1}$ is a function of only $\Delta_t$ and the indicator random variable $\mathbf{1}_{\{\|x_t\|_\infty \leq \frac{K}{2}\Delta_t\}}$. Also assume that both the encoder and decoder (controller) know $\Delta_0$. Then so long as the type I quantization $Q_K^{\Delta_t}(x_t)$ is sent over the channel, it is possible to synchronize knowledge of $\Delta_t$ between the quantizer and the controller since $\|x_t\|_\infty \leq \frac{K}{2}\Delta_t$ if and only if $Q_K^{\Delta_t}(x_t) \neq 0$.

The coding scheme is as follows. For $\{\Delta_t\}_{t=0}^\infty$ as above, we calculate the adaptive quantizer output $Q_K^{\Delta_t}(x_t)$ and the adaptive system error $e_t := x_t - Q_K^{\Delta_t}(x_t)$. Then for integer $N \geq 2$ we use a fixed type II quantizer $U_N$ with bin size $\Delta_{(N)}$ as in Section IV-A to calculate the fixed quantizer output $U_N(e_t)$. We then send $Q_K^{\Delta_t}(x_t)$ and $U_N(e_t)$ across the noiseless channel where the channel capacity is at least,

$$C = \log_2(K^n + 1) + \log_2((N+1)^n). \tag{24}$$

We estimate the state $x_t$ as $\hat{x}_t := Q_K^{\Delta_t}(x_t) + U_N(e_t)$. To mirror the fully observed case, the controller applies the control

$$u_t = -B^{-1}A\left(Q_K^{\Delta_t}(x_t) + U_N(e_t)\right) = -B^{-1}A\hat{x}_t.$$

The scheme is essentially exactly as illustrated in Figure 2 from the scalar case, where $-\frac{a}{b}$ is generalized to $-B^{-1}A$.

The controlled system dynamics resulting from this scheme are

$$\begin{aligned} x_{t+1} &= A(x_t - \hat{x}_t) + w_t \\ &= A(e_t - U_N(e_t)) + w_t. \end{aligned} \tag{25}$$

The update dynamics for $\{\Delta_t\}_{t=0}^\infty$ are nearly identical to the scalar case. We assume that $K > \|A\|_\infty$ and choose scalars $\frac{\|A\|_\infty}{K} < \alpha < 1$, $\rho > \|A\|_\infty$ and $L > 0$. We assume again that $\rho \geq K\alpha$. Choose $\Delta_0 \geq L$ arbitrarily, then for $t \geq 1$ the bin update is

$$\Delta_{t+1} = \begin{cases} \rho\Delta_t, & \text{if } \|x_t\|_\infty > \frac{K}{2}\Delta_t \\ \alpha\Delta_t, & \text{if } \|x_t\|_\infty \leq \frac{K}{2}\Delta_t, \Delta_t \geq L \\ \Delta_t, & \text{if } \|x_t\|_\infty \leq \frac{K}{2}\Delta_t, \Delta_t < L. \end{cases} \tag{26}$$

*Proposition 33:* With dynamics (25) and (26), the process $\{(x_t, \Delta_t)\}_{t=0}^\infty$ is a time-homogeneous Markov chain.

The motivation for this scheme is as in the scalar case. The adaptive type I quantizer $Q_K^{\Delta_t}$ will lead to stability in the sense of positive Harris recurrence, and the fixed type II quantizer $U_N$ will lead to order-optimal convergence of the system quadratic form $x^\top Qx$ under invariant measure as the fixed quantization rate $N$ grows large.

As in the scalar case, we impose Condition 17 so that the state space for $\Delta_t$ is countable. The state space for $\Delta_t$ is

$$\Omega_\Delta := \left\{\alpha^j \rho^k \Delta_0 : j, k \in \mathbb{Z}_{\geq 0}\right\}.$$

and the state space for the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^\infty$ is $\mathbb{R}^n \times \Omega_\Delta$. What remains is to specify additional constraints which complete our proposed scheme.

We assume that Condition 28 holds for some $\beta > 2$. For any $\varepsilon \in (0, \beta - 2)$ we finish our construction of Scheme $\mathrm{P}(\beta, \varepsilon)$ by requiring that $\rho > (\|A\|_\infty)^{\frac{\beta}{\varepsilon}}$ and specifying the dependence of $\Delta_{(N)}$ on $N$ as $\Delta_{(N)} = 2N^{-1+\frac{2}{\beta-\varepsilon}}$ (again, the constant 2 here is arbitrary).

We impose the following restriction on the minimum bin size which generalizes Condition 19 to the vector case.

*Condition 34:* The minimum adaptive bin size is at least,

$$\alpha L > \frac{\|A\|_\infty}{K\alpha - \|A\|_\infty}\Delta_{(N)}.$$

Finally, as in the scalar case, as $C \to \infty$ we keep $K$ fixed and let $N \to \infty$ to take advantage of fixed quantization results at high rates, to be presented shortly.

### D. Proof Program for Stability and Convergence

In this section, we outline the proof program for Theorem 29, i.e. that Scheme $\mathrm{P}(\beta, \varepsilon)$ achieves the exponential rate function $r(C) = 2^{\left(\frac{4}{\beta-\varepsilon}\right)\frac{1}{n}C}$. Many of the proofs of intermediate results are tedious and largely mechanical, so we have relegated these to the Appendix.

We first give an intermediate result on high-rate quantizer distortion and then present a high-level proof program of Theorem 29 with similar key arguments as those in Section III-B.

Let $\{X_N\}_{N=2}^\infty$ be a sequence of random vectors on $\mathbb{R}^n$ and consider the quantizer $U_N$ described in Section IV-A. We define for $N \geq 2$ the error vectors

$$Y_N := X_N - U_N(X_N).$$

*Lemma 35:* Suppose that

$$\sup_{N\geq 2} E\left[\|X_N\|_\infty^m\right] =: B_m < \infty$$

for some $m > 2$ (not necessarily integer). Set the bin size for the type II quantizer $U_N$ as $\Delta_{(N)} = 2N^{-1+\frac{2}{m}}$. Then for any positive semidefinite matrix $V \in \mathbb{R}^{n\times n}$ we have

$$\operatorname{tr}\left(VE\left[Y_N Y_N^\top\right]\right) = \mathcal{O}_N\left(N^{-2+\frac{4}{m}}\right).$$

The proof is mostly mechanical and can be found in the Appendix.

We denote the "in-view" set $\Lambda$[1] as,

$$\Lambda := \left\{(x,\Delta) \in \mathbb{R}^n \times \Omega_\Delta : \|x\|_\infty \leq \tfrac{K}{2}\Delta\right\},$$

i.e., the set of states $(x, \Delta)$ such that the adaptive quantizer $Q_K^\Delta(x)$ is non-zero.

We will show in Lemma 45 that states beginning in this set return to it very quickly, in that for constants $h > 0$ and $\xi > 1$ we have

$$P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \leq kT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right)$$

which decays very fast in integer $k \geq 1$. Essentially all of the following stability results will follow from repeated application of this inequality and Condition 28 on the noise process.

*Theorem 36:* Under Scheme $\mathrm{P}(\beta,\varepsilon)$ with $K > \|A\|_\infty$, $\{(x_t, \Delta_t)\}_{t=0}^\infty$ is positive Harris recurrent for every even $N \geq 2$. Therefore, for every even $N \geq 2$, Scheme $\mathrm{P}(\beta,\varepsilon)$ yields a unique invariant measure $\pi_N$ for the process $\{(x_t, \Delta_t)\}_{t=0}^\infty$.

*Sketch of Proof:* We establish $\varphi$-irreducibility and aperiodicity for $\{(x_t, \Delta_t)\}_{t=0}^\infty$, where $\varphi$ is the product of the Lebesgue and discrete measures on $\mathbb{R}^n \times \Omega_\Delta$. The logarithmic function $V(x,\Delta) = c\log_\alpha \Delta$ is shown to satisfy Condition 7 with $d(x, \Delta)$ constant and $f \equiv 1$ (i.e., in the form required by Lemma 10), leading to positive Harris recurrence. A complete proof is provided in the Appendix. □

We denote $(x_{*,N}, \Delta_{*,N}) \sim \pi_N$ as the state under invariant measure. This will also induce an invariant measure for the system adaptive error $e_t$, which we denote by $e_{*,N} \sim \pi_N^{\mathrm{err}}$.

We have the following ergodicity result, similar to the scalar case.

*Proposition 37:* The infinite-horizon second moment and the invariant second moment agree, that is

$$\lim_{T\to\infty}\frac{1}{T}E\left[\sum_{t=0}^{T-1} x_t^\top Q x_t\right] = E\left[(x_{*,N})^\top Q(x_{*,N})\right].$$

*Sketch of Proof:* We are able to show using the drift conditions of Section II-C that functions $g(x,\Delta)$ which are bounded by $\|x\|_\infty^{\beta-\varepsilon}$ satisfy the above ergodicity condition. The quadratic form $x^\top Qx$ is of the order $\|x\|_\infty^2$ and hence bounded by $\|x\|_\infty^{\beta-\varepsilon}$, since $\varepsilon < \beta - 2$. This will establish the result. A complete proof is provided in the Appendix. □

The following proposition is crucial to our proof program.

[1]In [12], this was referred to as the *perfectly zoomed* phase set.

*Proposition 38:* We have the following characterization of the invariant second moment.

$$\begin{aligned}&E\left[(x_{*,N})^\top Q(x_{*,N})\right] - \operatorname{tr}(Q\Sigma)\\ &= \operatorname{tr}\left(A^\top QA \cdot E\left[(e_{*,N} - U_N(e_{*,N}))(e_{*,N} - U_N(e_{*,N}))^\top\right]\right).\end{aligned}$$

*Proof:* The proof is by iterated expectations. Suppose that $(x_0, \Delta_0) \sim \pi_N$. Let $e_0 = x_0 - Q_K^{\Delta_0}(x_0)$ and $x_1 = A(e_0 - U_N(e_0)) + Z$, where $Z \sim \eta$ (recall $\eta$ is the distribution of $w_t$). Since we have applied the one-step transition kernel and $\pi_N$ is the invariant measure, the marginal distributions of $x_0$ and $x_1$ will be identical.

For brevity we denote $s_0 := e_0 - U_N(e_0)$ so that $x_1 = As_0 + Z$. Supposing that $E\left[(x_{*,N})^\top Q(x_{*,N})\right] < \infty$ (this follows from system moment results that will be stated shortly), we then have by invariance and iterated expectations that

$$\begin{aligned}E\left[x_0^\top Qx_0\right] &= E\left[x_1^\top Qx_1\right] = E\left[E\left[x_1^\top Qx_1 \mid s_0\right]\right]\\ &= E\left[E\left[(As_0+Z)^\top Q(As_0+Z) \mid s_0\right]\right]\\ &= E\left[s_0^\top A^\top QAs_0 + 2E[Z]^\top QAs_0 + Z^\top QZ\right]\\ &= E\left[s_0^\top A^\top QAs_0\right] + E\left[Z^\top QZ\right]\\ &= \operatorname{tr}\left(A^\top QA\cdot E\left[s_0s_0^\top\right]\right) + \operatorname{tr}\left(QE\left[ZZ^\top\right]\right)\\ &= \operatorname{tr}\left(A^\top QA\cdot E\left[(e_0 - U_N(e_0))(e_0 - U_N(e_0))^\top\right]\right)\\ &\quad + \operatorname{tr}(Q\Sigma).\end{aligned}$$

Rearranging the above equality completes the proof. Note that above we used the property that $Z \sim \eta$ is zero-mean. □

*Remark 39:* For the type of scheme we present here (that is, using two-stage adaptive and fixed uniform quantization), if one is able to show that

$$\begin{aligned}&\operatorname{tr}\left(A^\top QA\cdot E\left[(e_{*,N} - U_N(e_{*,N}))(e_{*,N} - U_N(e_{*,N}))^\top\right]\right)\\ &\qquad = \mathcal{O}_N\left(\frac{R(N)}{N^2}\right)\end{aligned}$$

for some function $R(N)$ then it follows from the above two propositions (and the fact that $N$ is a linear function of $2^{\frac{1}{n}C}$, recalling (24) and that $K$ is fixed) that the scheme achieves the rate function

$$r(C) = R\left((K^n+1)^{-\frac{1}{n}}2^{\frac{1}{n}C} - 1\right).$$

That is, we have

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}E\left[\sum_{t=0}^{T-1} x_t^\top Qx_t\right] - \operatorname{tr}(Q\Sigma)\\ &\qquad = \mathcal{O}_C\left(\frac{R\left((K^n+1)^{-\frac{1}{n}}2^{\frac{1}{n}C} - 1\right)}{2^{\frac{2}{n}C}}\right).\end{aligned} \tag{27}$$

Depending on the specific function $R(N)$ in question, this expression can be simplified (this will be the case in our analysis).

We have the following uniform stability result under invariant measure.

*Lemma 40:* Under Scheme $\mathrm{P}(\beta,\varepsilon)$, the invariant system error $e_{*,N}$ has finite $(\beta-\varepsilon)$th moment uniformly in $N \geq 2$. That is,

$$\sup_{N\geq 2} E\left[\|e_{*,N}\|_\infty^{\beta-\varepsilon}\right] < \infty. \tag{28}$$

*Sketch of Proof:* With Lyapunov functions $V(x,\Delta)$ and $d(x,\Delta)$ proportional to $\Delta^{\beta-\varepsilon}$, an appropriate "in-view" set $C$ and constant $b$ we show that these functions satisfy the drift condition (8). In particular, we do this for $f$ proportional to $\|x\|_\infty^{\beta-\varepsilon}$ and $f$ proportional to $\Delta^{\beta-\varepsilon}$ which, with the Lyapunov parameters independent of $N$ leads to results of the form (28) by Lemma 11 for the invariant state and adaptive bin size. A simple invariance argument finishes the result. A detailed proof is provided in the Appendix. $\square$

We note that $A^\top QA$ is positive semidefinite (by positive definiteness of $Q$). This allows us to use Lemma 35. Finally, we prove our ultimate result.

*Proof of Theorem 29:* Lemma 40 allows us to use Lemma 35 with the sequence of random vectors $\{e_{*,N}\}_{N=2}^\infty$ from which we obtain,

$$\operatorname{tr}\left(A^\top QA\cdot E\left[(e_{*,N}-U_N(e_{*,N}))(e_{*,N}-U_N(e_{*,N}))^\top\right]\right)$$
$$=\mathcal{O}_N\left(\frac{N^{\frac{4}{\beta-\varepsilon}}}{N^2}\right)$$

and so by the earlier remark with $R(N)=N^{\frac{4}{\beta-\varepsilon}}$, Scheme $\mathrm{P}(\beta,\varepsilon)$ achieves the rate function

$$r(C)=\left((K^n+1)^{-\frac{1}{n}}2^{\frac{1}{n}C}-1\right)^{\frac{4}{\beta-\varepsilon}}=\mathcal{O}_C\left(2^{\left(\frac{4}{\beta-\varepsilon}\right)\frac{1}{n}C}\right).$$

This proves Theorem 29 and completes our proof program. $\square$

## V. Simulation Results

In this section, we provide an example simulation to illustrate our results. Consider the linear scalar system,

$$x_{t+1}=1.2x_t+u_t+w_t.$$

Here the noise process $\{w_t\}_{t=0}^\infty$ is i.i.d. with the marginal distribution of $w_0=4Z$, where $Z$ is "Bucklew-Gallagher" distributed with pdf (20) with $\delta=2$. This marginal distribution is related to the typical Pareto distribution in that $\frac{1}{4}|w_t|+1\sim$ Pareto$(1,4)$.

We remark that $w_t$ admits finite moments $\beta$ only of order $\beta<4$, whereas all moments $\beta\geq 4$ are infinite. These random variables are thus badly behaved in the sense of having heavy tails. Despite this, it is clear that $\{w_t\}_{t=0}^\infty$ satisfies Condition 14 for all $\beta\in(2,4)$. We are interested in the minimization of

$$\limsup_{T\to\infty}\frac{1}{T}E\left[\sum_{t=0}^{T-1}x_t^2\right]$$

across a noiseless discrete channel of capacity $C$. This asymptotic performance has a lower bound of $E\left[w_0^2\right]=\frac{16}{3}$.

Let $K=2$ (the least even integer such that $K>|a|=1.2$). Then for even $N\geq 2$, the capacity of our channel is

$$C=\log_2(K+1)+\log_2(N+1)=\log_2 3+\log_2(N+1).$$

Let $\alpha=\frac{3}{4}$, $\rho=\left(\frac{4}{3}\right)^3$ and $L=9$ be the adaptive quantization parameters. Note that $\alpha^3\rho=1$ so that Condition 17 is satisfied. Let $\beta=3.95$ and $\varepsilon=0.95$. Finally, we let $\Delta_{(N)}=2N^{-1+\frac{2}{\beta-\varepsilon}}=2N^{-\frac{1}{3}}$ be the bin size for the fixed quantizer $U_N$ for all even $N\geq 2$.

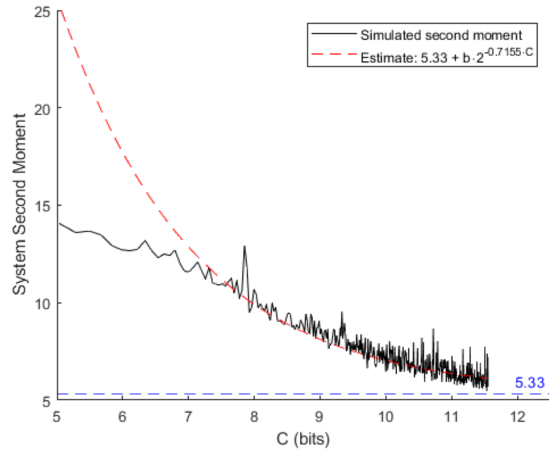


Fig. 3. Convergence to the optimum $E\left[w_0^2\right]=\frac{16}{3}$. The order of convergence is approximately $\mathcal{O}_C\left(2^{-0.7155C}\right)$, which is just slightly better than the expected convergence $\mathcal{O}_C\left(2^{-\frac{2}{3}C}\right)$. The constant $b$ is on the order of $2^8$.

Note that $\rho>(1.2)^{79/19}=|a|^{\frac{\beta}{\varepsilon}}$ and that $\rho\geq K\alpha=\frac{3}{2}$. Also note that Condition 19 is satisfied for all $N\geq 2$ since $\alpha L=\frac{27}{4}>\frac{8}{2^{1/3}}=\frac{|a|}{K\alpha-|a|}\Delta_{(2)}$.

Therefore, with $\beta-\varepsilon=3$, by employing Scheme $\mathrm{P}(\beta,\varepsilon)$ we are guaranteed by Theorem 15 that,

$$\limsup_{T\to\infty}\frac{1}{T}E\left[\sum_{t=0}^{T-1}x_t^2\right]-\frac{16}{3}=\mathcal{O}_C\left(2^{\left(-2+\frac{4}{\beta-\varepsilon}\right)C}\right)=\mathcal{O}_C\left(2^{-\frac{2}{3}C}\right).$$

Equivalently, in the sense of Definition 13 the scheme presented here achieves the exponential rate function $r(C)=2^{\frac{4}{3}C}$.

Using this scheme with the parameters above, the system was run for all $N\in\{10,12,\dots\ 1000\}$ and the average second moment was recorded. We remark that the settling time for the average second moment was observed to vary dramatically between small and large values of $N$. Thus, instead of a fixed simulation length $T$ for each trial of $N$, a variable stopping time was employed. Let

$$S_T:=\frac{1}{T}\sum_{t=0}^{T-1}x_t^2$$

be the average second moment up to time $T>0$. A small convergence threshold $\epsilon>0$ and an integer settling time $T^*>0$ are defined, and the simulation stops when for $T^*$ consecutive time stages we have $|S_{T+1}-S_T|<\epsilon$. That is, the average second moment varies no more than $\epsilon$ across a single time stage for $T^*$ consecutive time stages. For the simulations presented here, the convergence threshold used is $\epsilon=10^{-4}$ and the settling time is $T^*=10^4$.

The second moment achieved in each trial (i.e., for each $N$) is plotted against the capacity $C=\log_2 3+\log_2(N+1)$ below in Figure 3.

Since we expect that the optimality gap will converge to zero at a rate like $2^{-\delta C}$ for some $\delta>0$, it is reasonable to expect that the logarithm of the optimality gap is approximately linear in $C$ with slope $-\delta$. Therefore, an estimate of the order of convergence was obtained by performing a linear regression between $C$ and the logarithm of the optimality gap. This estimate on the order of convergence can be seen in

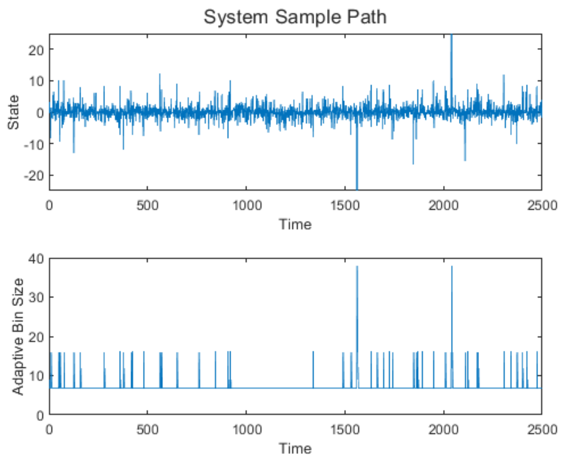


Fig. 4. A sample path for the system with $N = 100$ fixed quantizer bins. The system state and adaptive bin size are displayed over time.

Figure 3. Finally, a finite segment of a sample path for the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is displayed in Figure 4 for the case $N = 100$.

Finally, we remark on the convergence rate to the invariant measure $\pi_N$. In [51, Theorem IV.2] it is shown that in the scalar case with Gaussian noise, using only the adaptive part of the coding scheme, the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is "geometrically ergodic" in that for every $(x, \Delta) \in \mathbb{R} \times \Omega_\Delta$, the $n$-stage transition kernels $P^n((x, \Delta), \cdot)$ converge to $\pi_N(\cdot)$ under the total variation metric at a rate $r^n$ for some $r < 1$.

Much of the analysis needs to be re-derived in the more general case we consider here, but one can still follow the primary line of argument in the proof of [51, Theorem IV.2]. In particular, invoking [51, Theorem III.7] with the Lyapunov function $V(x, \Delta) = \Delta^2$ and using our super-geometric bound on the tail probabilities $P_{x,\Delta}(\tau_\Lambda \geq k + 1)$, Lemma 45, one can show that the scheme presented here also ensures that the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is geometrically ergodic for each $N \geq 2$ (in this proof, which we have not worked out in detail, a careful argument is needed to ensure that the geometric rate is uniformly bounded over all $N \geq 2$). This difficulty is of similar flavor that we resolved by a careful choice of Lyapunov parameters in Lemma 40).

## VI. Conclusion and Future Work

In conclusion, we have constructed joint coding and control schemes for networked control systems of the form (1) which are asymptotically optimal in the sense that, as the data rate grows without bound, the system second moment converges to the classical optimum with an explicit rate of convergence. The techniques in this paper build on prior work in this context, in particular by the use of random-time Lyapunov drift conditions to establish key stability results.

There are several potential directions for future work. It would be interesting to explore optimality of schemes for *non-linear* systems. The two-stage scheme approach seems applicable here, supposing that one can establish stability and ergodicity results by the adaptive part of the code. For instance, in [52, Theorem 5.1], a nonlinear system with "sub-linear" dynamics and additive Gaussian noise is shown to admit stability when subjected to an adaptive zooming quantization scheme similar to what is presented in [12] and [13] and the adaptive stage of the scheme presented here. It seems feasible that employing a two-stage (adaptive and fixed) uniform quantization scheme for systems of this type would, by some extra analysis, lead to convergence results similar to what we have presented here.

One direction is the relaxation of the invertibility assumption on the control matrix $B$. In general, it is possible to achieve stability in the sense of positive Harris recurrence and *finite* system moments with just the assumption that the pair $(A, B)$ is controllable [2, Theorem 2.2]. Controllability is a natural relaxation of the invertibility assumption to pursue for the kinds of linear systems considered here. Nonetheless, for optimality arguments on rates of decay in the distortion as the rate increases, further analysis is needed.

A further possibility is to consider stricter conditions on the noise process than Condition 28. For instance, one might assume that the noise tails are dominated by exponential decay (i.e., the noise has sub-exponential distribution), and seek to construct schemes which, in the sense of Definition 26, achieve rate functions that are much better than exponential (e.g., polynomial in $C$). The two-part coding scheme presented here is well suited to generalizations of this type, because the bound of Lemma 45 holds quite generally (e.g., with few assumptions on the noise process) and repeated use of this bound and Condition 28 leads to most of our key results. Thus, using this bound with a stricter condition on the noise process may lead to stronger stability and optimality results.

## Appendix: Proofs

*Proof of Proposition 1:* We show that in the fully observed setup, the optimal control policy which minimizes (2) is $u_t = -B^{-1}Ax_t$, achieving an optimal cost of $\mathrm{tr}(Q\Sigma)$, where $\mathrm{tr}(\cdot)$ is the trace operator. Let $v_t := Ax_t + Bu_t$ so that $x_{t+1} = v_t + w_t$, then under any policy $\gamma$ we have

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^\top Q x_t\right] \\
&\quad= \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_{t+1}^\top Q x_{t+1}\right] \\
&\quad= \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} (v_t + w_t)^\top Q (v_t + w_t)\right] \\
&\quad= \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} v_t^\top Q v_t + w_t^\top Q w_t\right] \\
&\quad\geq \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} w_t^\top Q w_t\right] \qquad (29) \\
&\quad= E\left[w_0^\top Q w_0\right] = \mathrm{tr}\left(QE\left[w_0 w_0^\top\right]\right) = \mathrm{tr}(Q\Sigma)
\end{aligned}$$

by positive definiteness of $Q$ and that $w_t$ is i.i.d. zero-mean. In the case $u_t = -B^{-1}Ax_t$ we have $v_t = 0$ and so (29) is an equality, establishing optimality. □

*Proof of Lemma 12:* Suppose that $b = 1$ without loss of generality so that we are considering control of the following system,

$$x_{t+1} = ax_t + u_t + w_t$$

across a discrete noiseless channel of capacity $C$ bits using an arbitrary joint coding and control scheme $\phi_C$. First, note we may assume that for all $C$ sufficiently large,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] < \infty. \tag{30}$$

If this fails to be true for the joint scheme $\phi_C$ then (12) will trivially hold. Note that (30) implies that

$$\liminf_{T\to\infty} E\left[x_T^2\right] < \infty. \tag{31}$$

We will follow the core arguments in the proof of [47, Theorem 11.3.2] with the key difference of not assuming that $\lim_{t\to\infty} E\left[x_t^2\right]$ exists.

Denote the received channel output at time $t \geq 0$ as $q_t'$. Let $D_t := E\left[(x_t + \frac{1}{a}u_t)^2\right]$, $d_t := E\left[x_t^2\right]$ and note that the system update gives us $D_t = \frac{1}{a^2}(d_{t+1} - \sigma^2)$.

We note that the discrete noiseless channel we consider is a special (noise-free discrete memoryless) case of "Class A" channels [47, Definition 8.5.1]. Such channels have their capacity characterized by limits of the directed mutual information between the channel input and output. In particular, following the beginning of the proof of [47, Theorem 8.5.2], we find that our channel capacity $C$ satisfies

$$C \geq \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} I(x_t; q_t' | q_{[0,t-1]}'), \tag{32}$$

where $I(x;y|z)$ is the conditional mutual information. Following the analysis in the proof of [47, Theorem 11.3.2] (essentially exactly that on page 392) and invoking a conditional version of the entropy-power inequality [47, Lemma 5.3.2] we are able to arrive at the following bound,

$$\begin{aligned}
&\frac{1}{T}\sum_{t=0}^{T-1} I(x_t; q_t' | q_{[0,t-1]}') \\
&\qquad\geq \frac{1}{T}\sum_{t=0}^{T-1} \frac{1}{2}\log_2\left(a^2 + \frac{\sigma^2}{D_t}\right) \\
&\qquad\quad + \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q_{[0,T-1]}')\right) \\
&\qquad\geq \frac{1}{2}\log_2\left(a^2 + \frac{\sigma^2}{\frac{1}{T}\sum_{t=0}^{T-1} D_t}\right) \\
&\qquad\quad + \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q_{[0,T-1]}')\right)
\end{aligned}$$

by the convexity of $x \mapsto \log(1 + \frac{1}{x})$. Above, $h(\cdot)$ and $h(\cdot \mid \cdot)$ are the regular and conditional differential entropies, respectively. Therefore, we arrive at the following bound on the channel capacity in view of (32),

$$\begin{aligned}
C &\geq \limsup_{T\to\infty}\left(\frac{1}{2}\log_2\left(a^2 + \frac{\sigma^2}{\frac{1}{T}\sum_{t=0}^{T-1} D_t}\right)\right. \\
&\qquad \left. + \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q_{[0,T-1]}')\right)\right) \\
&\geq \liminf_{T\to\infty} \frac{1}{2}\log_2\left(a^2 + \frac{\sigma^2}{\frac{1}{T}\sum_{t=0}^{T-1} D_t}\right) \\
&\qquad + \limsup_{T\to\infty} \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q_{[0,T-1]}')\right).
\end{aligned}$$

We will consider the two limits above separately. Let

$$d := \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} d_t = \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] < \infty$$

and since $D_t = \frac{1}{a^2}(d_{t+1} - \sigma^2)$ we have that $\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} D_t = \frac{1}{a^2}(d - \sigma^2)$.

Then since $\log\left(1 + \frac{1}{x}\right)$ is continuous and monotone decreasing in $x > 0$ we have

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{2}\log_2\left(a^2 + \frac{\sigma^2}{\frac{1}{T}\sum_{t=0}^{T-1} D_t}\right) \\
&\qquad = \frac{1}{2}\log_2\left(a^2 + \frac{\sigma^2}{\limsup_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} D_t}\right) \\
&\qquad = \frac{1}{2}\log_2\left(a^2 + \frac{a^2\sigma^2}{d - \sigma^2}\right) \\
&\qquad = \frac{1}{2}\log_2\left(\frac{a^2 d}{d - \sigma^2}\right).
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
C &\geq \frac{1}{2}\log_2\left(\frac{a^2 d}{d - \sigma^2}\right) \\
&\qquad + \limsup_{T\to\infty} \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q_{[0,T-1]}')\right)
\end{aligned}$$

and what remains is to bound this limit supremum. We then have,

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q_{[0,T-1]}')\right) \\
&\qquad = -\liminf_{T\to\infty} \frac{1}{T} h(ax_{T-1} + w_{T-1} \mid q_{[0,T-1]}') \\
&\qquad = -\liminf_{T\to\infty} \frac{1}{T} h(x_T - bu_{T-1} \mid q_{[0,T-1]}') \\
&\qquad = -\liminf_{T\to\infty} \frac{1}{T} h(x_T \mid q_{[0,T-1]}') && (33) \\
&\qquad \geq -\liminf_{T\to\infty} \frac{1}{T} h(x_T). && (34)
\end{aligned}$$

Above, (33) follows since $u_t$ is constant given $q_{[0,t]}'$ and differential entropy is translation invariant. (34) follows since conditioning reduces differential entropy.

We now show that $\liminf_{T\to\infty} \frac{1}{T} h(x_T) \leq 0$.

Note that since the noise process is added to the state at every time stage and has a pdf $\eta$ which is positive everywhere on $\mathbb{R}$, the state $x_t$ will also have a positive-everywhere density. Furthermore, the state $x_t$ will also have finite variance $E\left[x_t^2\right] - E\left[x_t\right]^2$ at each time stage.

It is known that for a distribution over $\mathbb{R}$ with specified variance $S$, the Gaussian distribution maximizes differential entropy at $\frac{1}{2}\log_2(2\pi e S)$. Therefore we have,

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T} h(x_T) \\
&\quad\le \liminf_{T\to\infty} \frac{1}{2T}\log_2\left(2\pi e\left(E\left[x_T^2\right] - E\left[x_T\right]^2\right)\right) \\
&\quad\le \liminf_{T\to\infty} \frac{1}{2T}\log_2\left(2\pi e E\left[x_T^2\right]\right) \\
&\quad\le \liminf_{T\to\infty} \frac{1}{T}\log_2\left(E\left[x_T^2\right]\right) = 0,
\end{aligned}$$

where the final limit is zero by (31).

Therefore, we have the following ultimate bound on the channel capacity,

$$C \ge \frac{1}{2}\log_2\left(\frac{a^2 d}{d-\sigma^2}\right).$$

Recall that $d = \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right]$. Rearranging the above inequality for the optimality gap $d-\sigma^2$ we find that

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] - \sigma^2 \ge \frac{a^2\sigma^2}{2^{2C} - a^2},$$

which completes the proof. □

*Proof of Lemma 35:* Define the event $F$ by

$$F = \left\{\|X_N\|_\infty \le \tfrac{1}{2}N\Delta_{(N)}\right\}.$$

To start, the $(i,j)$th component of $Y_N Y_N^\top$ is $Y_N^i Y_N^j$. Therefore,

$$\begin{aligned}
&\left|\left[E\left[Y_N Y_N^\top\right]\right]_{ij}\right| \\
&\quad= \left|E\left[Y_N^i Y_N^j\right]\right| = \left|E\left[Y_N^i Y_N^j \mathbf{1}_F\right] + E\left[Y_N^i Y_N^j \mathbf{1}_{F^C}\right]\right| \\
&\quad\le \left|E\left[Y_N^i Y_N^j \mathbf{1}_F\right]\right| + \left|E\left[Y_N^i Y_N^j \mathbf{1}_{F^C}\right]\right| \\
&\quad\le E\left[\left|Y_N^i Y_N^j\right| \mathbf{1}_F\right] + E\left[\left|Y_N^i Y_N^j\right| \mathbf{1}_{F^C}\right] \qquad (35)
\end{aligned}$$

by triangle inequality and Jensen's inequality. Since $F$ is such that $\left|X_N^k\right| \le \frac{1}{2}N\Delta_{(N)}$ for all $1 \le k \le n$, it follows by construction of $U_N$ that $\left|Y_N^k\right| \le \frac{1}{2}\Delta_{(N)}$. Therefore,

$$\begin{aligned}
E\left[\left|Y_N^i Y_N^j\right| \mathbf{1}_F\right] &\le E\left[\left(\tfrac{1}{2}\Delta_{(N)}\right)^2 \mathbf{1}_F\right] \\
&\le \left(\tfrac{1}{2}\Delta_{(N)}\right)^2 = \tfrac{1}{4}\Delta_{(N)}^2. \qquad (36)
\end{aligned}$$

Note that $\left|Y_N^k\right| \le \left|X_N^k\right|$, then for the second expectation of (35) we have by Hölder's inequality that

$$\begin{aligned}
&E\left[\left|Y_N^i Y_N^j\right| \mathbf{1}_{F^C}\right] \\
&\quad\le E\left[\left|X_N^i X_N^j\right| \mathbf{1}_{F^C}\right] \\
&\quad\le E\left[\left|X_N^i X_N^j\right|^{\frac{m}{2}}\right]^{\frac{2}{m}} E\left[\mathbf{1}_{F^C}\right]^{1-\frac{2}{m}} \\
&\quad= E\left[\left|X_N^i X_N^j\right|^{\frac{m}{2}}\right]^{\frac{2}{m}} P\left(\|X_N\|_\infty > \tfrac{1}{2}N\Delta_{(N)}\right)^{\frac{m-2}{m}}. \qquad (37)
\end{aligned}$$

Recall that we suppose $\sup_{N\ge 2} E\left[\|X_N\|_\infty^m\right] =: B_m < \infty$. It follows by Cauchy-Schwarz inequality that the expectation in (37) is bounded as

$$\begin{aligned}
E\left[\left|X_N^i X_N^j\right|^{\frac{m}{2}}\right]^{\frac{2}{m}} &\le E\left[\left|X_N^i\right|^m\right]^{\frac{1}{m}} E\left[\left|X_N^j\right|^m\right]^{\frac{1}{m}} \\
&\le E\left[\|X_N\|_\infty^m\right]^{\frac{2}{m}} \le (B_m)^{\frac{2}{m}}. \qquad (38)
\end{aligned}$$

Note that by Markov's inequality, we have for $u > 0$ that

$$\begin{aligned}
P(\|X_N\|_\infty > u) &= P(\|X_N\|_\infty^m > u^m) \\
&\le E\left[\|X_N\|_\infty^m\right] u^{-m} \le B_m u^{-m}
\end{aligned}$$

and so since $\frac{1}{2}N\Delta_{(N)} = N^{\frac{2}{m}}$, the tail probability in (37) can be bounded as,

$$\begin{aligned}
P\left(\|X_N\|_\infty > \tfrac{1}{2}N\Delta_{(N)}\right)^{\frac{m-2}{m}} &\le \left(B_m\left(N^{\frac{2}{m}}\right)^{-m}\right)^{\frac{m-2}{m}} \\
&= (B_m)^{\frac{m-2}{m}} N^{-2+\frac{4}{m}}.
\end{aligned}$$

Combining this with (37) and (38) we find that

$$\begin{aligned}
E\left[\left|Y_N^i Y_N^j\right| \mathbf{1}_{\{F^C\}}\right] &\le (B_m)^{\frac{2}{m}+\frac{m-2}{m}} N^{-2+\frac{4}{m}} \\
&= B_m N^{-2+\frac{4}{m}} = \mathcal{O}_N\left(\Delta_{(N)}^2\right),
\end{aligned}$$

which, combined with (36) yields that

$$\left|\left[E\left[Y_N Y_N^\top\right]\right]_{ij}\right| = \mathcal{O}_N\left(\Delta_{(N)}^2\right).$$

Now that we have demonstrated that each component of $E\left[Y_N Y_N^\top\right]$ satisfies the desired bound, the proof completes as follows. For any matrix $V \in \mathbb{R}^{n\times n}$ we have by triangle inequality and Jensen's inequality that

$$\begin{aligned}
\left|\operatorname{tr}\left(V E\left[Y_N Y_N^\top\right]\right)\right| &= \left|\sum_{i=1}^n \sum_{j=1}^n V_{ij}\left[E\left[Y_N Y_N^\top\right]\right]_{ji}\right| \\
&= \left|\sum_{i=1}^n \sum_{j=1}^n V_{ij} E\left[Y_N^i Y_N^j\right]\right| \\
&\le \sum_{i=1}^n \sum_{j=1}^n \left|V_{ij}\right| \left|E\left[Y_N^i Y_N^j\right]\right| \\
&\le \left(\max_{k,l} |V_{kl}|\right) \sum_{i=1}^n \sum_{j=1}^n \left|E\left[Y_N^i Y_N^j\right]\right| \\
&= \mathcal{O}_N\left(\Delta_{(N)}^2\right). \qquad (39)
\end{aligned}$$

The proof can be completed by noting that for any two positive semidefinite matrices $P, Q$ we have $\operatorname{tr}(PQ) \ge 0$. Note that for any random vector $X$, $E\left[XX^\top\right]$ is positive semidefinite. Therefore, for any positive semidefinite matrix $V$ it follows that

$$\operatorname{tr}\left(V E\left[Y_N Y_N^\top\right]\right) = \left|\operatorname{tr}\left(V E\left[Y_N Y_N^\top\right]\right)\right|$$

and so the proof concludes in view of (39). □

We now proceed with proving stability of $\{(x_t, \Delta_t)\}_{t=0}^\infty$ in the sense of both positive Harris recurrence as well as moment stability. To show positive Harris recurrence, we will need to demonstrate that our chain is irreducible and that an appropriate class of sets are small. The proofs of these two technical results (Propositions 41 and 42) are tedious and so we only provide proof sketches for brevity. For details, see [46].

*Proposition 41:* The process $\{(x_t,\Delta_t)\}_{t=0}^{\infty}$ is $\varphi$-irreducible and aperiodic, where $\varphi$ is the product of the Lebesgue and discrete measures on $\mathbb{R}^n \times \Omega_\Delta$.

*Sketch of Proof:* The condition of aperiodicity is strictly stronger than that of irreducibility, so it suffices to show only aperiodicity. Loosely, from any initial state $(x_0,\Delta_0)$ and a target set $B$ with $\varphi(B) > 0$ there exists some $n_0 > 0$ such that for every $n \geq n_0$, we can drive the system from $(x_0,\Delta_0)$ to $B$ in $n$ time stages with positive probability.

The reason this is possible is that we may "hold" the bin size $\Delta_t$ constant when it is less than $L$, and so by careful management we may direct the bin size $\Delta_n$ to anywhere in the state space $\Omega_\Delta$ we like (with a big enough $n_0$). In particular, we can direct the system to $B$ so long as the state $x_n$ falls into a subset of positive Lebesgue measure at time $n$. That this happens with positive probability is because the state is convolved with noise at every time stage that has a positive-everywhere pdf $\eta$. $\square$

We denote the in-view set as

$$\Lambda := \left\{(x,\Delta) \in \mathbb{R}^n \times \Omega_\Delta : \|x\|_\infty \leq \tfrac{K}{2}\Delta\right\} \tag{40}$$

and in light of Section II-C we will develop results concerning the return time $\tau_\Lambda$.

*Proposition 42:* For the Markov chain $\{(x_t,\Delta_t)\}_{t=0}^{\infty}$, bounded subsets of $\Lambda$ are small.

*Sketch of Proof:* We can show that subsets of $\Lambda$ containing only one bin size are 1-small. This follows mainly since such sets are bounded and $\Delta_1$ is known deterministically, so the next-stage pdf is bounded from below by a sub-probability measure in terms of the initial set and the noise pdf $\eta$.

It follows by aperiodicity that any finite union of small sets is small, and since bounded subsets of $\Lambda$ are the finite union of subsets with one bin size (which are 1-small), these sets are small (though generally not 1-small). $\square$

The following proposition will prove to be remarkably useful for the remainder of our proof program.

*Proposition 43:* Let $\{z_t\}_{t=0}^{\infty}$ be an i.i.d. sequence of non-negative random variables. For any $b > 0$ and integer $k \geq 1$ we have

$$P\left(\sum_{t=0}^{k-1} z_t > b\right) \leq kP\left(z_0 > \tfrac{b}{k}\right).$$

*Proof:* Since $\{z_t\}_{t=0}^{\infty}$ is identically distributed, we have for $k \geq 1$ that

$$\begin{aligned} P\left(\sum_{t=0}^{k-1} z_t > b\right) &\leq P\left(\bigcup_{t=0}^{k-1}\left\{z_t > \tfrac{b}{k}\right\}\right) \\ &\leq \sum_{t=0}^{k-1} P\left(z_t > \tfrac{b}{k}\right) = kP\left(z_0 > \tfrac{b}{k}\right). \end{aligned}$$

$\square$

*Corollary 44:* Let $\{z_t\}_{t=0}^{\infty}$ be an i.i.d. sequence of nonnegative random variables. Then for any (real) $m > 0$ and integer $k \geq 1$ we have

$$E\left[\left(\sum_{t=0}^{k-1} z_t\right)^m\right] \leq k^{m+1}E\left[z_0^m\right].$$

*Proof:* Using the tail formula for expectation of a nonnegative random variable and the previous proposition, we have

$$\begin{aligned} E\left[\left(\sum_{t=0}^{k-1} z_t\right)^m\right] &= \int_0^\infty P\left(\left(\sum_{t=0}^{k-1} z_t\right)^m > u\right) du \\ &= \int_0^\infty P\left(\sum_{t=0}^{k-1} z_t > u^{\frac{1}{m}}\right) du \\ &\leq k\int_0^\infty P\left(z_0 > \frac{u^{\frac{1}{m}}}{k}\right) \\ &= k\int_0^\infty P((kz_0)^m > u)\, du \\ &= kE\left[(kz_0)^m\right] = k^{m+1}E\left[z_0^m\right]. \end{aligned}$$

$\square$

*Lemma 45:* Define the constants

$$\xi := \frac{\rho}{\|A\|_\infty}, \quad h := \frac{K\alpha}{\rho}.$$

For $(x,\Delta) \in \Lambda$ we have for any $k \geq 1$ that

$$\begin{aligned} &P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \\ &\qquad \leq kT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right). \end{aligned} \tag{41}$$

*Proof:* Starting from $(x,\Delta) \in \Lambda$, let $e_0 := x - Q_K^\Delta(x)$. We define the map $S : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S(u) := u - U_N(u).$$

Note that $\|S(u)\|_\infty \leq \|u\|_\infty + \frac{\Delta_{(N)}}{2}$ for all $u \in \mathbb{R}^n$. The correction term $\frac{\Delta_{(N)}}{2}$ accounts for the possibility that $\|u\|_\infty < \frac{\Delta_{(N)}}{2}$.

We construct the following "zoom-out" process $y_t$. Let $y_0 = e_0$, and for $t \geq 0$ let

$$y_{t+1} = AS(y_t) + w_t. \tag{42}$$

Then the following holds for all $1 \leq t \leq \tau_\Lambda$:

$$y_t = x_t, \quad \Delta_t = \rho^{t-1}\alpha\Delta.$$

Therefore, it follows that for $k \geq 1$,

$$\begin{aligned} \{\tau_\Lambda \geq k+1\} &= \bigcap_{t=1}^{k}\{(x_t,\Delta_t) \notin \Lambda\} \\ &= \bigcap_{t=1}^{k}\left\{\|x_t\|_\infty > \tfrac{K}{2}\Delta_t\right\} \\ &= \bigcap_{t=1}^{k}\left\{\|y_t\|_\infty > \tfrac{\Delta}{2}K\alpha\rho^{t-1}\right\} \\ &\subseteq \left\{\|y_k\|_\infty > \tfrac{\Delta}{2}K\alpha\rho^{k-1}\right\} \\ &= \left\{\|y_k\|_\infty > \tfrac{\Delta}{2}h\rho^k\right\}. \end{aligned}$$

We may further relax this event as

$$\begin{aligned}
&\left\{\|y_k\|_\infty > \tfrac{\Delta}{2} h\rho^k\right\} \\
&\quad = \left\{\|AS(y_{k-1}) + w_{k-1}\|_\infty > \tfrac{\Delta}{2} h\rho^k\right\} \\
&\quad \subseteq \left\{\|AS(y_{k-1})\|_\infty + \|w_{k-1}\|_\infty > \tfrac{\Delta}{2} h\rho^k\right\} \quad (43)\\
&\quad \subseteq \left\{\|A\|_\infty \|S(y_{k-1})\|_\infty + \|w_{k-1}\|_\infty > \tfrac{\Delta}{2} h\rho^k\right\} \quad (44)\\
&\quad = \left\{\|S(y_{k-1})\|_\infty + \|A\|_\infty^{-1} \|w_{k-1}\|_\infty > \tfrac{\Delta}{2} h\xi\rho^{k-1}\right\} \\
&\quad \subseteq \left\{\|S(y_{k-1})\|_\infty + \|w_{k-1}\|_\infty > \tfrac{\Delta}{2} h\xi\rho^{k-1}\right\} \quad (45)\\
&\quad \subseteq \left\{\|y_{k-1}\|_\infty + \|w_{k-1}\|_\infty + \tfrac{\Delta_{(N)}}{2} > \tfrac{\Delta}{2} h\xi\rho^{k-1}\right\}. \quad (46)
\end{aligned}$$

Above, (43) follows by triangle inequality, (44) holds in light of (23), (45) holds since $\|A\|_\infty \geq |\lambda| \geq 1$ and (46) holds since $\|S(u)\|_\infty \leq \|u\|_\infty + \frac{\Delta_{(N)}}{2}$.

The steps (43) through (45) can be repeated $k-1$ more times to ultimately obtain that the event (45) implies

$$\left\{\|y_0\|_\infty + \sum_{t=0}^{k-1}\left(\|w_t\|_\infty + \tfrac{\Delta_{(N)}}{2}\right) > \tfrac{\Delta}{2} h\xi^k\right\}. \quad (47)$$

Using the fact that $y_0 = e_0$ and that starting in-view, $\|e_0\|_\infty \leq \frac{\Delta}{2}$, we find that

$$\{\tau_\Lambda \geq k+1\} \subseteq \left\{\sum_{t=0}^{k-1}\left(\|w_t\|_\infty + \tfrac{\Delta_{(N)}}{2}\right) > \tfrac{\Delta}{2}\left(h\xi^k - 1\right)\right\}.$$

Therefore,

$$\begin{aligned}
&P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \\
&\quad \leq P_{x,\Delta}\left(\sum_{t=0}^{k-1}\left(\|w_t\|_\infty + \tfrac{\Delta_{(N)}}{2}\right) > \tfrac{\Delta}{2}\left(h\xi^k - 1\right)\right).
\end{aligned}$$

Note that since $\alpha > \frac{\|A\|_\infty}{K}$ we have $h\xi^k > 1$ for $k \geq 1$. Therefore, $\frac{\Delta}{2}\left(h\xi^k - 1\right) > 0$ so we may apply Proposition 43 to the sequence of i.i.d. random variables $\left\{\|w_t\|_\infty + \frac{\Delta_{(N)}}{2}\right\}_{t=0}^\infty$ to find that

$$\begin{aligned}
&P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \\
&\quad \leq kP_{x,\Delta}\left(\|w_0\|_\infty + \tfrac{\Delta_{(N)}}{2} > \frac{\frac{\Delta}{2}\cdot\left(h\xi^k - 1\right)}{k}\right) \\
&\quad = kP_{x,\Delta}\left(\|w_0\|_\infty > \frac{\Delta}{2}\cdot\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{2}\right) \\
&\quad = kT_{w_0}\left(\frac{\Delta}{2}\cdot\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{2}\right).
\end{aligned}$$

Finally, note that with $\Delta \geq \alpha L$ we have,

$$\begin{aligned}
\frac{\Delta}{2}\cdot\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{2} &= \frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\Delta}\right) \\
&\geq \frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right),
\end{aligned}$$

which, since $T_{w_0}(\cdot)$ is nonincreasing, proves the claimed bound (41).

Briefly, we justify that the argument to $T_{w_0}(\cdot)$ in (41) is *strictly positive* for all $k \geq 1$. First, note that Condition 34 is equivalent to,

$$\frac{\Delta_{(N)}}{\alpha L} < h\xi - 1. \quad (48)$$

Since $\alpha > \frac{\|A\|_\infty}{K}$ and $\rho \geq K\alpha$ we have $h \in (\frac{1}{\xi}, 1]$. It is relatively straightforward (if tedious) to verify that for $\xi > 1$ and $h \in (\frac{1}{\xi}, 1]$ that the function

$$s \mapsto \frac{h\xi^s - 1}{s}$$

is monotone increasing for real $s > 0$ (e.g. by careful analysis of its derivative). In particular, it follows that $\frac{h\xi^k - 1}{k} \geq h\xi - 1$ for integer $k \geq 1$. Therefore, it follows from (48) (i.e., from Condition 34) that

$$\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right) > \frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - (h\xi - 1)\right) \geq 0.$$

□

*Corollary 46:* For $(x,\Delta) \in \Lambda$, the in-view return time satisfies

$$\sup_{(x,\Delta)\in\Lambda} E_{x,\Delta}\left[\tau_\Lambda\right] < \infty$$
$$\lim_{\Delta\to\infty} E_{x,\Delta}\left[\tau_\Lambda - 1\right] = 0.$$

*Proof:* Note that by Condition 28, we have $E\left[\|w_0\|_\infty^2\right] < \infty$ (this follows since $\beta > 2$). Therefore by Markov's inequality we have for $u > 0$ that

$$T_{w_0}(u) \leq E\left[\|w_0\|_\infty^2\right] u^{-2}.$$

We use the tail formula for the expectation of a nonnegative discrete random variable and Lemma 45 to then find that for $(x,\Delta) \in \Lambda$,

$$\begin{aligned}
&E_{x,\Delta}\left[\tau_\Lambda\right] \\
&= \sum_{k=1}^\infty P_{x,\Delta}\left(\tau_\Lambda \geq k\right) \\
&= 1 + \sum_{k=1}^\infty P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \\
&\leq 1 + \sum_{k=1}^\infty kT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right) \\
&\leq 1 + \sum_{k=1}^\infty kE\left[\|w_0\|_\infty^2\right]\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right)^{-2} \\
&= 1 + \Delta^{-2}\cdot\sum_{k=1}^\infty 4E\left[\|w_0\|_\infty^2\right]k^3\left(h\xi^k - 1 - \frac{\Delta_{(N)}}{\alpha L}k\right)^{-2},
\end{aligned}$$

where the series converges because each term is $\mathcal{O}_k\left(k^3\xi^{-2k}\right)$ for $\xi > 1$. We remark that $\Delta \geq \alpha L$ implies that the above is uniformly bounded (when one replaces $\Delta$ by $\alpha L$) and that as $\Delta \to \infty$ the above is $1 + \mathcal{O}_\Delta\left(\Delta^{-2}\right)$, which proves both claims. □

Finally, we prove positive Harris recurrence.

*Proof of Theorem 36:* Following the remark after Condition 7, we will satisfy the drift condition (8) for Lemma 10.

In this case, we set the following.

$$\begin{aligned}&V(x,\Delta)=c\log_\rho\Delta,\\&d(x,\Delta)\equiv d=\sup_{(x,\Delta)\in\Lambda}E_{x,\Delta}\left[\tau_\Lambda\right],\\&f(x,\Delta)\equiv 1,\\&C=\Lambda\cap\left\{(x,\Delta)\in\mathbb{R}^n\times\Omega_\Delta:\Delta\le D\right\},\\&b=c(d-1)+c\log_\rho\alpha+d,\end{aligned}$$

where $d$ is finite by Corollary 46, $c\ge\frac{2d}{\log_\rho(\frac{1}{\alpha})}$ and $D$ is such that $\Delta>D$ implies that $E_{x,\Delta}\left[\tau_\Lambda-1\right]\le\frac{1}{2}\log_\rho(\frac{1}{\alpha})$ (such a $D$ exists by Corollary 46). We note that $C$ is a small set by Proposition 42.

First, notice that by construction we have for any $(x,\Delta)\in\Lambda$ that $E_{x,\Delta}\left[\tau_\Lambda\right]\le d$, so the second inequality of (8) is satisfied. We now show that the first inequality also holds, which in this case is that for any $(x,\Delta)\in\Lambda$,

$$E_{x,\Delta}\left[V(x_{\tau_\Lambda},\Delta_{\tau_\Lambda})\right]-V(x,\Delta)\le -d+b\mathbf{1}_{\{(x,\Delta)\in C\}}. \quad (49)$$

First, note that for $1\le t\le\tau_\Lambda$ we have that $\Delta_t=\rho^{t-1}\alpha\Delta$, so,

$$\begin{aligned}V(x_{\tau_\Lambda},\Delta_{\tau_\Lambda})&=c\log_\rho\Delta_{\tau_\Lambda}=c\log_\rho\left(\rho^{\tau_\Lambda-1}\alpha\Delta\right)\\&=c(\tau_\Lambda-1)+c\log_\rho\alpha+c\log_\rho\Delta\\&=c(\tau_\Lambda-1)+c\log_\rho\alpha+V(x,\Delta).\end{aligned}$$

Thus we have that

$$E_{x,\Delta}\left[V(x_{\tau_\Lambda},\Delta_{\tau_\Lambda})\right]-V(x,\Delta)=cE_{x,\Delta}\left[\tau_\Lambda-1\right]+c\log_\rho\alpha. \quad (50)$$

First, suppose $\Delta>D$ (so that $(x,\Delta)\notin C$), then by construction of $D$ and $c$, (50) becomes

$$\begin{aligned}cE_{x,\Delta}\left[\tau_\Lambda-1\right]+c\log_\rho\alpha&\le\tfrac{1}{2}c\log_\rho(\tfrac{1}{\alpha})+c\log_\rho\alpha\\&=-\tfrac{1}{2}c\log_\rho(\tfrac{1}{\alpha})\\&\le-\frac{1}{2}\left(\frac{2d}{\log_\rho(\frac{1}{\alpha})}\right)\log_\rho(\tfrac{1}{\alpha})\\&=-d=-d+b\mathbf{1}_{\{(x,\Delta)\in C\}},\end{aligned}$$

which satisfies (49). Next, suppose that $\Delta\le D$, then by construction of $b$ and $d$, (50) becomes

$$\begin{aligned}cE_{x,\Delta}\left[\tau_\Lambda-1\right]+c\log_\rho\alpha&\le c(d-1)+c\log_\rho\alpha\\&=-d+b=-d+b\mathbf{1}_{\{(x,\Delta)\in C\}}\end{aligned}$$

and so (49) is satisfied over $\Lambda$. It follows then by Lemma 10 that $\{(x_t,\Delta_t)\}_{t=0}^\infty$ is positive Harris recurrent, provided that we can show that $P_{x,\Delta}\left(\tau_C<\infty\right)=1$ for all $(x,\Delta)\in\mathbb{R}^n\times\Omega_\Delta$. We do this to complete the proof.

Since we have proven the drift condition (8) holds at all $(x,\Delta)\in\Lambda$, it follows from the proof of Lemma 10 that for all $(x,\Delta)\in\Lambda$ we have

$$E_{x,\Delta}\left[\tau_C\right]\le V(x,\Delta)+b<\infty$$

so we must have that $P_{x,\Delta}\left(\tau_C<\infty\right)=1$ for all $(x,\Delta)\in\Lambda$. From here, it suffices to verify that all states return to $\Lambda$ in finite time. This is automatic for initial $(x,\Delta)\in\Lambda$ by Corollary 46. Now suppose that $(x,\Delta)\in\Lambda^C$. By nearly identical arguments to those of Lemma 45, we can show that

$$\begin{aligned}&P_{x,\Delta}\left(\tau_\Lambda\ge k+1\right)\\&\quad\le P_{x,\Delta}\left(\sum_{t=0}^{k-1}\left(\|w_t\|_\infty+\tfrac{\Delta_{(N)}}{2}\right)>\tfrac{\Delta}{2}K\xi^k-\|x\|_\infty\right).\end{aligned}$$

For $k$ sufficiently large we have $\frac{\Delta}{2}K\xi^k>\|x\|_\infty$, so we may apply Proposition 43 which yields that for $k$ sufficiently large,

$$P_{x,\Delta}\left(\tau_\Lambda\ge k+1\right)\le kT_{w_0}\left(\frac{\frac{\Delta}{2}K\xi^k-\|x\|_\infty}{k}-\frac{\Delta_{(N)}}{2}\right),$$

which converges to zero as $k$ grows large ($T_{w_0}(u)=\mathcal{O}_u\left(u^{-2}\right)$, as in the proof of Corollary 46). Therefore, we must have $P_{x,\Delta}\left(\tau_\Lambda=\infty\right)=0$, which completes the proof. □

*Proof of Proposition 37:* First, we note that if $\lambda^*$ is the maximum eigenvalue of $Q$ then the following holds for arbitrary $x\in\mathbb{R}^n$,

$$x^\top Qx\le\lambda^*\|x\|_2^2\le\lambda^*\left(\sqrt{n}\,\|x\|_\infty\right)^2=n\lambda^*\|x\|_\infty^2, \quad (51)$$

where the first bound follows by properties of positive semidefinite matrices and the second follows by (4). Since $\varepsilon<\beta-2$ it follows that $x^\top Qx$ is bounded by $\|x\|_\infty^{\beta-\varepsilon}$ in the sense of Lemma 11.

We will show in the proof of Proposition 48 that Condition 7 holds in the form required by Lemma 11 with the function $f(x,\Delta)=C\|x\|_\infty^{\beta-\varepsilon}$, for a constant $C>0$. Since $x^\top Qx$ is bounded by $\|x\|_\infty^{\beta-\varepsilon}$, the proof completes in view of the ergodicity result (9). □

To complete the proof program, we must show the moment condition of Lemma 40 holds. The rest of the Appendix is dedicated to this proof. We first have some intermediate results.

*Proposition 47:* Under Scheme P$(\beta,\varepsilon)$, for $(x,\Delta)\in\Lambda$ the return time $\tau_\Lambda$ satisfies the following independently of $N$.

$$\sup_{(x,\Delta)\in\Lambda}E\left[\left(\rho^{\beta-\varepsilon}\right)^{\tau_\Lambda-1}\right]<\infty$$

$$\lim_{\Delta\to\infty}E\left[\left(\rho^{\beta-\varepsilon}\right)^{\tau_\Lambda-1}\right]=1.$$

*Proof:* Note that by Condition 28 with $\beta>2$ and Markov's inequality we have for $u>0$ that

$$T_{w_0}(u)\le E\left[\|w_0\|_\infty^\beta\right]u^{-\beta}.$$

Let $r:=\rho^{\beta-\varepsilon}$ for brevity. Then we have by Lemma 45 that

$$\begin{aligned}&E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]\\&=\sum_{k=0}^\infty P_{x,\Delta}\left(\tau_\Lambda=k+1\right)r^k\\&\le 1+\sum_{k=1}^\infty P_{x,\Delta}\left(\tau_\Lambda\ge k+1\right)r^k\\&\le 1+\sum_{k=1}^\infty kT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k-1}{k}-\frac{\Delta_{(N)}}{\alpha L}\right)\right)r^k\end{aligned}$$

$$\leq 1+\sum_{k=1}^{\infty} kr^k E\left[\|w_0\|_\infty^\beta\right]\left(\frac{\Delta}{2}\left(\frac{h\xi^k-1}{k}-\frac{\Delta_{(N)}}{\alpha L}\right)\right)^{-\beta}$$
$$= 1$$
$$+\Delta^{-\beta}\cdot\sum_{k=1}^{\infty} 2^\beta E\left[\|w_0\|_\infty^\beta\right] k^{\beta+1} r^k\left(h\xi^k-1-\frac{\Delta_{(N)}}{\alpha L}k\right)^{-\beta}.$$

Above, the series converges since the summand is $\mathcal{O}_k\left(k^{\beta+1}\left(r\xi^{-\beta}\right)^k\right)$, where $r\xi^{-\beta}<1$ (this is ensured by the assumption that $\rho>(\|A\|_\infty)^{\frac{\beta}{\varepsilon}}$). Then, we have shown that

$$E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]=1+\mathcal{O}_\Delta\left(\Delta^{-\beta}\right),$$

which (in light of the fact that $\Delta\geq\alpha L$) completes the proof. □

*Proposition 48:* Under Scheme P$(\beta,\varepsilon)$, the invariant state $x_{*,N}$ has finite $(\beta-\varepsilon)$th moment uniformly in $N\geq 2$. That is,

$$\sup_{N\geq 2} E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right]<\infty.$$

*Proposition 49:* Under Scheme P$(\beta,\varepsilon)$, the invariant adaptive bin size $\Delta_{*,N}$ has finite $(\beta-\varepsilon)$th moment uniformly in $N\geq 2$. That is,

$$\sup_{N\geq 2} E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right]<\infty.$$

We will prove these two results shortly via Lyapunov drift arguments, but using them we will first provide a short proof of Lemma 40.

*Proof of Lemma 40:* Let $(x_{*,N},\Delta_{*,N})\sim\pi_N$ and let $e_{*,N}=x_{*,N}-Q_K^{\Delta_{*,N}}(x_{*,N})$. It follows that $e_{*,N}$ is distributed as the invariant system adaptive error. Note that for any $(x,\Delta)$ we have the inequality $\left\|x-Q_K^{\Delta}(x)\right\|_\infty\leq\|x\|_\infty+\frac{\Delta}{2}$, where the extra $\frac{\Delta}{2}$ term accounts for the possibility that $\|x\|_\infty<\frac{\Delta}{2}$. Therefore, we have that

$$\|e_{*,N}\|_\infty=\left\|x_{*,N}-Q_K^{\Delta_{*,N}}(x_{*,N})\right\|_\infty \leq\|x_{*,N}\|_\infty+\frac{\Delta_{*,N}}{2}$$

and so,

$$E\left[\|e_{*,N}\|_\infty^{\beta-\varepsilon}\right] \leq E\left[\left(\|x_{*,N}\|_\infty+\frac{\Delta_{*,N}}{2}\right)^{\beta-\varepsilon}\right] \leq 2^{\beta-\varepsilon}E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right]+E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right]. \tag{52}$$

Above, the second bound follows from the general fact that for any $\theta>0$ and nonnegative random variables $X,Y$ we have

$$E\left[(X+Y)^\theta\right]\leq 2^\theta\left(E\left[X^\theta\right]+E\left[Y^\theta\right]\right), \tag{53}$$

which follows by convexity of $u\mapsto u^\theta$ when $\theta\geq 1$ (via Jensen's inequality) and by sub-additivity of $u\mapsto u^\theta$ when $\theta<1$.

Therefore we have by (52) that

$$\sup_{N\geq 2} E\left[\|e_{*,N}\|_\infty^{\beta-\varepsilon}\right] \leq 2^{\beta-\varepsilon}\sup_{N\geq 2} E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right]+\sup_{N\geq 2} E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right] <\infty,$$

which is finite by Propositions 48 and 49. □

All that remains is to prove Propositions 48 and 49, which we do here via Lyapunov drift arguments.

*Proof of Proposition 48:* Recall that we must show that

$$\sup_{N\geq 2} E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right]<\infty.$$

We will do this via random-time Lyapunov drift arguments, in particular Lemma 11 using the drift condition (8). Let $r:=\rho^{\beta-\varepsilon}$ as in Proposition 47. We set the following:

$$V(x,\Delta)=\Delta^{\beta-\varepsilon},$$
$$d(x,\Delta)=s\Delta^{\beta-\varepsilon},$$
$$f(x,\Delta)=cs\left(\tfrac{2}{K}\|x\|_\infty\right)^{\beta-\varepsilon},$$
$$C=\Lambda\cap\{(x,\Delta)\in\mathbb{R}^n\times\Omega_\Delta:\Delta\leq D\},$$
$$b=D^{\beta-\varepsilon}\left(\alpha^{\beta-\varepsilon}\sup_{x,\Delta\in\Lambda}E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]-1+s\right),$$

where $s\in(0,1-\alpha^{\beta-\varepsilon})$ is arbitrary, $D>0$ is such that

$$(x,\Delta)\in\Lambda \text{ and } \Delta>D \implies \alpha^{\beta-\varepsilon}E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]-1+s\leq 0$$

(such a $D$ exists by Proposition 47), $b$ is finite by Proposition 47, and $c>0$ is a sufficiently small constant (which will be specified shortly). We will show that the drift condition (8) holds for the choices above.

First, note that for $(x,\Delta)\in\Lambda$ we have that $\Delta_{\tau_\Lambda}=\rho^{\tau_\Lambda-1}\alpha\Delta$, so

$$E_{x,\Delta}\left[V(x_{\tau_\Lambda},\Delta_{\tau_\Lambda})\right]-V(x,\Delta)+d(x,\Delta) =E_{x,\Delta}\left[\left(\rho^{\tau_\Lambda-1}\alpha\Delta\right)^{\beta-\varepsilon}\right]-\Delta^{\beta-\varepsilon}+s\Delta^{\beta-\varepsilon} =\Delta^{\beta-\varepsilon}\left(\alpha^{\beta-\varepsilon}E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]-1+s\right). \tag{54}$$

For $\Delta\leq D$, by construction of $b$ we have that (54) is bounded by $b$. For $\Delta>D$, by construction of $D$ we have that (54) is nonpositive. In either case, the first drift inequality is satisfied.

Next we show that the second inequality of (8) holds, when $c$ is sufficiently small. To start, we have that for $(x,\Delta)\in\Lambda$ that

$$E_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda-1}f(x_t,\Delta_t)\right] =f(x,\Delta)+E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}f(x_t,\Delta_t)\right] =cs\left(\tfrac{2}{K}\|x\|_\infty\right)^{\beta-\varepsilon}+cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon}E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\|x_t\|_\infty^{\beta-\varepsilon}\right]$$

$$\leq cs\Delta^{\beta-\varepsilon} + cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\|x_t\|_\infty^{\beta-\varepsilon}\right]$$
$$= cd(x,\Delta) + cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\|x_t\|_\infty^{\beta-\varepsilon}\right]. \quad (55)$$

The remainder of the proof will be dedicated to showing that there exists a constant $M>0$ so that uniformly over $(x,\Delta)\in\Lambda$ we have,

$$E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\|x_t\|_\infty^{\beta-\varepsilon}\right] \leq M\Delta^{\beta-\varepsilon} \quad (56)$$

so that we may bound (55) as

$$(55) \leq cd(x,\Delta) + cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon} M\Delta^{\beta-\varepsilon}$$
$$= c\left(1+\left(\tfrac{2}{K}\right)^{\beta-\varepsilon}M\right)d(x,\Delta).$$

It then suffices to take $c=(1+(\frac{2}{K})^{\beta-\varepsilon}M)^{-1}$, and the second drift inequality holds. Therefore, we dedicate the remainder of this proof to establishing (56).

As in the proof of Lemma 45, the "zoom-out" process $y_t$ defined by $y_0=e_0$, $y_{t+1}=AS(y_t)+w_t$ agrees with $x_t$ for $1\leq t\leq\tau_\Lambda$. Therefore, we have that

$$\begin{aligned}E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\|x_t\|_\infty^{\beta-\varepsilon}\right] &= E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\|y_t\|_\infty^{\beta-\varepsilon}\right]\\ &= E_{x,\Delta}\left[\sum_{t=1}^{\infty}\mathbf{1}_{\{\tau_\Lambda\geq t+1\}}\|y_t\|_\infty^{\beta-\varepsilon}\right]\\ &= \sum_{t=1}^{\infty}E_{x,\Delta}\left[\mathbf{1}_{\{\tau_\Lambda\geq t+1\}}\|y_t\|_\infty^{\beta-\varepsilon}\right],\end{aligned} \quad (57)$$

where the exchange of summation and expectation is justified by the monotone convergence theorem [43, Theorem 1.26].

Let $a := \|A\|_\infty$ for brevity. Now, choose $q\in\left(\frac{\beta}{\varepsilon},\ \frac{\beta}{\beta-\varepsilon}\log_a\xi\right)$ arbitrarily (this is a valid interval by the condition that $\rho > a^{\frac{\beta}{\varepsilon}}$). We let $p$ be the Hölder conjugate of $q$ (i.e., $\frac{1}{p}+\frac{1}{q}=1$) and apply Hölder's inequality to (57).

$$(57) \leq \sum_{t=1}^{\infty}P_{x,\Delta}(\tau_\Lambda\geq t+1)^{\frac{1}{q}}\,E_{x,\Delta}\left[\|y_t\|_\infty^{p(\beta-\varepsilon)}\right]^{\frac{1}{p}}. \quad (58)$$

We will consider each factor of the summand separately. First, by Lemma 45 and familiar arguments we have

$$\begin{aligned}&P_{x,\Delta}(\tau_\Lambda\geq t+1)\\ &\leq tT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^t-1}{t}-\frac{\Delta_{(N)}}{\alpha L}\right)\right)\\ &\leq tE\left[\|w_0\|_\infty^\beta\right]\left(\frac{\Delta}{2}\left(\frac{h\xi^t-1}{t}-\frac{\Delta_{(N)}}{\alpha L}\right)\right)^{-\beta}\\ &\leq E\left[\|w_0\|_\infty^\beta\right]2^\beta(\alpha L)^{-\beta}\cdot t^{\beta+1}\left(h\xi^t-1-\frac{\Delta_{(N)}}{\alpha L}t\right)^{-\beta},\end{aligned}$$

so that for series convergence,

$$P_{x,\Delta}(\tau_\Lambda\geq t+1)^{\frac{1}{q}} = \mathcal{O}_t\left(t^{\frac{\beta+1}{q}}\left(\xi^{-\frac{\beta}{q}}\right)^t\right) \quad (59)$$

and this term is $\mathcal{O}_\Delta(1)$. Next, we consider the second summand factor of (58). For brevity, let $m := p(\beta-\varepsilon)$. Since $\|Av\|_\infty\leq a\|v\|_\infty$ and $\|S(v)\|_\infty\leq\|v\|_\infty+\frac{\Delta_{(N)}}{2}$, we have for $t\geq1$ that

$$\begin{aligned}\|y_t\|_\infty &= \|AS(y_{t-1})+w_{t-1}\|_\infty\\ &\leq a\|y_{t-1}\|_\infty+\|w_{t-1}\|_\infty+a\cdot\tfrac{\Delta_{(N)}}{2}.\end{aligned}$$

Repeating this argument $t-1$ times yields

$$\|y_t\|_\infty \leq a^t\left(\|y_0\|_\infty+\sum_{i=0}^{t-1}a^{-i}\left(\frac{\|w_i\|_\infty}{a}+\frac{\Delta_{(N)}}{2}\right)\right)$$

and since $a>1$ and $\|y_0\|_\infty=\|e_0\|_\infty\leq\frac{\Delta}{2}$ we find that

$$\|y_t\|_\infty \leq a^t\left(\frac{\Delta}{2}+\sum_{i=0}^{t-1}\left(\|w_i\|_\infty+\frac{\Delta_{(N)}}{2}\right)\right).$$

Therefore, we have by Corollary 44 and repeated application of (53) that

$$\begin{aligned}&E_{x,\Delta}\left[\|y_t\|_\infty^m\right]\\ &\leq a^{mt}E_{x,\Delta}\left[\left(\frac{\Delta}{2}+\sum_{i=0}^{t-1}\left(\|w_i\|_\infty+\frac{\Delta_{(N)}}{2}\right)\right)^m\right]\\ &\leq a^{mt}\left(\Delta^m+2^mE\left[\left(\sum_{i=0}^{t-1}\left(\|w_i\|_\infty+\frac{\Delta_{(N)}}{2}\right)\right)^m\right]\right)\\ &\leq a^{mt}\left(\Delta^m+2^mt^{m+1}E\left[\left(\|w_0\|_\infty+\frac{\Delta_{(N)}}{2}\right)^m\right]\right)\\ &\leq a^{mt}\left(\Delta^m+t^{m+1}4^m\left(E\left[\|w_0\|_\infty^m\right]+\left(\frac{\Delta_{(N)}}{2}\right)^m\right)\right)\\ &\leq \Delta^ma^{mt}\left(1+t^{m+1}\left(\tfrac{4}{\alpha L}\right)^m\left(E\left[\|w_0\|_\infty^m\right]+\left(\frac{\Delta_{(N)}}{2}\right)^m\right)\right)\\ &= \Delta^ma^{mt}\mathcal{O}_t\left(t^{m+1}\right)\\ &= \Delta^ma^{mt}\mathcal{O}_t\left(t^{\beta+1}\right),\end{aligned}$$

where $q>\frac{\beta}{\varepsilon}$ implies that $\beta>m=p(\beta-\varepsilon)$. Note that the term $\mathcal{O}_t\left(t^{\beta+1}\right)$ is $\mathcal{O}_\Delta(1)$. Then finally we have,

$$E_{x,\Delta}\left[\|y_t\|_\infty^{p(\beta-\varepsilon)}\right]^{\frac{1}{p}} \leq \Delta^{\beta-\varepsilon}\mathcal{O}_t\left(t^{\frac{\beta+1}{p}}a^{(\beta-\varepsilon)t}\right), \quad (60)$$

which in combination with (59) yields that (58) is bounded by

$$(58) \leq \Delta^{\beta-\varepsilon}\sum_{t=1}^{\infty}\mathcal{O}_t\left(t^{\beta+1}\left(a^{\beta-\varepsilon}\xi^{-\frac{\beta}{q}}\right)^t\right),$$

where the series above converges to a constant $M>0$ when $a^{\beta-\varepsilon}\xi^{-\frac{\beta}{q}}<1$. This is ensured by the condition $q<\frac{\beta}{\beta-\varepsilon}\log_a\xi$, since then we have

$$\begin{aligned}a^{\beta-\varepsilon}\xi^{-\frac{\beta}{q}} &< a^{\beta-\varepsilon}\xi^{-\beta\left(\frac{\beta-\varepsilon}{\beta}\right)\frac{1}{\log_a\xi}}\\ &= a^{\beta-\varepsilon}\xi^{-(\beta-\varepsilon)\log_\xi a}\\ &= a^{\beta-\varepsilon}a^{-(\beta-\varepsilon)}=1.\end{aligned}$$

This establishes (56) and completes the proof. □

*Proof of Proposition 49:* Recall that we must show that

$$\sup_{N\geq2}E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right]<\infty.$$

We will do this again via random-time Lyapunov drift arguments. In particular, we will use most of the same Lyapunov parameters as in the previous proof. To be precise, with $r := \rho^{\beta-\varepsilon}$ we once again set

$$\begin{aligned} &V(x,\Delta) = \Delta^{\beta-\varepsilon}, \\ &d(x,\Delta) = s\Delta^{\beta-\varepsilon}, \\ &C = \Lambda \cap \{(x,\Delta) \in \mathbb{R}^n \times \Omega_\Delta : \Delta \le D\}, \\ &b = D^{\beta-\varepsilon}\left(\alpha^{\beta-\varepsilon} \sup_{x,\Delta\in\Lambda} E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right] - 1 + s\right), \end{aligned}$$

where $s \in (0, 1-\alpha^{\beta-\varepsilon})$ is arbitrary, $D > 0$ is such that $(x,\Delta) \in \Lambda$ and $\Delta > D$ imply

$$\alpha^{\beta-\varepsilon} E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right] - 1 + s \le 0$$

(such a $D$ exists by Proposition 47), and $b$ is finite by Proposition 47.

Instead of $f$ proportional to $\|x\|_\infty^{\beta-\varepsilon}$, we set $f$ proportional to $\Delta^{\beta-\varepsilon}$. That is, we set:

$$f(x,\Delta) = cs\Delta^{\beta-\varepsilon},$$

where

$$c = \left(1 + \frac{\alpha^{\beta-\varepsilon}}{r-1}\left(\sup_{(x,\Delta)\in\Lambda} E\left[r^{\tau_\Lambda-1}\right] - 1\right)\right)^{-1}$$

is well-defined by Proposition 47. We will show that these Lyapunov parameters satisfy the drift condition (8) and invoke Lemma 11 to complete the proof.

To begin, note that the first drift inequality of (8) does not involve $f$, and with otherwise identical Lyapunov parameters we showed in the proof of Proposition 48 that this drift inequality holds. Therefore, what remains is to show only that the second inequality in (8) holds. Explicitly, we must show that

$$csE_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda-1} \Delta_t^{\beta-\varepsilon}\right] \le s\Delta^{\beta-\varepsilon},$$

which is equivalent to,

$$\begin{aligned} &E_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda-1}\left(\frac{\Delta_t}{\Delta}\right)^{\beta-\varepsilon}\right] \\ &\quad \le c^{-1} = 1 + \frac{\alpha^{\beta-\varepsilon}}{r-1}\left(\sup_{(x,\Delta)\in\Lambda} E\left[r^{\tau_\Lambda-1}\right] - 1\right). \end{aligned}$$

We establish this now. Note that $\Delta_0 = \Delta$ and for $1 \le t \le \tau_\Lambda$ we have $\Delta_t = \alpha\Delta\rho^{t-1}$ so,

$$\begin{aligned} &E_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda-1}\left(\frac{\Delta_t}{\Delta}\right)^{\beta-\varepsilon}\right] \\ &\quad = 1 + \alpha^{\beta-\varepsilon} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\left(\rho^{t-1}\right)^{\beta-\varepsilon}\right] \\ &\quad = 1 + \alpha^{\beta-\varepsilon} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1} r^{t-1}\right] \\ &\quad = 1 + \alpha^{\beta-\varepsilon} \cdot \frac{E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right] - 1}{r-1} \\ &\quad = 1 + \frac{\alpha^{\beta-\varepsilon}}{r-1}\left(E_{x,\Delta}\left[r^{\tau_\Lambda}\right] - 1\right) \le c^{-1} \end{aligned}$$

by construction of $c$. Therefore the drift condition (8) is satisfied for choice of $b$ and $f$ independent of $N$, which by Lemma 11 completes the proof. □

## References


[1] J. Keeler, T. Linder, and S. Yüksel, "An asymptotically optimal two-part coding scheme for networked control under fixed-rate constraints," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2022, pp. 1360–1365.

[2] A. P. Johnston and S. Yüksel, "Stochastic stabilization of partially observed and multi-sensor systems driven by unbounded noise under fixed-rate information constraints," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 792–798, Mar. 2014.

[3] D. Bertsekas, *Dynamic Programming and Optimal Control*, vol. 1. Nashua, NH, USA: Athena Sci., 2005.

[4] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints—Part II: Stabilization with limited information feedback," *IEEE Trans. Autom. Control*, vol. 42, pp. 1294–1299, Mar. 1997.

[5] J. Baillieul, "Feedback designs for controlling device arrays with communication channel bandwidth constraints," in *Proc. 4th ARO Workshop Smart Struct.*, Aug. 1999, pp. 1–4.

[6] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1056–1068, Jul. 2004.

[7] L. V. J. Hespanha and A. Ortega, "Towards the control of linear systems with minimum bit-rate," in *Proc. Int. Symp. Math. Theory Netw. Syst. (MTNS)*, 2002.

[8] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM J. Control Optim.*, vol. 43, no. 2, pp. 413–436, Jan. 2004.

[9] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link—Part I: Scalar systems," *IEEE Trans. Inf. Theory*, vol. 52, no. 8, pp. 3369–3395, Aug. 2006.

[10] N. C. Martins, M. A. Dahleh, and N. Elia, "Feedback stabilization of uncertain systems in the presence of a direct link," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 438–447, Mar. 2006.

[11] A. S. Matveev and A. V. Savkin, "An analogue of Shannon information theory for detection and stabilization via noisy discrete communication channels," *SIAM J. Control Optim.*, vol. 46, no. 4, pp. 1323–1367, Jan. 2007.

[12] S. Yüksel, "Stochastic stabilization of noisy linear systems with fixed-rate limited feedback," *IEEE Trans. Autom. Control*, vol. 55, no. 12, pp. 2847–2853, Dec. 2010.

[13] S. Yüksel and S. P. Meyn, "Random-time, state-dependent stochastic drift for Markov chains and application to stochastic stabilization over erasure channels," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 47–59, Jan. 2013.

[14] V. Kostina, Y. Peres, G. Ranade, and M. Sellke, "Exact minimum number of bits to stabilize a linear system," *IEEE Trans. Autom. Control*, vol. 67, no. 10, pp. 5548–5554, Oct. 2022.

[15] O. Sabag, V. Kostina, and B. Hassibi, "Stabilizing dynamical systems with fixed-rate feedback using constrained quantizers," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2020, pp. 2855–2860.

[16] R. Zamir and M. Feder, "On universal quantization by randomized uniform/lattice quantizers," *IEEE Trans. Inf. Theory*, vol. 38, no. 2, pp. 428–436, Mar. 1992.

[17] E. I. Silva, M. S. Derpich, and J. Ostergaard, "A framework for control system design subject to average data-rate constraints," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1886–1899, Aug. 2011.

[18] C. D. Charalambous, P. A. Stavrou, and N. U. Ahmed, "Nonanticipative rate distortion function and relations to filtering theory," *IEEE Trans. Autom. Control*, vol. 59, no. 4, pp. 937–952, Apr. 2014.

[19] V. Kostina, "Data compression with low distortion and finite blocklength," *IEEE Trans. Inf. Theory*, vol. 63, no. 7, pp. 4268–4285, Jul. 2017.

[20] V. Kostina and B. Hassibi, "Rate-cost tradeoffs in control," *IEEE Trans. Autom. Control*, vol. 64, no. 11, pp. 4525–4540, Nov. 2019.

[21] J. Østergaard, "Stabilizing error correction codes for controlling LTI systems over erasure channels," in *Proc. IEEE Conf. Decis. Control (CDC)*, Mar. 2021, pp. 1–12.

[22] E. I. Silva, M. S. Derpich, J. Østergaard, and M. A. Encina, "A characterization of the minimal average data rate that guarantees a given closed-loop performance level," *IEEE Trans. Autom. Control*, vol. 61, no. 8, pp. 2171–2186, Aug. 2016.

[23] P. A. Stavrou, J. Østergaard, and C. D. Charalambous, "Zero-delay rate distortion via filtering for vector-valued Gaussian sources," *IEEE J. Sel. Topics Signal Process.*, vol. 12, no. 5, pp. 841–856, Oct. 2018.

[24] T. C. Cuvelier, T. Tanaka, and R. W. Heath Jr., "Time-invariant prefix coding for LQG control," 2022, *arXiv:2204.00588*.

[25] J. L. Massey, "Causality, feedback and directed information," in *Proc. IEEE Int. Symp. Inf. Theory*, Waikiki, HI, USA, May 1990, pp. 303–305.

[26] A. Gorbunov and M. Pinsker, "Nonanticipatory and prognostic epsilon entropies and message generation rates," *Problemy Peredachi Informatsii*, vol. 9, no. 3, pp. 12–21, 1973.

[27] R. Bansal and T. Başar, "Simultaneous design of measurement and control strategies for stochastic systems with feedback," *Automatica*, vol. 25, no. 5, pp. 679–694, Sep. 1989.

[28] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channels," *IEEE Trans. Autom. Control*, vol. 49, no. 8, pp. 1549–1561, Oct. 2004.

[29] T. Tanaka, "Semidefinite representation of sequential rate-distortion function for stationary Gauss–Markov processes," in *Proc. IEEE Conf. Control Appl. (CCA)*, Sep. 2015, pp. 1217–1222.

[30] T. Tanaka, K. K. Kim, P. A. Parrilo, and S. K. Mitter, "Semidefinite programming approach to Gaussian sequential rate-distortion trade-offs," *IEEE Trans. Autom. Control*, vol. 62, no. 4, pp. 1896–1910, Apr. 2017.

[31] T. Tanaka, P. M. Esfahani, and S. K. Mitter, "LQG control with minimum directed information: Semidefinite programming approach," *IEEE Trans. Autom. Control*, vol. 63, no. 1, pp. 37–52, Jan. 2018.

[32] M. S. Derpich and J. Ostergaard, "Improved upper bounds to the causal quadratic rate-distortion function for Gaussian stationary sources," *IEEE Trans. Inf. Theory*, vol. 58, no. 5, pp. 3131–3152, May 2012.

[33] P. A. Stavrou, T. Charalambous, C. D. Charalambous, and S. Loyka, "Optimal estimation via nonanticipative rate distortion function and applications to time-varying Gauss–Markov processes," *SIAM J. Control Optim.*, vol. 56, no. 5, pp. 3731–3765, Jan. 2018.

[34] P. Stavrou and M. Skoglund, "Asymptotic reverse waterfilling algorithm of NRDF for certain classes of vector Gauss–Markov processes," *IEEE Trans. Autom. Control*, vol. 67, no. 6, pp. 3196–3203, Jun. 2022.

[35] P. A. Stavrou, T. Tanaka, and S. Tatikonda, "The time-invariant multidimensional Gaussian sequential rate-distortion problem revisited," *IEEE Trans. Autom. Control*, vol. 65, no. 5, pp. 2245–2249, May 2020.

[36] S. Yüksel, "Jointly optimal LQG quantization and control policies for multi-dimensional linear Gaussian sources," *IEEE Trans. Autom. Control*, vol. 59, no. 6, pp. 1612–1617, Oct. 2014.

[37] S. Yüksel, "A note on the separation of optimal quantization and control policies in networked control," *SIAM J. Control Optim.*, vol. 57, no. 1, pp. 773–782, Jan. 2019.

[38] T. Linder and S. Yüksel, "On optimal zero-delay quantization of vector Markov sources," *IEEE Trans. Inf. Theory*, vol. 60, no. 10, pp. 2975–5991, Dec. 2014.

[39] M. Ghomi, T. Linder, and S. Yüksel, "Zero-delay lossy coding of linear vector Markov sources: Optimality of stationary codes and near optimality of finite memory codes," *IEEE Trans. Inf. Theory*, vol. 68, no. 5, pp. 3474–3488, May 2022.

[40] B.-H. Juang and A. Gray, "Multiple stage vector quantization for speech coding," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, May 1982, pp. 597–600.

[41] T. Berger, "Information rates of Wiener processes," *IEEE Trans. Inf. Theory*, vol. IT-16, no. 2, pp. 134–139, Mar. 1970.

[42] A. Sahai, "Coding unstable scalar Markov processes into two streams," in *Proc. Int. Symp. Inf. Theory*, Jun. 2004, p. 462.

[43] W. Rudin, *Real and Complex Analysis*, 3rd ed. New York, NY, USA: McGraw-Hill, 1987.

[44] E. Nummelin, "A splitting technique for Harris recurrent Markov chains," *Zeitschrift Wahrscheinlichkeitstheorie Verwandte Gebiete*, vol. 43, no. 4, pp. 309–318, 1978.

[45] K. B. Athreya and P. Ney, "A new approach to the limit theory of recurrent Markov chains," *Trans. Amer. Math. Soc.*, vol. 245, pp. 493–501, Jan. 1978.

[46] J. Keeler, "An asymptotically optimal two-part fixed-rate coding scheme for networked control," M.S. thesis, Dept. Math. Statist., Queen's Univ., Kingston, ON, Canada, 2022.

[47] S. Yüksel and T. Başar, *Stochastic Networked Control Systems: Stabilization and Optimization Under Information Constraints*. New York, NY, USA: Springer, 2013.

[48] D. Hui and D. L. Neuhoff, "Asymptotic analysis of optimal fixed-rate uniform scalar quantization," *IEEE Trans. Inf. Theory*, vol. 47, no. 3, pp. 957–977, Mar. 2001.

[49] C. Kawan and S. Yüksel, "Metric and topological entropy bounds for optimal coding of stochastic dynamical systems," *IEEE Trans. Autom. Control*, vol. 65, no. 6, pp. 2466–2479, Jun. 2020.

[50] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 2013.

[51] R. Zurkowski, S. Yuksel, and T. Linder, "On rates of convergence for Markov chains under random time state dependent drift criteria," *IEEE Trans. Autom. Control*, vol. 61, no. 1, pp. 145–155, Jan. 2016.

[52] S. Yüksel, "Stationary and ergodic properties of stochastic non linear systems controlled over communication channels," *SIAM J. Control Optim.*, vol. 54, no. 5, pp. 2844–2871, Jan. 2016.

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