STUDENT NUMBER

NAME (optional)

QUEEN'S UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 111
MIDYEAR EXAMINATION **SOLUTIONS**

DEC 2011
PROF. PETER TAYLOR

INSTRUCTIONS: This examination is THREE HOURS in length. Please answer all questions in the space provided. If you need more room, use the back of the previous page. The following aids are allowed and can be brought into the examination

Casio fx-991 calculator (or equivalent)
One 8.5×11 sheet with writing on both sides
One water bottle

PLEASE NOTE: "Proctors may be unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."

Check that your question paper has 13 pages. There are 87 total marks but the paper was marked out of 70.

Question	TOTAL MARKS	MARK GIVEN
1	32	
2	9	
3	10	
4	8	
5	20	
6	8	
TOTAL	87	

1. [32 marks] For this question, **only the answer will be marked**. **2 marks for each**

(a) Find the inverse of the matrix $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$.

$$\frac{1}{7}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

(b) Calculate the matrix product $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$ if you are given that $a\begin{bmatrix} 3 \\ 2 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$$

(c) Find the equation of the plane passing through the point (1, 0, 0) and perpendicular to the line:
$x = 3$
$y = 2+t$
$z = 1-2t$.

The line is parallel to the vector [0 1 –2] so this is normal to the plane which will have equation $y - 2z = d$ for some d. To find d plug the point (1, 0, 0) into the equation and get $0 = d$. Hence the plane is:

$$y - 2z = 0.$$

(d) Find the projection of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the plane $x+y+z = 0$.

The projection of the vector on the normal to the plane is $\dfrac{[1,2,3]\bullet[1,1,1]}{[1,1,1]\bullet[1,1,1]}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{6}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

Hence the projection on the plane is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(e) Find the point where the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ intersects the plane $x + 2y = 0$.

Put the equations into the plane: $(1+3t) + 2(2+t) = 0$.
Solve for t: $5t + 5 = 0$ so that $t = -1$.
The point is $(x, y, z) = (-2, 1, -1)$

(f) Let P be the plane spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, i.e. P is the set of all vectors of the form $s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. Suppose that the projection of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ on P is $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

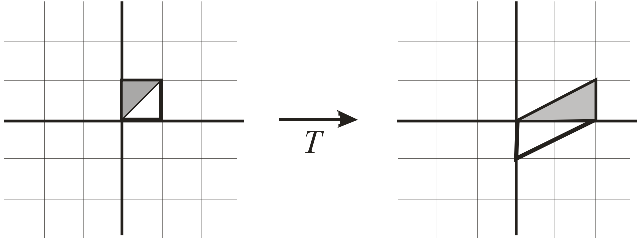
Find the least squares approximate solution to the system: $\begin{aligned} x - y &= a \\ x + 2y &= b \\ x + y &= c \end{aligned}$.

$x =$ ___2___
$y =$ ___1___

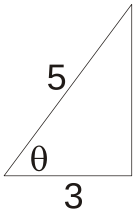
(g) Define a transformation T from R^3 to R^3 by the formula $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ x - y \\ z + 2 \end{bmatrix}$. Show that T is affine by producing a matrix A and vector $\mathbf{b}$ such that $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$$

(h) The linear transformation T multiplies each vector in R^2 by the matrix $\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$. On the set of axes at the right draw the image of the "half-shaded square" drawn on the left-hand set of axes. Be sure to include the half-shading.



(i) The point Q = (1, –3) is rotated about the origin in a positive (counterclockwise) direction through the angle θ defined by the right-angled triangle in the diagram. Find the coordinates of the rotated point.



The matrix that rotates through θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

Apply this to the vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and we get the point (3, –1).

(j) A point Q is on the line segment running between the points A(1, 1, –2) and B(4, –5, 4). It is twice as far from B as it is from A. Find the coordinates of Q.

Q = (2/3)A + (1/3)B = = (2/3)(1, 1, –2) + (1/3)(4, –5, 4) = (2, –1, 0).

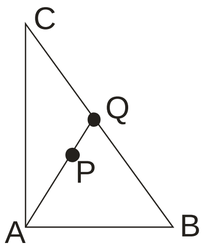
(k) Write $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as an affine combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(l) Suppose that P is the following affine combination of A, B and C:

$\mathrm{P} = \dfrac{5\mathrm{A} + 4\mathrm{B} + 6\mathrm{C}}{15}$

Let Q be the intersection of the line AP extended with the line BC. Write P as an affine combination of A and Q.



$$\mathrm{P} = \frac{5\mathrm{A} + 10\mathrm{Q}}{15} = \frac{1}{3}\mathrm{A} + \frac{2}{3}\mathrm{Q}$$

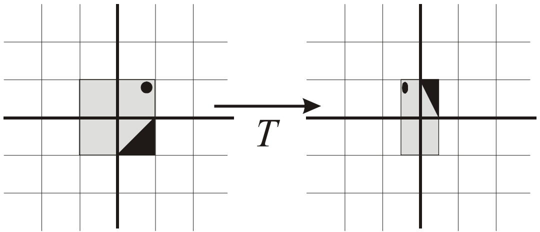
(m) Let T be an affine transformation on R^2 for which

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

Using the structure theorem for affine transformations,

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(n) The linear transformation T maps the 2×2 square centred at the origin to the 2×1 rectangle drawn below at the right. Find a general formula for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$. [Check your work with a suitable point, e.g. (1, –1).]

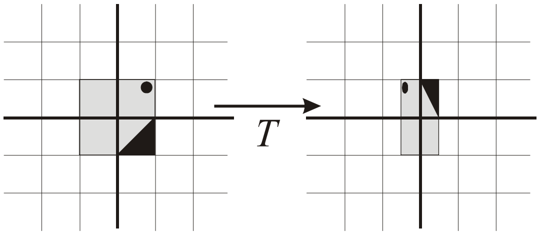


From the diagram,

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}$. Thus T has matrix $A = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix}$. Then

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y/2 \\ x \end{bmatrix}$$

(o) Suppose that over the time interval $0 \le t \le 1$ every point $\begin{bmatrix} x \\ y \end{bmatrix}$ moves *in a straight line at constant speed* to its image $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ where T is the transformation depicted in (n) above. Find a formula for the position of $\begin{bmatrix} x \\ y \end{bmatrix}$ at any time t.



$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (1-t)\begin{bmatrix} x \\ y \end{bmatrix} + t\begin{bmatrix} -y/2 \\ x \end{bmatrix}$$

(p) In the 7-hat game, players 3 and 5 have blue hats and the other players have red. Which player (or players) guesses a hat colour, and what is that colour?

If player 6 had a blue hat we would have a losing configuration. Hence player 6 guesses RED.

2. (a) [6] Find the least squares approximation to the solution of the equation

$$A\boldsymbol{\beta} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$$

Check by calculating the error $\boldsymbol{\varepsilon} = \mathbf{b} - A\boldsymbol{\beta}$^ and noting that it is orthogonal to the columns of A. :

Solution. The equation for the least-squares solution $\boldsymbol{\beta}$ is $A^{\mathrm{T}}A\boldsymbol{\beta} = A^{\mathrm{T}}\mathbf{b}$. We calculate:

$$A^T A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

Then:

$$\boldsymbol{\beta} = (A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -6 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -6 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 18 \\ -8 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 9 \\ -4 \end{bmatrix} \quad \text{or} \quad \begin{matrix} \alpha = 9/7 \\ \beta = -4/7 \end{matrix}$$

The error is:

$$\boldsymbol{\varepsilon} = \mathbf{b} - A\boldsymbol{\beta} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 9/7 \\ -4/7 \end{bmatrix} = \frac{1}{7}\left(\begin{bmatrix} 7 \\ -21 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ -17 \\ 13 \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$$

and it is orthogonal to the columns of A.

(b) [2] Find vectors **u**, **v** and **w** such that the question of (a) can be interpreted as asking for the projection of **w** on the plane spanned by **u** and **v**, and show how this projection can be deduced from the solution $\boldsymbol{\beta}$.

Solution.

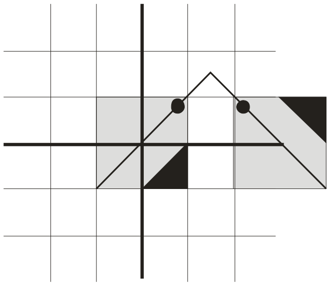
The vectors **u** and **v** are the columns of A and **w** is the vector **b**. Thus

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$$

The projection of **w** on the plane is

$$A\boldsymbol{\beta} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 9/7 \\ -4/7 \end{bmatrix} = \begin{bmatrix} 5 \\ -17 \\ 13 \end{bmatrix}.$$

3. In the diagram below, the 2×2 shaded square centred at the origin is bounded by the lines $x = \pm 1$ and $y = \pm 1$. Thus the grid lines are 1 unit apart. Let T denote the transformation in $\mathbb{R}^2$ that rotates through 90º about the point (1.5, 1.5).

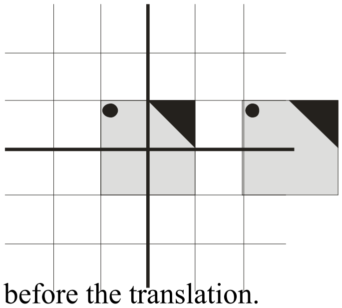


[3] (a) On the diagram below, sketch the image of the shaded square under the transformation T.

The image is displayed at the right.

[3] (b) Use the structure theorem to find a matrix A and a vector **b**, for which $T(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ for all vectors **v**. Then simplify your expression to get a formula of the form:

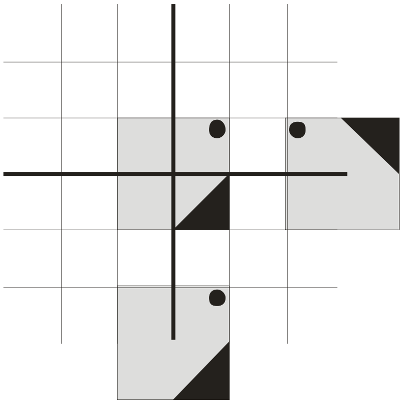
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by + c \\ dx + ey + f \end{bmatrix}.$$



The $T(\mathbf{v})$ expression in (b) decomposes T as a linear transformation (a rotation about the origin through 90º) followed by a translation (by **b**). The rotation is given by the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and from the diagram, the translation is through $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Thus: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -y+3 \\ x \end{bmatrix}$.

The diagram at the right shows the configuration after the rotation but before the translation.

[4] (c) Show that T can also be obtained as a translation first followed by a rotation about the origin. Provide an algebraic expression for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ that clearly displays the translation and the rotation. Use the diagram below to illustrate the two steps.



The fast way to do this is to work with the diagram. Rotate the final image about the origin by *minus* 90 to get the position after the translation. We see that it is 3 units below the original. Thus the initial translation needs to be by $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$. Following that we rotate through 90º. Putting those together:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

Check that this gives us the same answer as (b):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-3 \end{bmatrix} = \begin{bmatrix} -y+3 \\ x \end{bmatrix}.$$

4. [8] Use the structure theorem to find the affine transformation that reflects each point in the plane $x + y + z = 2$. Be sure to check your final formula with a point that lies on the plane (which must reflect into itself).

Solution. We need the T-images of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{0}$. Thus we need to reflect these in the plane. The normal to the plane is $\mathbf{n} = [1, 1, 1]$, so the line through $\mathbf{e}_1$ orthogonal to the plane is:

$$\mathbf{e}_1 + t\mathbf{n} = \begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1+t\\t\\t\end{bmatrix}$$

This intersects the plane when $1+3t = 2$, so that $t = 1/3$. The reflected point will then be at $t = 2/3$, so is $[5, 2, 2]/3$. Symmetry will give us the remaining $\mathbf{e}_i$:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}5\\2\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}2\\5\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}2\\2\\5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}4\\4\\4\end{bmatrix}$$

The last formula comes from the equation:

$$\mathbf{0} + t\mathbf{n} = t\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}t\\t\\t\end{bmatrix}$$

And this intersects the plane when $3t = 2$, so that $t = 2/3$. The reflected point will then be at $t = 4/3$, so is $[4, 4, 4]/3$. The structure theorem for affine transformations gives us:

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) - T(\mathbf{0}) & T(\mathbf{e}_2) - T(\mathbf{0}) & \cdots & T(\mathbf{e}_n) - T(\mathbf{0}) \end{bmatrix}\mathbf{x} + T(0)$$

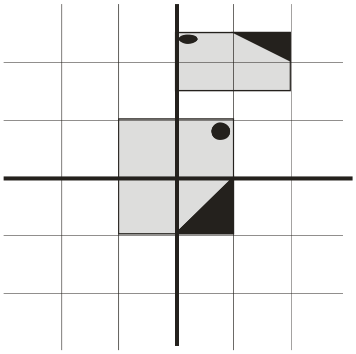
$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}1 & -2 & -2\\-2 & 1 & -2\\-2 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} + \frac{1}{3}\begin{bmatrix}4\\4\\4\end{bmatrix}$$

Check with $\mathbf{x} = [1, 1, 0]$ which lies on the plane

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}1 & -2 & -2\\-2 & 1 & -2\\-2 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}4\\4\\4\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-1\\-1\\-4\end{bmatrix} + \frac{1}{3}\begin{bmatrix}4\\4\\4\end{bmatrix} = \frac{1}{3}\begin{bmatrix}3\\3\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

as expected.

5. In the diagram at the right, the grid lines are 1 unit apart. Let T denote the affine transformation in R^2 that maps the 2×2 shaded square centred at the origin onto the 1×2 rectangle in the upper right corner of the diagram.



(a) [5] Describe T as the composition of a dilation, D, followed by a rotation R, about the origin, followed by a translation M. Find expressions for each of these three transformations and put them together to get a formula for $T(\mathbf{v})$ of the form $A\mathbf{v} + \mathbf{b}$. As a check, use your formula to calculate $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$ and compare with the diagram.

You should show that you have done this check.

D is multiplication of x by 0.5. R is rotation through 90º. M is translation by [1, 2].

$$D\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}0.5x\\y\end{bmatrix},\quad R\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix},\quad M\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}$$

Putting these together:

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=M\circ R\circ D\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0.5x\\y\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0&-1\\0.5&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}.$$

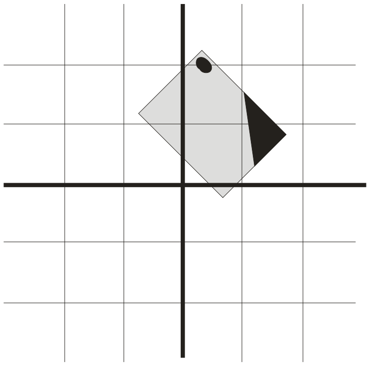
Check:

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}0&-1\\0.5&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\2.5\end{bmatrix}$$ and that corresponds with the diagram.

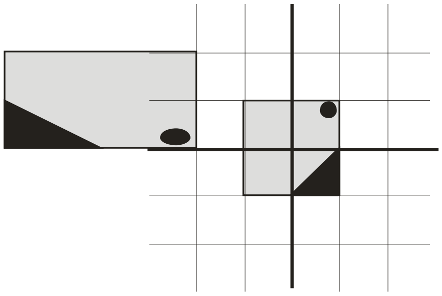
(b) [3] Construct a family of transformations $T_t(\mathbf{x})$ $(0\le t\le 1)$ which implements the transformation T continuously such that at any time t, $T_t(\mathbf{x})$ maps the initial image to the image at time t, and so that the dilation, the rotation and the translation happen at a constant rate.

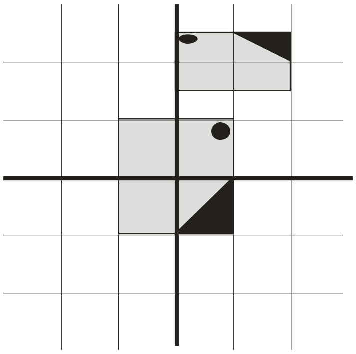
$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\cos(90t)&-\sin(90t)\\\sin(90t)&\cos(90t)\end{bmatrix}\begin{bmatrix}(1-t/2)x\\y\end{bmatrix}+\begin{bmatrix}t\\2t\end{bmatrix}$$

(c) [3] On the diagram at the right, sketch the image of the shaded square under the transformation $T_t(\mathbf{x})$ at time $t = ½$.



(d) [4] Here we continue to work with the transformation *T* of part (a).
Your job now is to draw *on the diagram below right* a shaded square which will transform under the action of *T* to the original shaded square centred at the origin. That is, the shaded rectangle that you draw, when dilated, rotated and translated suitable amounts, will then coincide with the shaded square centred at the origin. Alternatively stated, what you will draw is the image of the shaded square centred at the origin under the *inverse* of the transformation *T*.





(e) [5] Here we continue to work with the transformation T of part (a). The *inverse* of a transformation T is the transformation S that when applied to $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$ gives me $\begin{bmatrix}x\\y\end{bmatrix}$ back again. That is:

$S\left(T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)\right)=\begin{bmatrix}x\\y\end{bmatrix}$. Another way to say this is:

$$\text{When } T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}u\\v\end{bmatrix} \text{ then } S\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix}.$$

Find an algebraic expression for the inverse of T of the form $S\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}au+bv+c\\du+ev+f\end{bmatrix}$.

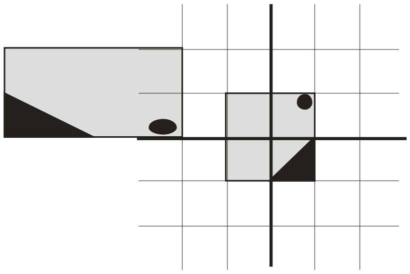
There are two ways you might do this:
Geometric: work from your diagram of (d), decomposing the transformation as a composition of dilation, rotation and translation in some order.
Algebraic: Let $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}u\\v\end{bmatrix}$ and solve for x and y in terms of u and v.

Do it both ways and check that you get the same answer.

Geometric:
In the diagram of (c) we have to transform the central square into the 2×4 rectangle on the left.
We can do this if we multiply v by 2, rotate about the origin by –90º and then translate by $\begin{bmatrix}-4\\1\end{bmatrix}$.

This gives us:

$$S\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}u\\2v\end{bmatrix}+\begin{bmatrix}-4\\1\end{bmatrix}=\begin{bmatrix}2v-4\\-u+1\end{bmatrix}.$$



Algebraic:

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0.5x\\y\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-y+1\\0.5x+2\end{bmatrix}=\begin{bmatrix}u\\v\end{bmatrix}$$

Solving for x and y:
$y=1-u$
$0.5x=v-2$
$x=2v-4$
This gives the same answer as above.

6. [8] In the graphs at the right, the grid lines are 1 unit apart.
The linear transformation T is a composition of two transformations: a horizontal shear with matrix of the form

$$S = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

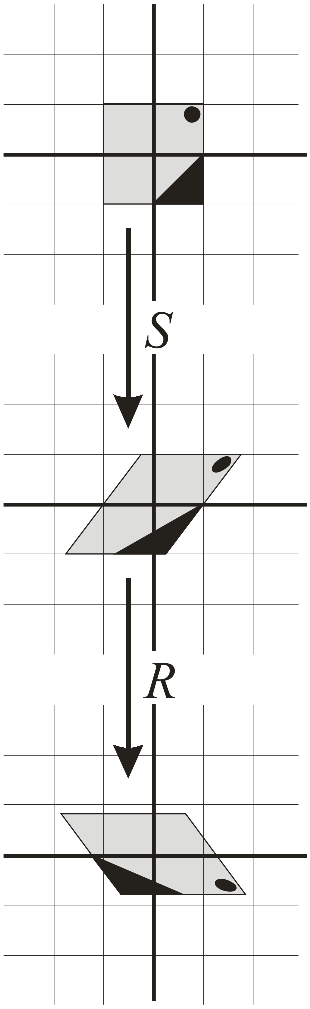
followed by a rotation with matrix of the form

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Assume that after the first step (the shear), the point (–1, 1) has been shifted to (–1/4, 1) and after the second step (the rotation), the vertical edges of the original shaded square have become horizontal (parallel to the x-axis), as depicted in the diagram.

Using this information, find the matrices S and R and put these together to obtain the matrix of T.
When forming the matrix of *T*, take care with the order of the matrices *S* and *R*.

Check your answer by calculating $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and compare with the diagram. **You should show that you have done this check**



Solution:
There are two possible approaches.

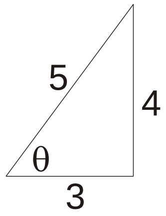
Algebraic: The point (–1, 1) has been shifted a distance 3/4 and thus s = 3/4. The matrix of T is then:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 3/4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & (3/4)\cos\theta - \sin\theta \\ \sin\theta & (3/4)\sin\theta + \cos\theta \end{bmatrix}$$

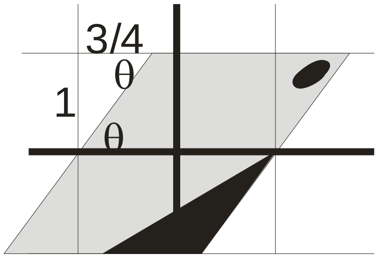
Since the original vertical lines are horizontal after T, T must map $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and hence the second column of the matrix of T must be a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus (3/4)sin$\theta$ + cosθ = 0 and hence

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{4}{3}.$$

Thus we can get sinθ and cosθ from the 3-4-5 triangle drawn at the right: sinθ = –4/5, and cosθ = 3/5. The negative sign on sin comes from the fact that we can see from the diagram that the angle is negative.



Geometric: We can see the angle of rotation in the middle graph—it's $-\theta$ in the diagram at the right. If we multiply the sides of the white triangle by 4, we get the 3-4-5 triangle above and again $\sin\theta = -4/5$, and $\cos\theta = 3/5$.



From either of these arguments, the rotation matrix is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}.$$

Finally the matrix of T is:

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \frac{1}{20}\begin{bmatrix} 12 & 25 \\ -16 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -13 \\ -16 \end{bmatrix}$$

And that fits with the information in the final diagram.