Unit #4 - Interpreting Derivatives, Local Linearity, Marginal Rates

Section 3.5

TEST PREPARATION PROBLEMS

44. $y = 5 + 4.9\cos\left(\frac{\pi}{6}t\right)$

 (a) $\frac{dy}{dt} = -4.9\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$. This represents the rate at which the depth of the water is changing, in feet (units of y) per hour (units of t).

 (b) $\frac{dy}{dt} = 0$ when $\sin = 0$, or when

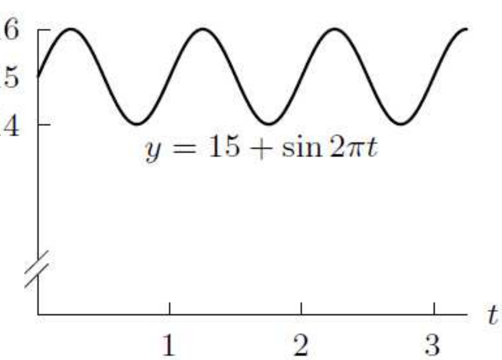
$$\frac{\pi}{6}t = 0, \pi, 2\pi, \text{ etc.}$$
$$t = \frac{6}{\pi} \times \{0, \pi, 2\pi, \ldots\}$$
$$t = 0, 6, 12, 18, 24 \text{ hours}$$

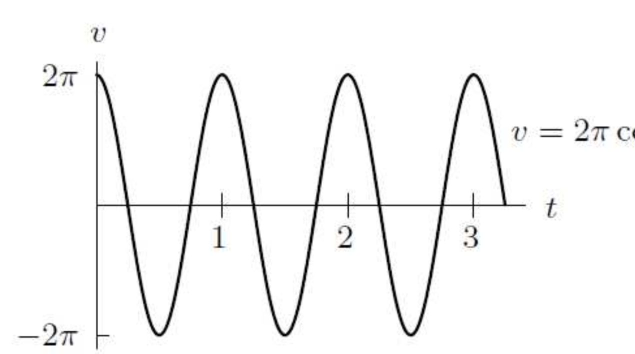
 in the period $0 \leq t \leq 24$. At these times, the depth of water is not changing, and is frequently associated with high- or low-tide. (We will study this question further in our study of optimization problems.)

45. $y = 15 + \sin(2\pi t)$

 (a) The vertical velocity is given by $v = \frac{dy}{dt} = 2\pi\cos(2\pi t)$.

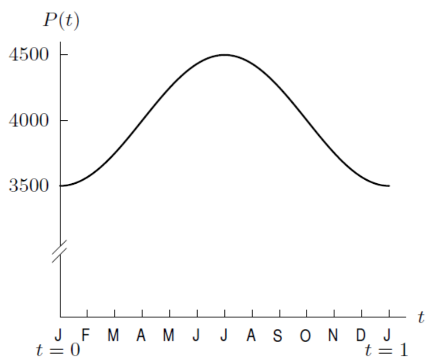
 (b)





48. $P(t) = 4000 + 500\sin\left(2\pi t - \frac{\pi}{2}\right)$

(a) The period is $\frac{2\pi}{2\pi} = 1$ year. The sine wave "starts" when $\left(2\pi t - \frac{\pi}{2}\right) = 0$ or $t = \frac{1}{4}$ of a year.
The population average is 4000, and its amplitude is 500.



(b) The population reaches its maximum at $t = 1/2$ year, and that max population is 4500. The minimum occurs at $t = 0$ years, and represents a population of 3500.

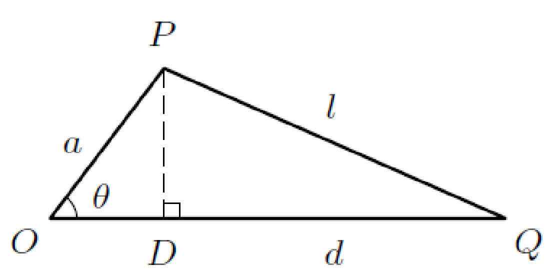
(c) The population is growing fastest at $t = 1/4$, and is decreasing fastest at $t = 3/4$ years.

(d) On July 1st, $t = 1/2$, and at that point, the population is at its peak. This means that $P'(1/2) = 0$ (instantaneous rate of change is zero). We can confirm this with the derivative,

$$\begin{aligned} P'(t) &= 500(2\pi)\cos\left(2\pi t - \frac{\pi}{2}\right) \\ \text{so } P'(1/2) &= 500(2\pi)\cos\left(2\pi\frac{1}{2} - \frac{\pi}{2}\right) \\ &= 500(2\pi)\cos\left(\frac{\pi}{2}\right) \\ &= 0 \end{aligned}$$

49. (a) From the diagram below,

$$\frac{OD}{a} = \cos(\theta) \qquad \text{so } OD = a\cos(\theta)$$
$$\frac{PD}{a} = \sin(\theta) \qquad \text{so } PD = a\sin(\theta)$$



Since we have a right-angle triangle,

$$(PD)^2 + d^2 = l^2$$
$$\text{or } a^2 \sin^2(\theta) + d^2 = l^2$$
$$\text{so } d = \sqrt{l^2 - a^2 \sin^2(\theta)}$$

Finally, we can write the relationship between x and θ:

$$\begin{aligned} x &= OD + DQ \\ &= a\cos(\theta) + \sqrt{l^2 - a^2 \sin^2(\theta)} \end{aligned}$$

(b) The derivative $\frac{dx}{dt}$ can be carefully computed using the chain rule, understanding that θ is also a function of time.

$$\begin{aligned} \frac{dx}{dt} &= -a\sin(\theta)\frac{d\theta}{dt} + \frac{1}{2}\left(l^2 - a^2\sin^2(\theta)\right)^{-1/2}\left(-a^2\left(2\sin(\theta)\cos(\theta)\frac{d\theta}{dt}\right)\right) \\ &= -a\sin(\theta)\frac{d\theta}{dt} - \frac{a^2\sin(\theta)\cos(\theta)\frac{d\theta}{dt}}{\sqrt{l^2 - a^2\sin^2(\theta)}} \end{aligned}$$

(i) Using the values given, $\frac{d\theta}{dt} = 2$ rad/s and $\theta = \pi/2$,

$$\left.\frac{dx}{dt}\right|_{\theta=\pi/2} = -a\sin(\pi/2)(2) - \frac{a^2\sin(\pi/2)\cos(\pi/2)(2)}{\sqrt{l^2 - a^2\sin^2(\theta)}}$$

but since $\cos(\pi/2) = 0$, $\left.\frac{dx}{dt}\right|_{\theta=\pi/2} = -2a$ cm/s

(ii) Using the values given, $\frac{d\theta}{dt} = 2$ rad/s and $\theta = \pi/4$,

$$\begin{aligned} \left.\frac{dx}{dt}\right|_{\theta=\pi/4} &= -a\sin(\pi/4)(2) - \frac{a^2\sin(\pi/4)\cos(\pi/4)(2)}{\sqrt{l^2 - a^2\sin^2(\pi/4)}} \\ &= -2a\frac{1}{\sqrt{2}} - \frac{a^2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)(2)}{\sqrt{l^2 - a^2\left(\frac{1}{\sqrt{2}}\right)^2}} \\ &= \frac{-2a}{\sqrt{2}} - \frac{a^2}{\sqrt{l^2 - \frac{a^2}{2}}} \text{ cm/s} \end{aligned}$$

50. $f(x) = \sin(x)$ so $f'(x) = \cos(x)$.

- At $x = 0$, the tangent line is defined by $f(0) = 0$ and $f'(0) = 1$, so

$$y = 1(x - 0) + 0 = x$$

is the tangent line to $f(x)$ at $x = \frac{\pi}{3}$.

- At $x = \dfrac{\pi}{3}$, the tangent line is defined by $f(\pi/3) = \frac{\sqrt{3}}{2}$ and $f'(\pi/3) = \frac{1}{2}$, so

$$y = \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$$

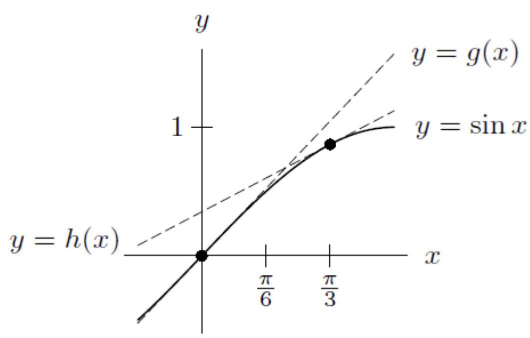
  is the tangent line to $f(x)$ at $x = 0$.

The estimates of each tangent line at the point $x = \pi/6$ would be

- Based on $x = 0$ tangent line, $f(x) \approx x$, so $f(\pi/6) \approx \pi/6 \approx 0.5236$.
- Based on $x = \pi/3$ tangent line, $f(x) \approx \dfrac{1}{2}\left(x - \dfrac{\pi}{3}\right) + \dfrac{\sqrt{3}}{2}$,
  so $f(\pi/6) \approx \dfrac{1}{2}\left(\pi/6 - \dfrac{\pi}{3}\right) + \dfrac{\sqrt{3}}{2} \approx 0.6042$
- The **actual** value of $f(x) = \sin(x)$ at $x = \pi/6$ is $\sin(\pi/6) = 0.5$.

From these calculations, the estimate obtained by using the tangent line based at $x = 0$ gives the more accurate prediction for $f(x)$ at $x = \pi/6$

A sketch might help explain these results.



In the interval $x \in [0, \pi/6]$, the function stays very close to linear (i.e. does not curve much), which means that the tangent line stays a good approximation for a relatively long time.

The function is most curved/least linear around its peak, so the linear approximation around $x = \pi/3$ is less accurate even over the same Δx.

51. Let $f(x) = \sin(x)$ and $g(x) = ke^{-x}$. They intersect when $f(x) = g(x)$, and they are tangent at that intersection if $f'(x) = g'(x)$ as well. Thus we must have

$$\sin(x) = ke^{-x} \qquad \text{and } \cos(x) = -ke^{-x}$$

We can't solve either equation on its own, but we can divide one by the other:

$$\frac{sin(x)}{\cos(x)} = \frac{ke^{-x}}{-ke^{-x}}$$
$$\tan(x) = -1$$
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$$

Since we only need one value of k, we try the first value, $x = 3\pi/4$.

$$\begin{aligned} \sin(3\pi/4) &= ke^{-3\pi/4} \\ \frac{1}{\sqrt{2}}e^{3\pi/4} &= kk \qquad \approx 7.46 \end{aligned}$$

We confirm our answer by verifying both the values and derivatives are equal at $x = 3\pi/4$,

$$\sin(3\pi/4) = 7.46e^{-3\pi/4} \approx 0.7071 \qquad \text{and} \quad \cos(3\pi/4) = -7.46e^{-3\pi/4} \approx -0.7071$$

The actual point of tangency is at $(x, y) = \left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$. A sketch is shown below.