Week #6 - Taylor Series, Derivatives and Graphs

Section 10.1

From "Calculus, Single Variable" by Hughes-Hallett, Gleason, McCallum et. al.

QUIZ PREPARATION PROBLEMS

1.

$$\begin{aligned} f(x) &= (1-x)^{-1} & f(0) &= 1 \\ f'(x) &= -(1-x)^{-2}(-1) = (1-x)^{-2} & f'(0) &= 1 \\ f''(x) &= -2(1-x)^{-3}(-1) = (1-x)^{-3} & f''(0) &= 2 \\ f'''(x) &= 2\cdot 3(1-x)^{-4} & f'''(0) &= 6 \\ &\vdots \\ f^{(n)}(x) &= n!(1-x)^{-n-1} & f^{(n)}(0) &= n! \end{aligned}$$

Using this information,

$$\begin{aligned} P_3(x) &= 1 + (x-0) + \frac{2}{2}(x-0)^2 + \frac{6}{3!}(x-0)^3 \\ &= 1 + x + x^2 + x^3 \end{aligned}$$

Similarly,

$$\begin{aligned} P_5(x) &= 1 + x + \frac{2}{2}x^2 + \frac{6}{3!}x^3 + \frac{4!}{4!}x^4 + \frac{5!}{5!}x^5 \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 \end{aligned}$$

Following the same pattern,

$$P_7(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$$

3.

$$
\begin{aligned}
f(x) &= (1+x)^{1/2} & f(0) &= 1 \\
f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(0) &= \frac{1}{2} \\
f''(x) &= \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(1+x)^{-3/2} & f''(0) &= \frac{-1}{4} \\
f'''(x) &= \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(1+x)^{-5/2} & f'''(0) &= \frac{3}{8} \\
f''''(x) &= \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(1+x)^{-7/2} & f''''(0) &= \frac{-15}{16}
\end{aligned}
$$

Using this information,

$$
\begin{aligned}
P_2(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 \\
&= 1 + \frac{1}{2}x + \frac{-1}{8}x^2
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
P_3(x) &= 1 + \frac{1}{2}x - \frac{1}{4}\frac{1}{2}x^2 + \frac{3}{8}\frac{1}{3!}x^3 \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \\
P_4(x) &= 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{8}\frac{1}{3!}x^3 - \frac{15}{16}\frac{1}{4!}x^4 \\
&= 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4
\end{aligned}
$$

8. Recall the useful identities that $\sin(0) = 0$ and $\cos(0) = 1$.

$$
\begin{aligned}
f(x) &= \tan(x) & f(0) &= 0 \\
f'(x) &= (\cos x)^{-2} & f'(0) &= 1 \\
f''(x) &= -2(\cos x)^{-3}(-\sin x) = 2(\cos x)^{-3}(\sin x) & f''(0) &= 0 \\
f^{(3)}(x) &= -6(\cos x)^{-4}(-\sin x)(\sin x) + 2(\cos x)^{-3}\cos(x) & & \\
&= 6(\cos x)^{-4}(\sin x)^2 + 2(\cos x)^{-2} & f^{(3)}(0) &= 2 \\
f^{(4)}(x) &= -24(\cos x)^{-5}(-\sin x)(\sin x)^2 + 12(\cos x)^{-4}(\sin x)(\cos x) - 4(\cos x)^{-3}(-\sin x) & & \\
&= 24(\cos x)^{-5}(\sin x)^3 + 16(\cos x)^{-3}(\sin x) & f^{(4)}(0) &= 0
\end{aligned}
$$

Because the fourth derivative is zero, both $P_3(x)$ and $P_4(x)$ will be equal:

$$
\begin{aligned}
P_4(x) &= 0 + 1 \cdot x + 0\frac{1}{2}x^2 + 2\frac{1}{3!}x^3 + 0\frac{1}{4!}x^4 \\
&= x + \frac{1}{3}x^3 \\
P_3(x) &= x + \frac{1}{3}x^3
\end{aligned}
$$

11.

$$
\begin{aligned}
f(x) &= e^x & f(1) &= e \\
f'(x) &= e^x & f'(1) &= e \\
f''(x) &= e^x & f''(1) &= e \\
f'''(x) &= e^x & f'''(1) &= e \\
f''''(x) &= e^x & f''''(1) &= e
\end{aligned}
$$

Recall, e is just a number, with a value close to 2.7182. We build the Taylor polynomial using the usual formula.

$$
\begin{aligned}
P_4(x) &= e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4 \\
&= e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4
\end{aligned}
$$

12.

$$
\begin{aligned}
f(x) &= (1+x)^{1/2} & f(1) &= \sqrt{2} \\
f'(x) &= \left(\frac{1}{2}\right)(1+x)^{-1/2} & f'(1) &= \frac{1}{2\sqrt{2}} \\
f''(x) &= \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(1+x)^{-3/2} & f''(1) &= \frac{-1}{4 \cdot 2^{3/2}} = \frac{-1}{8\sqrt{2}} \\
f'''(x) &= \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(1+x)^{-5/2} & f'''(1) &= \frac{3}{8 \cdot 2^{5/2}} = \frac{3}{32\sqrt{2}}
\end{aligned}
$$

$$
P_3(x) = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) - \frac{1}{8\sqrt{2} \cdot 2!}(x-1)^2 + \frac{3}{32\sqrt{2} \cdot 3!}(x-1)^3
$$

13. Recall from the graphs of sin and cos that $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$.

$$\begin{aligned} f(x) &= \sin x & f(\pi/2) &= 1 \\ f'(x) &= \cos x & f'(\pi/2) &= 0 \\ f''(x) &= -\sin x & f''(\pi/2) &= -1 \\ f'''(x) &= -\cos x & f'''(\pi/2) &= 0 \\ f''''(x) &= \sin x & f''''(\pi/2) &= 1 \end{aligned}$$

$$\begin{aligned} P_4(x) &= 1 + 0(x-\pi/2) - \frac{1}{2!}(x-\pi/2)^2 + \frac{0}{3!}(x-\pi/2)^3 + \frac{1}{4!}(x-\pi/2)^3 \\ &= 1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 \end{aligned}$$

14. $\pi/4$ radians is 45^o. From the standard triangles, this gives $\sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$.

$$\begin{aligned} f(x) &= \cos x & f(\pi/4) &= \frac{1}{\sqrt{2}} \\ f'(x) &= -\sin x & f'(\pi/4) &= -\frac{1}{\sqrt{2}} \\ f''(x) &= -\cos x & f''(\pi/4) &= -\frac{1}{\sqrt{2}} \\ f'''(x) &= \sin x & f'''(\pi/4) &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$P_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x-\pi/4) - \frac{1}{\sqrt{2}\cdot 2!}(x-\pi/4)^2 + \frac{1}{\sqrt{2}\cdot 3!}(x-\pi/4)^3$$

15.

$$\begin{aligned} f(x) &= \ln(x^2) & f(1) &= 0 \\ f'(x) &= \frac{1}{x^2}2x = 2x^{-1} & f'(1) &= 2 \\ f''(x) &= -2x^{-2} & f''(1) &= -2 \\ f'''(x) &= 4x^{-3} & f'''(1) &= 4 \\ f''''(x) &= -12x^{-4} & f''''(1) &= -12 \end{aligned}$$

$$\begin{aligned} P_3(x) &= 0 + 2(x-1) - \frac{2}{2!}(x-1)^2 + \frac{4}{3!}(x-1)^3 - \frac{12}{3!}(x-1)^4 \\ &= 2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{2}(x-1)^4 \end{aligned}$$

16. Recall from the graphs of sin and cos that $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$.

$$\begin{aligned} f(x) &= \sin(2x) & f(\pi/4) &= \sin(\pi/2) = 1 \\ f'(x) &= 2\cos(2x) & f'(\pi/4) &= 0 \\ f''(x) &= -4\sin(2x) & f''(\pi/4) &= -4 \\ f'''(x) &= -8\cos(2x) & f'''(\pi/4) &= 0 \\ f''''(x) &= 16\sin(2x) & f''''(\pi/4) &= 16 \end{aligned}$$

$$\begin{aligned} P_4(x) &= 1 + 0(x-\pi/4) - \frac{4}{2!}(x-\pi/4)^2 + \frac{0}{3!}(x-\pi/4)^3 + \frac{16}{4!}(x-\pi/4)^4 \\ &= 1 - 2(x-\pi/4)^2 + \frac{2}{3}(x-\pi/4)^4 \end{aligned}$$

18. Since $P_2(x)$ is the second degree Taylor polynomial for $f(x)$ about $x = 0$, we know from the definition that

$$\begin{aligned} P(0) &= a = f(0) & &\text{y-intercept} \\ P'(0) &= b = f'(0) & &\text{slope at } x = 0 \\ P''(0) &= \frac{c}{2} = f''(0) & &\text{concavity at } x = 0 \end{aligned}$$

As with the previous problem, $a > 0$, $b < 0$ and $c < 0$.

20. As with the original problem, $a < 0$, $b < 0$ and $c > 0$.

21. Using the fact that

$$f(x) \approx P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

and identifying coefficients with those given for $P_2(x)$, we obtain the following:

(a) $f(0) =$ constant term = 5,

(b) $f'(0) =$ coefficient of x = -7,

(c) $\dfrac{f''(0)}{2} =$ coefficient of $x^2 = 8$, so $f''(0) = 16$

25.

$$\begin{aligned} f(x) &= 4x^2 - 7x + 2 & f(0) &= 2 \\ f'(x) &= 8x - 7 & f'(0) &= -7 \\ f''(x) &= 8 & f''(0) &= 8, \end{aligned}$$

so $P_2(x) = 2 + (-7)x + \frac{8}{2}x^2 = 4x^2 - 7x + 2$. We notice that $f(x) = P_2(x)$ in this case.

26. $f'(x) = 3x^2 + 14x - 5$, $f''(x) = 6x + 14$, $f'''(x) = 6$. Thus, about $a = 0$,

$$\begin{aligned} P_3(x) &= 1 + \frac{-5}{1!}x + \frac{14}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 - 5x + 7x^2 + x^3 \\ &= f(x). \end{aligned}$$

Once again, we notice that the Taylor polynomial of a function which already is a polynomial gives us the same polynomial back.

27. (a) We'll make the following conjecture:
"If $f(x)$ is a polynomial of degree n, i.e.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n;$$

then $P_n(x)$, the n^{th} degree Taylor polynomial for $f(x)$ about $x = 0$, is $f(x)$ itself."

(b) All we need to do is to calculate $P_n(x)$, the n^{th} degree Taylor polynomial for f about $x = 0$ and see if it is the same as $f(x)$.

$$\begin{aligned} f(0) &= a_0; \\ f'(0) = \left(a_1 + 2a_2 x + \cdots + n a_n x^{n-1}\right)\Big|_{x=0} & \\ &= a_1; \\ f''(0) = \left(2a_2 + 3\cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2}\right)\Big|_{x=0} & \\ &= 2!\ a_2. \end{aligned}$$

If we continue doing this, we'll see in general

$$f^{(k)}(0) = k!\ a_k, \quad k = 1, 2, 3, \ldots, n.$$

Therefore,

$$\begin{aligned} P_n(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\ &= f(x). \end{aligned}$$

33. (a) $f(x) = e^{x^2}$.
$f'(x) = 2xe^{x^2}$, $f''(x) = 2(1 + 2x^2)e^{x^2}$, $f'''(x) = 4(3x + 2x^3)e^{x^2}$,
$f^{(4)} = 4(3 + 6x^2)e^{x^2} + 4(3x + 2x^3)2xe^{x^2}$.
The Taylor polynomial about $x = 0$ is

$$\begin{aligned} P_4(x) &= 1 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 \\ &= 1 + x^2 + \frac{1}{2}x^4. \end{aligned}$$

(b) $f(x) = e^z$. The Taylor polynomial of degree 2 is

$$Q_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} = 1 + x + \frac{1}{2}x^2.$$

If we substitute x^2 for x in the Taylor polynomial for e^x of degree 2, we will get $P_4(x)$, the Taylor polynomial for e^{x^2} of degree 4:

$$\begin{aligned} Q_2(x^2) &= 1 + x^2 + \frac{1}{2}(x^2)^2 \\ &= 1 + x^2 + \frac{1}{2}x^4 \\ &= P_4(x). \end{aligned}$$

(c) Let $Q_{10}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{10}}{10!}$ be the Taylor polynomial of degree 10 for e^x about $x = 0$. Then

$$\begin{aligned} P_{20}(x) &= Q_{10}(x^2) \\ &= 1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \cdots + \frac{(x^2)^{10}}{10!} \\ &= 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots + \frac{x^{20}}{10!}. \end{aligned}$$

(d) Let $e^x \approx Q_5(x) = 1 + \frac{x}{1!} + \cdots + \frac{x^5}{5!}$. Then

$$\begin{aligned} e^{-2x} &\approx Q_5(-2x) \\ &= 1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} \\ &= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5. \end{aligned}$$