Today’s main lesson:

- To learn two ways to describe the “powers” of players in a weighted voting system.
  - the Banzhaf power index
  - the Shapley-Shubik power index

**Dictators, dummies and players with veto power**

In a weighted voting system with the quota $q$,

- a **dictator** has absolute power over the outcome of the voting. Mathematically speaking, a dictator is a player whose weight is greater than or equal to $q$.

- a **dummy** has no power over the outcome of the voting. Mathematically speaking, a dummy is a player who cannot be a critical player for any winning coalition.

- a player that is not a dictator is said to have **veto power** if a motion cannot pass unless the player votes in favour of the motion. Mathematically speaking, a player, say $P$, is determined to have veto power (or to be a player with veto power) if
  - $P$ is not a dictator
  - a coalition consisting of all players except $P$ is a losing coalition.

**How to calculate the Banzhaf power index for each player**

Suppose that there are $N$ players ($P_1$ through $P_N$) in a weighted voting system.

- **Step 1**: List out all possible winning coalitions.
- **Step 2**: For each winning coalition, determine who are the critical players: For each winning coalition, remove a player (say $P$) from the coalition and see if the resulting coalition is a winning coalition or a losing coalition. If the resulting coalition is a losing coalition, then $P$ is a critical player.
- **Step 3**: For player $P_1$, count the number of times that $P_1$ becomes a critical player. Call the number $B_1$. Repeat the same process for all other players $P_2$ through $P_N$ to find $B_2$ through $B_N$.
- **Step 4**: Calculate $T = B_1 + B_2 + \cdots + B_N$.
- **Step 5**: Calculate $\beta_1 = B_1/T$. This $\beta_1$ is called the Banzhaf power index of $P_1$. Similarly, calculate $\beta_2 = B_2/T, \ldots, \beta_N = B_N/T$ to find the Banzhaf power indices for $P_2, \ldots, P_N$, respectively. The complete list of $\beta_1, \beta_2, \ldots, \beta_N$ is called the **Banzhaf power distribution** of this weighted voting system.
How to calculate the Shapley-Shubik power index for each player

Suppose that there are \( N \) players (\( P_1 \) through \( P_N \)) in a weighted voting system.

\( \diamond \) **Step 1:** List out all possible sequential coalitions involving all \( N \) players.

\( \diamond \) **Step 2:** For each sequential coalition, determine the **pivotal player**: In each sequential coalition, add the weights of the players one by one from left to right. The first player in the sequential coalition that causes this total to be greater than or equal to the quota is called the pivotal player.

\( \diamond \) **Step 3:** For player \( P_1 \), count the number of times that \( P_1 \) becomes the pivotal player. Call the number \( S_1 \). Repeat the same process for all other players \( P_2 \) through \( P_N \) to find \( S_2 \) through \( S_N \).

\( \diamond \) **Step 4:** Calculate \( T = S_1 + S_2 + \cdots + S_N \) (In fact, \( T \) is equal to the number of sequential coalitions involving \( N \) players).

\( \diamond \) **Step 5:** Calculate \( \sigma_1 = S_1/T \). This \( \sigma_1 \) is called the **Shapley-Shubik power index** of \( P_1 \). Similarly, calculate \( \sigma_2 = S_2/T, \ldots, \sigma_N = S_N/T \) to find the Shapley-Shubik power indices for \( P_2, \ldots, P_N \), respectively. The complete list of \( \sigma_1, \sigma_2, \ldots, \sigma_N \) is called the **Shapley-Shubik power distribution** of this weighted voting system.

**Weekly Assignment 3 (Due: January 30th, 2008)**

You may hand in your complete assignment at the next class (January 29th), or the following day at the math department office (Jeffery 310). Assignments should be stapled and clearly labeled with your full name, student number and the class number.

1. Consider the weighted voting system \([6 : 5, 5, 1]\). Determine which players, if any, (a) are dictators; (b) have veto power; (c) are dummies.

2. Question 9 (a),(d) of Chapter 2; 6th edition (Question 9 (a),(d) of Chapter 2; 5th edition)

3. Question 14 of Chapter 2; 6th edition (Question 14 of Chapter 2; 5th edition)

4. Question 15 of Chapter 2; 6th edition (Question 15 of Chapter 2; 5th edition)

5. Question 20 of Chapter 2; 6th edition (Question 20 of Chapter 2; 5th edition)

6. Question 24 of Chapter 2; 6th edition (Question 24 of Chapter 2; 5th edition)

7. Question 25 of Chapter 2; 6th edition (Question 25 of Chapter 2; 5th edition)

8. Question 55 of Chapter 2; 6th edition (Question 49 of Chapter 2; 5th edition)