# Unit #1 - Transformation of Functions, Exponentials and Logarithms

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Note: This unit, being review of pre-calculus has **substantially more practice problems** than later units. Don't be intimidated by the length, and just do as many problems as you need to refresh your skills.

Functions

1. What are the domain and range of $f(x)$ shown in the graph below?
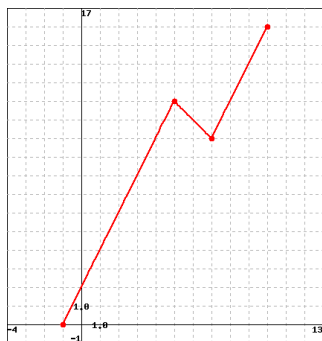
2. Which of the following graphs is the graph of a function?
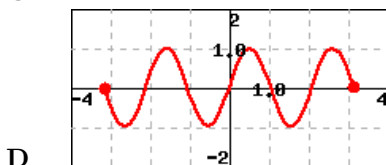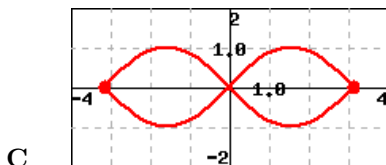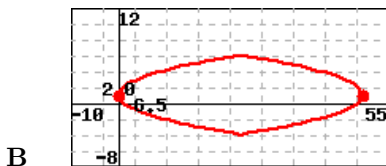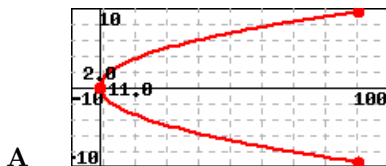
3. Consider the function $g(x) = \dfrac{(4 - 4x^2)}{(4x^2 + 3x)}$.

   (a) Find the domain of $g(x)$.
   (b) Find the values of x that satisfy $g(x) = 0$.

4. Let $f(t) = \sqrt{t^2 - 36}$.

   a) Find all values of t for which $f(t)$ is a real number.
   b) Find the value(s) of t such that $f(t) = 8$.

5. Give the domain and range of the function $y = \dfrac{1}{x^2 + 5}$.

6. Give the domain and range of the function $y = x^2 + 1$.

7. Determine the domain of the function
$$f(x) = \frac{1}{x + 1}$$

8. The monthly charge for a waste collection service is 1475 dollars for 100 kg of waste and 1895 dollars for 130 kg of waste.

   (a) Find a linear model for the cost, C, of waste collection as a function of the number of kilograms, w.

   (b) What is the slope of the line found in part (a)?

   Give units for the units for the slope.

   (c) What is the value of the vertical intercept of the line found in part (a)? Give the units of the intercept.

9. For tax purposes, you may have to report the value of your assets, such as cars or refrigerators. The value you report drops with time. "Straight-line depreciation" assumes that the value is a linear function of time. If a 1100 dollar refrigerator depreciates completely in 14 years, find a formula for its value as a function of time, x, in years.

10. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount monthly plus a charge for each cubic foot of water used. A household using 1500 cubic feet was billed $55, while one using 2100 cubic feet was billed $73.

    (a) What is the charge per cubic foot?

    (b) Write an equation for the total cost, C, of a resident's water as a function of cubic feet, x, of water used.

    (c) How many cubic feet of water used would lead to a bill of $100?

11. The value of a car, $V = f(a)$, in thousands of dollars, is a function of the age of the car, a, in years.

    (a) Interpret the statement $f(5) = 6$.

    (b) Sketch a possible graph of V against a. Is f an increasing or decreasing function?

(c) In terms of the value of the car, what does the horizontal intercept signify? What does the vertical intercept signify?

12. The force, F, between two atoms depends on the distance r separating them. See the graph below. A positive F represents a repulsive force; a negative F represents an attractive force.

(a) What happens to the force if the atoms start with $r = a$ and are:

(i) Pulled slightly further apart?

(ii) Pushed slightly closer together?

(b) The atoms are said to be in stable equilibrium if the force between them is zero and the atoms tend to return to the equilibrium after a minor disturbance. Is the distance $r = a$ a stable equilibrium for the two
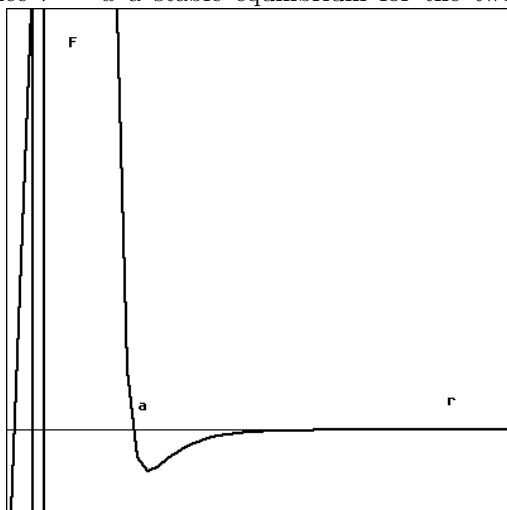
particles?

13. The volume of a sphere, V, is proportional to the cube of its radius, r. Write a formula representing V in terms of r.

14. The average velocity, v, for a trip over a fixed distance, d, is inversely proportional to the time of travel, t. Write a formula representing v in terms of t and d.

15. The strength, S, of a beam is proportional to the square of its thickness, h. Write a formula representing S in terms of h.

16. The energy, E, expended by a swimming dolphin is proportional to the cube of the speed, v, of the dolphin. Write a formula representing E in terms of v.

17. The number of animal species, N, of a certain body length, l, is inversely proportional to the square of l. Write a formula representing N in terms of l.

18. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen.

Below are the results from two experiments dropping an object, with negligible air resistance.

Table A					
Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

Table B					
Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

a) What does the experimental data in **Table A** tell you about Galileo's hypothesis?

b) What alternative hypothesis is suggested by the two sets of data in **Table A** and **Table B**?

Exponentials

19. A town has a population of 1600 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.

a) The population increases by 62 people per year.
b) The population increases by 9% per year.

20. a) Transform the function $P = P_0 e^{0.8t}$ into the form $P = P_0 a^t$.

b) Does the function represent exponential growth or exponential decay?

21. The table below shows some values of a linear function f and an exponential function g. Fill in exact values

(not decimal approximations) for each of the missing entries.

x	0	1	2	3	4
$f(x)$	15			225	
$g(x)$	15			225	

22. Match the functions $f(x)$, $g(x)$ and $h(x)$, whose values are given in the table below, with the formulas

$$y = a(1.3)^x, \qquad y = b(1.08)^x, \qquad y = c(1.02)^x,$$

assuming a, b and c are constants. Note that the func-

tion values in the table have been rounded to two decimal places.

x	$f(x)$	x	$g(x)$	x	$h(x)$
2	3.38	0	4	1	1.08
3	4.39	1	4.08	2	1.17
4	5.71	2	4.16	3	1.26
5	7.43	3	4.24	4	1.36
6	9.65	4	4.33	5	1.47

23. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time, t, goes on.

    a) Which graph best reflects the above situation?
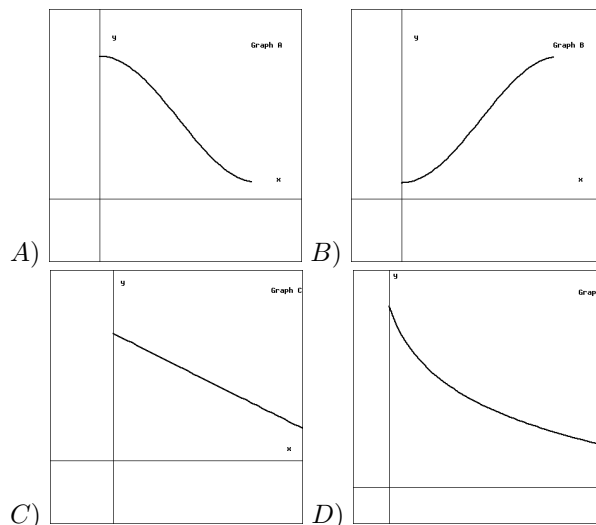

A) B)

C) D)
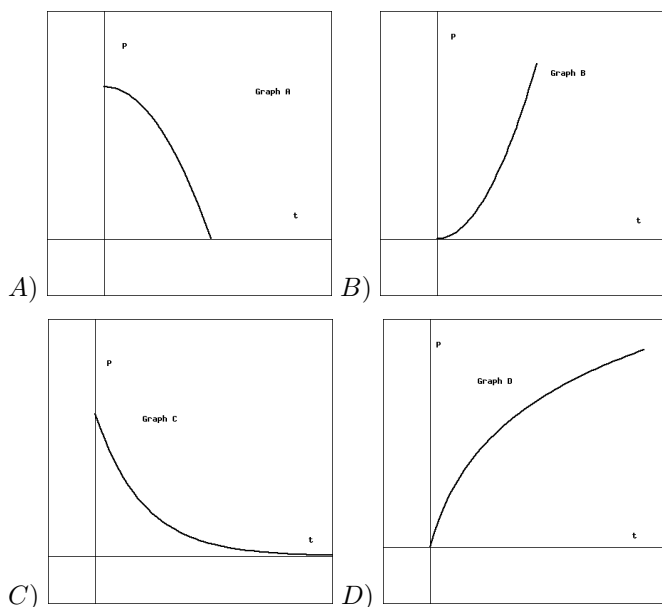
    b) What do you know about the concavity of the graph?

24. Choose the graph that best reflects the given situation:

    a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.

    b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.

A) B)

C) D)

25. When the Olympic Games were held outside Mexico City in 1968, there was much discussion about the effect the high altitude (7340 feet) would have on the athletes. Assuming air pressure decays exponentially by 0.4% every 100 feet, by what percentage is air pressure reduced by moving from sea level to Mexico City?

26. During April 2006, Zimbabwe's inflation rate averaged 0.67% a **day**. This means that, on average, prices went up by 0.67% from one day to the next.

    (a) By what percentage did prices in Zimbabwe increase in April of 2006?
    (b) Assuming the same rate all year, what was Zimbabwe's annual inflation rate during 2006?

27. (a) The half-life of radium-226 is 1620 years. Write a formula for the quantity, Q, of radium left after t years, if the initial quantity is q.

    (b) What percentage of the original amount of radium is left after 600 years?

28. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and was absorbed into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 2002?

Function Transformations

29. Are the following functions invertible?

    (a) $f(t)$ is the total accumulated rainfall in inches t minutes into a sudden rainstorm in July, 2005.

    (b) $g(w)$ is the cost of mailing a letter weighing w grams.

30. Let $f(x)$ be the temperature ($^\circ C$) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(22)$ in practical terms?
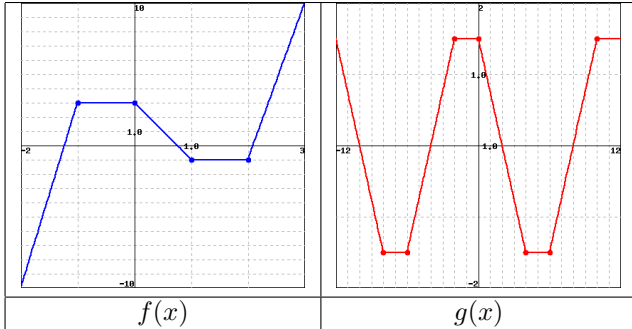
31. For the function $f(x) = 2e^x$ and $g(x) = x^6$, find the following:
    (a) $f(g(1))$
    (b) $g(f(1))$

(c) $f(g(x))$
(d) $g(f(x))$
(e) $f(t)g(t)$

32.

| $f(x)$ | $g(x)$ |

Use the figures above to find the following values. Note that you can find exact values.
(a) $f(g(3))$
(b) $g(f(2))$
(c) $g(g(9))$

33. Calculate the composite functions $f \circ g$ and $g \circ f$ given that $f(x) = \cos(x)$, $g(x) = 6x^3 + 9x^2 - 3$

34. Calculate the composite functions $f \circ g$ and $g \circ f$, given that $f(x) = 6^x$, $g(x) = x^7$
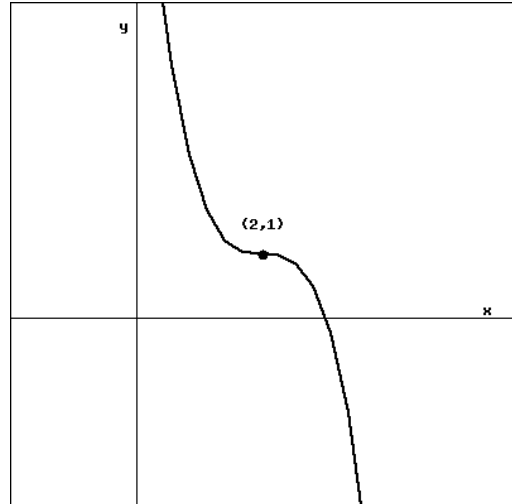
35. Relative to the graph of

$$y = x^2$$

the graphs of the following equations have been changed in what way?

(a) $y = (5x)^2$
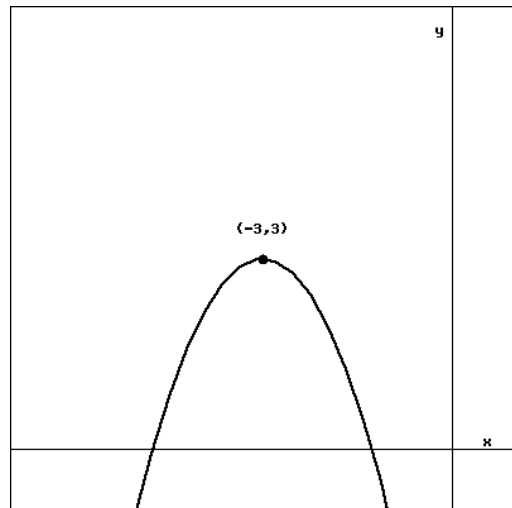(b) $y = (x + 5)^2$
(c) $y = (x - 5)^2$
(d) $y = (x/5)^2$

36. How does the graph of $Q = S(1 - e^{-kt})$ relate to the graph of the exponential decay function, $y = Se^{-kt}$?

37.

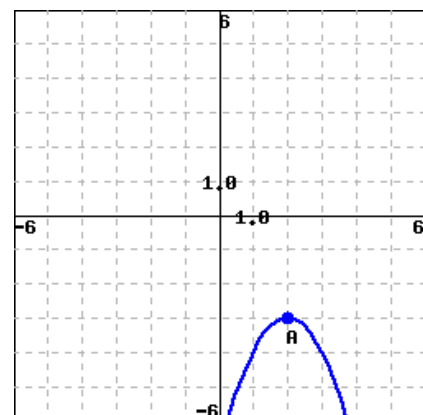Give a possible formula for the function shown in the graph using shifts of x^2 or x^3.

38.

Give a possible formula for the function shown in the graph using shifts of x^2 or x^3.

39. The graph of a function $f(x)$ is given below.
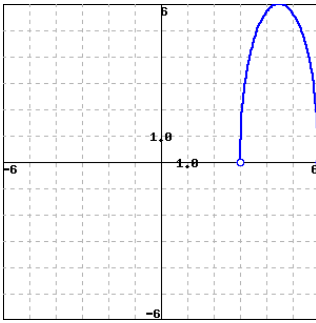

Find a possible formula for $f(x)$ whose graph is illustrated above by using a shift of either x^2 or x^3. Point A has coordinates $(2, -3)$.
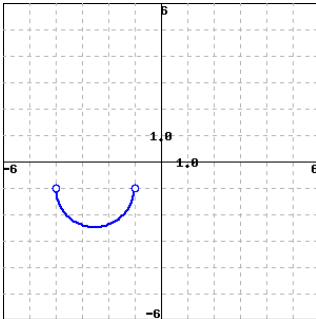
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40. The function $f(x) = \sqrt{3x - x^2}$ is given graphed below:
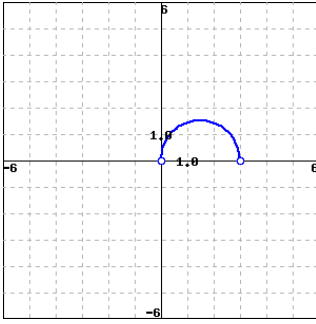
(a) Starting with the formula for $f(x)$, find a formula for $g(x)$, which is graphed below:

(b) Starting with the formula for $f(x)$, find a formula for $h(x)$, which is graphed below:

41. A spherical balloon is growing with radius $r = 3t + 4$, in centimeters, for time t in seconds. Find the volume of the balloon at 3 seconds.

42. A tree of height y meters has, on average, B branches, where $B = y - 1$. Each branch has, on average, n leaves where $n(B) = 2B^2 - B$. Find the average number of leaves of a tree as a function of height.

43. The cost of producing q articles is given by the function $C = f(q) = 125 + 3q$.

(a) Find a formula for the inverse function, $f^{-1}(C)$.
(b) In practical terms, what does the inverse function tell you?

44. A kilogram weighs about 2.2 pounds.

(a) Write a formula for the function, f, which gives an object's mass in kilograms, k, as a function of its weight in pounds, p:

(b) Find a formula for the inverse function of f:

(c) In practical terms, what does inverse function tell you?

45. Consider the graph f below:
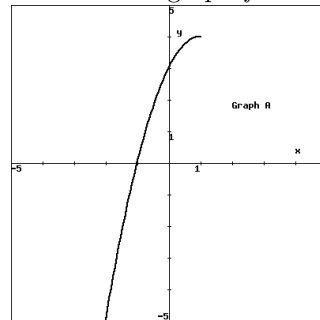
a) Which graph could be a representation of f^{-1}?

A)   B)

C)   D)
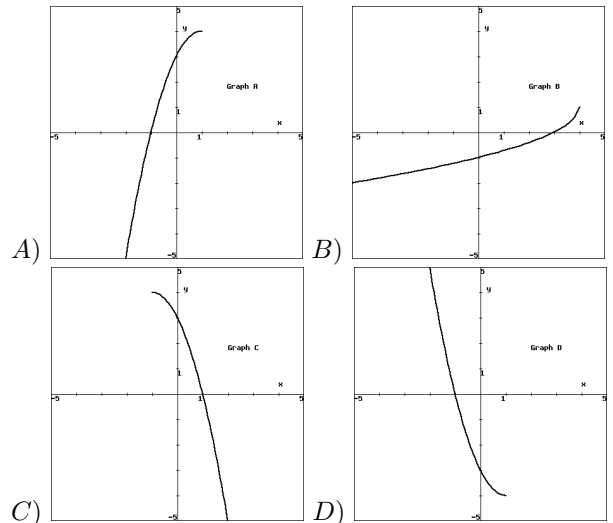
b) Using your graph choice from part a), estimate $f^{-1}(1)$.

Logarithms

46. Find the exact value for each expression:

(a) $\log_{3125} 5$.
(b) $\ln(e^{\sqrt{9}})$.

47. Solve each equation for x:

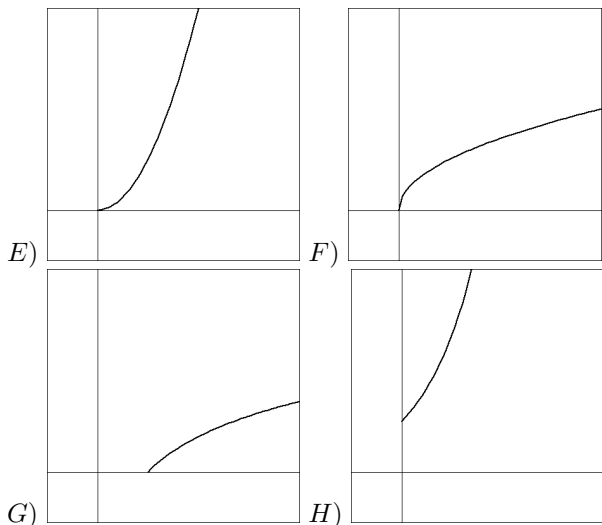(a) Solve $\ln(8x - 1) = -2$ for x.
(b) Solve $e^{(4x+3)} = 6$ for x.

48. Classify each of the following as either *True* or *False*.

(a) $(\ln a^b) = b \ln a$

(b) $\ln \frac{P}{Q} = \ln P - \ln Q$

(c) $\ln a b = b \ln a$

(d) $\log_2 PQR = \log_2 P + \log_2 Q + \log_2 R$

49. Without a calculator or computer, match the functions e^x, $\ln x$, x^2, and $x^{1/2}$ to their graphs.

(a) e^x

(b) $\ln x$

(c) x^2

(d) $x^{1/2}$

50. The exponential function $y(x) = Ce^{\alpha x}$ satisfies the conditions $y(0) = 3$ and $y(1) = 5$.

a) Find the constant C.

b) Find the constant α.

c) What is $y(2)$?

51. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on:

a) The y-intercept?

b) The x-intercept?

52. If $g(x) = \ln(ax + 2)$, where a is a positive quantity, what is the effect of increasing a on:

(a) The y-intercept?

(b) The x-intercept?

53. If $f(x) = a \ln(x + 2)$, what is the effect of increasing a on the vertical asymptote?

54. There are currently 21 frogs in a (large) pond. The frog population grows exponentially, tripling every 7 days.

(a) How long will it take (in days) for there to be 210 frogs in the pond?

(b) The pond's ecosystem can support 1900 frogs. How long until the situation becomes critical?

55. At time t hours after taking the cough suppressant hydrocodone bitartrate, the amount, A, in mg, remaining in the body is given by $A = 10(0.83)^t$.

(a) What was the initial amount taken?

(b) What percentage of the drug leaves the body each hour?

(c) How much of the drug is left in the body 8 hours after the dose is administered?

(d) How long is it until only 1 mg of the drug remains in the body?

56. A cup of coffee contains 110 mg of caffeine, which leaves the body at a **continuous** rate of 18% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.

(b) Use logarithms to find the half-life of caffeine.

57. In 2000, there were about 212 million vehicles (cars and trucks) and about 283 million people in the US. The number of vehicles has been growing at 6% a year, while the population has been growing at 1% a year.

If the growth rates remain constant, when is there, to the nearest year, one vehicle per person?

58. The air in a factory is being filtered so that the quantity of a pollutant, P (in mg/liter), is decreasing according to the function $P = P_0 e^{-kt}$, where t is time in hours. If 15% of the pollution is removed in the first 6 hours:

(a) What percentage of the pollution is left after 12 hours?

(b) How long is it before the pollution is reduced to 70% of its original level?

59. A scientist places 20 cells in a petri dish. She knows the cells grow at an exponential rate, doubling in number every hour.

(a) How long will it take (in hours) for there to be 1900 cells in the dish?

(a) How long will it take to reach 230 cells?

60. A picture supposedly painted by Vermeer (1632-1675) contains 99.5% of its carbon-14 (half-life 5730 years).

(a) What percentage of the painting's original carbon would be left today if the picture had been created in 1675?

(b) From this information, decide whether the picture is a fake.