Unit #1 - Transformation of Functions; Exponential and Logarithms

Section 1.4

Some material from "Calculus, Single and MultiVariable" by Hughes-Hallett, Gleason, McCallum et. al.

SUGGESTED PROBLEMS

1. $\mathrm{e}^{\ln(1/2)} = \frac{1}{2}$

2. $10^{\log(AB)} = AB$

3. $5\mathrm{e}^{\ln\left(A^2\right)} = 5A^2$

4.
$$\ln\left(\mathrm{e}^{2AB}\right) = 2AB$$

5.
$$\begin{aligned}&\ln(1/\mathrm{e}) + \ln(AB)\\ &= \ln\left(\mathrm{e}^{-1}\right) + \ln(A) + \ln(B)\\ &= -1 + \ln(A) + \ln(B)\end{aligned}$$

6.
$$2\ln(e^A) + 3\ln(B^e) = 2A + 3e\ln(B)$$

7.
$$\begin{aligned}3^x &= 11\\ \ln(3^x) &= \ln(11)\\ x\ln(3) &= \ln(11)\\ x &= \frac{\ln(11)}{\ln(3)}\end{aligned}$$

8.
$$\begin{aligned}17^x &= 2\\ \ln(17^x) = x\ln(17) &= \ln(2)\\ x &= \frac{\ln(2)}{\ln(17)}\end{aligned}$$

9.
$$\begin{aligned}20 &= 50(1.04)^x\\ \ln(20) = \ln[50(1.04)^x] &= \ln(50) + x\ln(1.04)\\ x &= \frac{\ln(20) - \ln(50)}{\ln(1.04)} \approx -23.3624\end{aligned}$$

10.
$$\begin{aligned}4\cdot 3^x &= 7\cdot 5^x\\ \frac{3^x}{5^x} = \left(\frac{3}{5}\right)^x &= \frac{7}{4}\\ x\ln\left(\frac{3}{5}\right) &= \ln\left(\frac{7}{4}\right)\\ x &= \frac{\ln(7/4)}{\ln(3/5)} \approx -1.0955\end{aligned}$$

11.
$$\begin{aligned}7 &= 5e^{0.2x}\\ \ln(7) &= \ln(5) + 0.2x\\ x &= \frac{\ln(7) - \ln(5)}{0.2} \approx 1.6824\end{aligned}$$

12.
$$\begin{aligned}2^x &= e^{x+1}\\ \text{Take ln of both sides}: x\ln(2) &= x + 1\\ x\ln(2) - x &= 1\\ x(\ln(2) - 1) &= 1\\ x &= \frac{1}{\ln(2) - 1} \approx -3.2589\end{aligned}$$

13.
$$\begin{aligned}50 &= 600e^{-0.4x}\\ \ln(50) &= \ln(600) - 0.4x\\ x &= \frac{\ln(50) - \ln(600)}{-0.4} \approx 6.212\end{aligned}$$

14.
$$\begin{aligned}2^{3x} &= 4^{5x}\\ 3x\ln(2) &= 5x\ln(4)\\ x(3\ln(2)-5\ln(4)) &= 0\\ x &= 0\end{aligned}$$

15.
$$\begin{aligned}7^{x+2} &= e^{17x}\\ (x+2)\ln(7) &= 17x\\ x\ln(7)-17x &= -2\ln(7)\\ x[\ln(7)-17] &= -2\ln(7)\\ x &= \frac{-2\ln(7)}{\ln(7)-17} \approx .2585\end{aligned}$$

16.
$$\begin{aligned}10^{x+3} &= 5e^{7-x}\\ (x+3)\ln(10) &= \ln(5) + \text{(7 - x)}\\ \ln(10)x + x &= -3\ln(10)+\ln(5) + 7\\ \text{x} &= \frac{-3\ln(10)+\ln(5) + 7}{\ln(10)+1} \approx .5153\end{aligned}$$

17.
$$\begin{aligned}2x-1 = e^{\ln(x^2)} &= x^2\\ x^2-2x+1 &= 0\\ (x-1)(x-1) &= 0\\ x &= 1\end{aligned}$$

18.
$$\begin{aligned}4e^{2x-3}-5 &= e\\ \text{Move - to RHS:}4e^{2x-3} &= e+5\\ \ln(4)+(2x-3) &= \ln(e+5)\\ x &= \frac{\ln(e+5)-\ln(4)+3}{2} \approx 1.829\end{aligned}$$

19.
$$\begin{aligned}a &= b^t\\ \ln(a) &= t\ln(b)\\ t &= \frac{\ln(a)}{\ln(b)}\end{aligned}$$

20.
$$\begin{aligned}P &= P_0 a^t\\ \ln(P) &= \ln(P_0) + \text{t } \ln(a)\\ t &= \frac{\ln(P)-\ln(P_0)}{\ln(a)}\end{aligned}$$

21.
$$\begin{aligned}Q &= Q_0 a^{nt}\\ \ln(Q) &= \ln(Q_0)+nt\ln(a)\\ a &= \frac{\ln(Q)-\ln(Q_0)}{n\ln(a)}\end{aligned}$$

22.
$$\begin{aligned}P_0 a^t &= Q_0 b^t\\ \frac{P_0}{Q_0} = \frac{b^t}{a^t} &= \left(\frac{b}{a}\right)^t\\ \ln\left(\frac{P_0}{Q_0}\right) &= t\ln\left(\frac{b}{a}\right)\\ t = \frac{\ln\left(\frac{P_0}{Q_0}\right)}{\ln\left(\frac{b}{a}\right)} &= \frac{\ln(P_0)-\ln(Q_0)}{\ln(b)-\ln(a)}\\ \text{or } &= \frac{\ln(Q_0)-\ln(P_0)}{\ln(a)-\ln(b)}\end{aligned}$$

23.
$$\begin{aligned}a &= be^t\\ \ln(a) &= \ln(b)+t\\ t &= \ln(a)-\ln(b)\end{aligned}$$

24.
$$\begin{aligned}P &= P_0 e^{kt}\\ \ln(P) &= \ln(P_0)+kt\\ t &= \frac{\ln(P)-\ln(P_0)}{k}\end{aligned}$$

41. If prices are increasing at 5% per year, prices follow the formula

$$P = P_0(1.05)^t$$

We find the doubling time by solving for t when prices reach twice the initial price, or $P = 2P_0$:

$$\begin{aligned}2P_0 &= P_0(1.05)^t\\ \ln(2) &= t\ln(1.05)\\ t &= \frac{\ln(2)}{\ln(1.05)} \approx 14.2\end{aligned}$$

Thus prices will double in roughly 14 years.

42. It takes 5 hours to double, so the model for bacteria population can be written as

$$P = P_0(2)^{t/5}$$

Solving for when P reaches 3 times the initial population, or $P = 3P_0$,

$$\begin{aligned} 3P_0 &= P_0(2)^{t/5} \\ \ln(3) &= \frac{t}{5}\ln(2) \\ t &= \frac{5\ln(3)}{\ln(2)} \approx 7.92 \end{aligned}$$

The population will triple in roughly 7.9 hours (is reasonable, relative to the 5 hours to double in population).

43. 100 kg is reduced to 40 kg in 10 years. Using the standard exponential decay model,

$$M = M_0 e^{-kt}$$

we are given $M_0 = 100$, and can use the 10 year data point to solve for k:

$$\begin{aligned} 40 &= 100e^{-10k} \\ \ln(40) &= \ln(100) - 10k \\ k &= \frac{\ln(40) - \ln(100)}{-10} \approx 0.0916 \end{aligned}$$

After 20 years,

$$\begin{aligned} M &= 100e^{-k(20)} \\ &= 16 \end{aligned}$$

Alternative: We could also have used the information given to develop the model $M = 100(0.4)^{t/10}$, based on the reduction to 40% of the original mass in 10 years.

44. Since reducing by 30% in 20 hours means that 70% is left after that time, one model for the mass of substance would be

$$M = M_0(0.7)^{t/20}$$

Find the half-life by setting $M = \frac{M_0}{2}$:

$$\begin{aligned} \frac{M_0}{2} &= M_0(0.7)^{t/20} \\ \ln(1/2) &= (t/20)\ln(0.7) \\ t &= 20\frac{\ln(1/2)}{\ln(0.7)} \approx 38.9 \text{ hours} \end{aligned}$$

TEST PREPARATION PROBLEMS

32. From the first lecture.
A is e^x, because $e^0 = 1$.
B is x^2, because $0^2 = 0$, and the parabola opens upwards.
C is $x^{1/2} = \sqrt{x}$, because the parabola opens sideways/inverse of x^2.
D is $\ln(x)$, because $\ln(1) = 0$, and it is the inverse of e^x.

34. If $y(0) = 2$, then $2 = Ce^0$ or $C = 2$.
If $y(1) = 1$, then $1 = 2e^{\alpha \cdot 1}$, so $\dfrac{1}{2} = e^{\alpha}$, so $\alpha = \ln\left(\dfrac{1}{2}\right)$.
$y(2) = 2e^{\ln(1/2)\cdot 2} = 2\left(\dfrac{1}{2}\right)^2 = \dfrac{1}{2}$

35. For $h(x) = \ln(x + a)$, increasing a shifts the graph to the **left**. (This question is best answered with sketches of the shifted graphs.)

    (a) The y intercept will move upwards as the graph shifts left. Can also be seen using the formula: at the y intercept, $x = 0$, so $y = \ln(0 + a) = \ln(a)$. Increating a increases y.
    (b) The x intercept will move left, along with the entire graph.

36. For $g(x) = \ln(ax+2)$, increasing a (assumed positive in the assignment page) will increase the horizontal compression. Note that the vertical asymptote will occur when $ax+2 = 0$, or at $x = -2/a$. Sketching may or may not help as much for this problem.

    (a) The y intercept is defined at $x = 0$, or at $y = \ln(0 + 2)$. This means that the y intercept isn't changed at all by changing a.
    (b) The x intercept will move to the right. The intercept occurs when $y = 0 = \ln(ax+2)$, or at

$$\begin{aligned} e^0 &= e^{\ln ax+2} = ax + 2 \\ ax &= -1 \\ x &= -1/a \end{aligned}$$

    As a increases, the size of x decreases, meaning it moves towards the origin, and because x will be negative, that means the origin is to the right.

37. For $f(x) = a\ln(x + 2)$, increasing a stretches the graph vertically. This will move the intercept away from the origin. Finding the exact y value for the intercept, $y = a\ln(0 + 2) = a\ln(2)$. Increasing a clearly increases y.

47. $A = 10(0.82)^5$.

    (a) The initial amount taken was $A(0) = 10(0.82)^0 = 10$ mg.
    (b) After one hour, there is $10(0.82)^1 = 8.2$ mg remaining. This 82% of the original amount, so 18% left the body.
    (c) After 6 hours, $A(6) = 10(0.82)^5 \approx 3.04$ mg will remain in the body.
    (d) Set $A = 1 = 10(0.82)^t$:

$$\begin{aligned} \frac{1}{10} &= (0.82)^t \\ \text{Take ln of both sides: } \ln\left(\frac{1}{10}\right) &= \ln(0.82)^t \\ \ln\left(\frac{1}{10}\right) &= t\ln(0.82) \\ t &= \frac{1}{\ln(0.82)}\ln\left(\frac{1}{10}\right) \approx 11.6 \text{ hours} \end{aligned}$$

    After approximately 11.6 hours, 1 mg of the drug will remain in the body.

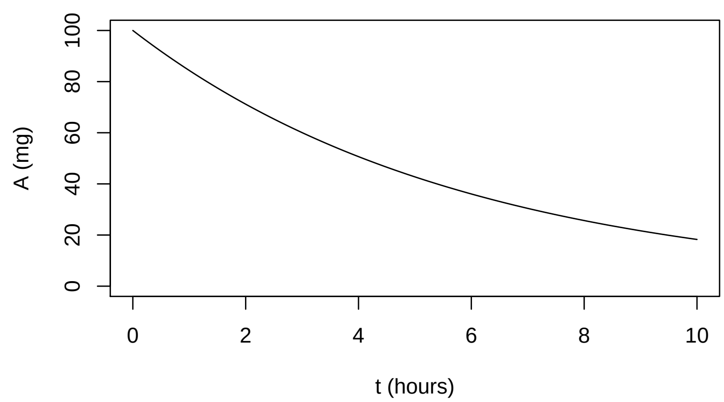
48. (a) Initial amount = 100 (mg). That amount is leaving (decaying) at a rate of 17% per hour. Just as "increasing continuously at a rate of 17%" would have the form $e^{.17t}$, "leaving at 17%" has the same structure, but the sign is negative:

$$A(t) = 100e^{(-.17t)}$$

NOTE: this question is ambiguous, in that the phrase "rate 17% per hour" can also be interpreted as "83% is left when $t = 1$". This would produce the function $A_2(t) = (1 - 0.71)^t = 0.83^t$. The two versions are similar since $A(t) = e^{(-.17)t} \simeq (0.843)^t$). Obviously, these two similar formulas produce similar, but not identical, answers to (b) and (c). For convention in this course, the phrase "continuous rate of r" will always indicate the use of e^{rt}.

(b) Use a few of points to help draw the graph of exponential decay:

$$A(0) = 100$$
$$A(1) = 100e^{-.17} \simeq 84.4$$
$$A(2) = 100e^{-.17 \cdot 2} \simeq 71.2$$



(c) If you know a half-life formula, feel free to use it. A more general way to calculate the half-life is by setting $A(t_2) = \frac{1}{2}A(0)$ and solving for t_2:

$$100e^{-.17 \cdot t_2} = \frac{1}{2}100e^0$$
$$e^{-.17 \cdot t_2} = \frac{1}{2}$$

Take ln of both sides: $$-.17 \cdot t_2 = \ln(1/2)$$
$$t_2 = \frac{\ln(1/2)}{-.17} \simeq 4.08 \text{ hours}$$

49. NOTE: this question has an interpretation difficulty similar to #48, in that "4% per year" could mean a growth function of either $e^{0.04t}$ or $(1.04)^t$. Here, because the phrase

"continuous rate" was absent, we use the more common-place interpretation: after one year, the number of cars has gone up by exactly 4%, given by the formula $(1.04)^t$.

Let t = number of years since 2000. Then the number of vehicles, V, in millions, at time t is given by

$$V = 213(1.04)^t$$

and the number of people, P, in millions, at time t is given by

$$P = 281(1.01)^t.$$

There is an average of one vehicle per person when $V = P$. Thus we must solve

$$213(1.04)^t = 281(1.01)^t$$

As with all log problems like this, you may proceed in several different ways. One way is shown below.

$$\begin{aligned} 213(1.04)^t &= 281(1.01)^t \\ \left(\frac{1.04}{1.01}\right)^t &= \frac{281}{213} \\ \text{Taking log of both sides:} \qquad t \log\left(\frac{1.04}{1.01}\right) &= \log\left(\frac{281}{213}\right) \\ t &= \frac{\log\left(\frac{281}{213}\right)}{\log\left(\frac{1.04}{1.01}\right)} \simeq 9.4656 \end{aligned}$$

To check, ensure that both populations (cars and people) are at the same level near that year.

$$\begin{aligned} V(9.4656) &= 213(1.04)^{(9.4656)} \simeq 308.75 \\ P(9.4656) &= 281(1.01)^{(9.4656)} \simeq 308.75 \end{aligned}$$

This confirms that our estimate that at $t \approx 9.4656$, or mid-year in 2009, the car and human populations would equal each other in this model.

50. (a) We know the decay follows the equation

$$P = P_0 e^{-kt}$$

and that 10% of the pollution is removed after 5 hours (meaning that 90% is left). Therefore,

$$0.90P_0 = P_0 e^{-k\cdot 5}$$
$$k = -\frac{1}{5}\ln(0.90)$$

Thus after 10 hours,

$$\begin{aligned} P &= P_0 e^{10\frac{1}{5}\ln(0.9)} \\ &= P_0 e^{2\ln(0.9)} \\ &= P_0(0.9)^2 \\ &= 0.81P_0 \end{aligned}$$

or 81% of the original amount of pollutant.
Note that we can simplify our formula for P to make it easier to use:

$$\begin{aligned} P &= P_0 e^{\frac{1}{5}\ln(0.9)t} \\ &= P_0\left(e^{\ln(0.9)}\right)^{t/5} \\ &= P_0(0.9)^{t/5} \end{aligned}$$

Using this formula, we can see that waiting 10 hours would leave the following amount of contaminant:

$$\begin{aligned} P(10) &= P_0(0.9)^{10/5} \\ &= P_0(0.9)^2 \\ &= P_0(0.81), \text{ or } 81\% \text{ of the original amount.} \end{aligned}$$

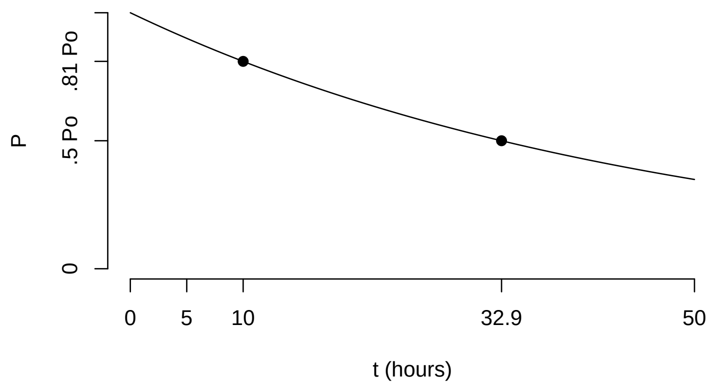
(b) Using our easier formula, we want to solve for t when

$$0.5P_0 = P_0(0.9)^{t/5}$$

Canceling P_o, taking logs: $$\log(0.5) = \frac{t}{5}\log(0.9)$$

Solving for t: $$t = 5\,\frac{\log(0.5)}{\log(0.9)} \simeq 32.9 \text{ hours}$$

(c) Graph below.



(d) When highly polluted air is filtered, there is more pollutant per liter of air to remove. As the air gets cleaner and cleaner, the filter will collect pollutant at a slower rate.

53. Carbon 14 has a half-life of 5730 years, which means the proportion/fraction left after t years is $A(t) = \left(\frac{1}{2}\right)^{t/5730}$. Some students might also use the half-life formula for the natural exponential, $k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -1.2097 \times 10^{-4}$, so $A(t) \approx e^{-1.2097\times 10^{-4}\cdot t}$, but the first formula is identical and clearly easier to use in practice.

If the painting were a Vermeer, and painted in his lifetime, it would be roughly 2000-1675 or 325 years old. After this amount of time, there would be $A(325) = \left(\frac{1}{2}\right)^{325/5730} \approx .9614$ or 96% of the original carbon-14 left in the material. The fact that the actual painting has a much higher percentage, 99.5%, of its carbon-14 left indicates that it is likely much younger than Vermeer, and so likely a fake.