Unit #4 - Inverse Trig, Interpreting Derivatives, Newton's Method

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Computing Inverse Trig Derivatives

1. Starting with the inverse property that $\sin(\arcsin(x)) = x$, find the derivative of $\arcsin(x)$.

   You will need to use the trig identity $\sin^2(x) + \cos^2(x) = 1$.

We begin with:

$$\sin(\arcsin(x)) = x$$

Take the x derivative of both sides:

$$\frac{d}{dx}\left[\sin(\arcsin(x))\right] = \frac{d}{dx}\left[x\right]$$

$$\cos(\arcsin(x))\left(\frac{d}{dx}\arcsin(x)\right) = 1$$

Solve for $\frac{d}{dx}\arcsin(x)$:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$$

This is true, but not helpful.

Recall: $\sin^2(\theta) + \cos^2(\theta) = 1$

so $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$

Using this formula in the arcsin derivative formula,

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 - [\sin(\arcsin(x))]^2}}$$

But $\sin(\arcsin(x)) = x$, so

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 - x^2}}$$

2. Starting with the inverse property that $\cos(\arccos(x)) = x$, find the derivative of $\arccos(x)$.

   You will need to use the trig identity $\sin^2(x) + \cos^2(x) = 1$.

We begin with:

$$\cos(\arccos(x)) = x$$

Take the x derivative of both sides:

$$\frac{d}{dx}\left[\cos(\arccos(x))\right] = \frac{d}{dx}\left[x\right]$$

$$-\sin(\arccos(x))\left(\frac{d}{dx}\arccos(x)\right) = 1$$

Solve for $\frac{d}{dx}\arccos(x)$:

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sin(\arccos(x))}$$

This is true, but not helpful.

Recall: $\sin^2(\theta) + \cos^2(\theta) = 1$

so $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$

Using this formula in the arccos derivative formula,

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1 - [\cos(\arccos(x))]^2}}$$

But $\cos(arccos(x)) = x$, so

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1 - x^2}}$$

3. Starting with the inverse property that $\tan(\arctan(x)) = x$, find the derivative of $\arctan(x)$.

   You will need to use the trig identity $\sin^2(x) + \cos^2(x) = 1$, or its related form, dividing each term by $\cos^2(x)$,

$$\tan^2(x) + 1 = \sec^2(x)$$

We begin with:

$$\tan(\arctan(x)) = x$$

Take the x derivative of both sides:

$$\frac{d}{dx}\,[\tan(\arctan(x))] = \frac{d}{dx}\,[x]$$

$$\sec^2(\arctan(x))\left(\frac{d}{dx}\,\arctan(x)\right) = 1$$

Solve for $\frac{d}{dx}\,\arctan(x)$:

$$\frac{d}{dx}\,\arctan(x) = \frac{1}{\sec^2(\arctan(x))}$$

This is true, but not helpful.

Recall: $\tan^2(\theta) + 1 = \sec^2(\theta)$

so $\sec^2(\theta) = 1 + \tan^2(\theta)$

Using this formula in the arctan derivative formula,

$$\frac{d}{dx}\,\arctan(x) = \frac{1}{1 + [\tan(\arctan(x))]^2}$$

But $\tan(\arctan(x)) = x$, so

$$\frac{d}{dx}\,\arctan(x) = \frac{1}{1+x^2}$$

Interpreting Derivatives

4. The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of 5 at two points. Find the coordinates of the points.

$y = x^3 - 9x^2 - 16x + 1$ has slopes given by $y' = 3x^2 - 18x - 16$. The original curve will have a slope of 5 at points where $y' = 5$:

$$\begin{aligned} \text{Solve for x:} \quad 5 &= 3x^2 - 18x - 16 \\ 0 &= 3x^2 - 18x - 21 \\ 0 &= x^2 - 6x - 7 \\ 0 &= (x-7)(x+1) \\ x &= 7, -1 \end{aligned}$$

The function will have slopes of 5 at $x = 7$ and $x = -1$. The coordinates of these points are (using $f(x)$), (-1, 7) and (7, -209).

5. Determine coefficients a and b such that $p(x) = x^2 + ax + b$ satisfies $p(1) = 3$ and $p'(1) = 1$.

Let $p(x) = x^2 + ax + b$ satisfy $p(1) = 3$ and $p'(1) = 1$. Since $p'(x) = 2x + a$, this implies $3 = p(1) = 1 + a + b$ and $1 = p'(1) = 2 + a$; i.e., $a = -1$ and $b = 3$.

6. A ball is thrown up in the air, and its height over time is given by

$$f(t) = -4.9t^2 + 25t + 3$$

where t is in seconds and $f(t)$ is in meters.

(a) What is the average velocity of the ball during the first two seconds? Include units in your answer.

(b) Find the instantaneous velocity of the ball at $t = 2$.

(c) Compute the acceleration of the ball at $t = 2$.

(d) What is the highest height reached by the ball?

(e) How long is the ball in the air?

(a) The **average** velocity is not found directly with derivatives, but rather by taking total distance traveled, divided by time taken.

$$
\begin{aligned}
f(0) &= 3 \text{ meters} \\
f(2) &= 33.4 \text{ meters} \\
&\text{so avg speed is} \\
&= \frac{\Delta f}{\Delta t} = \frac{f(2)-f(0)}{2-0} = \frac{(33.4-3)\text{ meters}}{2 \text{ seconds}} = 15.2 \text{ m/s}
\end{aligned}
$$

(b) The **instantaneous** velocity at $t = 2$ *is* given by the derivative.

$$
\begin{aligned}
f'(t) &= -9.8t + 25 \\
\text{at } t = 2, \quad f'(2) &= 5.4 \text{ m/s}
\end{aligned}
$$

(c) The acceleration at $t = 2$ is given by the **second** derivative of position,

$$
\begin{aligned}
f''(t) &= -9.8 \\
\text{so at } t = 2 \text{ (and all } t \text{ values)}, \quad f''(2) &= -9.8 \text{ m/s}^2
\end{aligned}
$$

(d) You can use parabola analysis to find vertex of the parabola. A more calculus-oriented method is to find when the velocity of the ball is zero: that will be at the top of its arc.

$$
\begin{aligned}
f'(t) &= -9.8t + 25 \\
\text{Setting } f' \text{ to zero}, \quad 0 &= -9.8t + 25 \\
t &\approx 2.551
\end{aligned}
$$

at the top of its arc. The height at this point is $f(2.551) \approx 34.89$ m.

(e) Knowing that the height is defined by a parabola, and given that the ball rose for $t \approx 2.551$ seconds means it will also take ≈ 2.551 seconds to return to earth, so it spends $2 \cdot (2.551) \approx 5.102$ seconds in the air. **Note:** this is not exactly correct, though, because the ball was launched ($t = 0$) from height $f(0) = 3$, but the impact occurs when $f(t) = 0$ later. It will take 5.102 second to return to height 3 m, but just a little bit longer to actually impact the ground at height 0 m.

A more accurate approach to find the impact time is to use the quadratic formula to find the two roots of $f(t)$, the larger of which will be the landing time:

$$
\begin{aligned}
f(t) = -4.9t^2 + 25t + 3 &\text{ equals zero when} \\
t &= \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(3)}}{2(-4.9)} \\
&= 5.2193 \text{ and } -0.1173
\end{aligned}
$$

The impact will be at exactly $t = 5.2193$ seconds, or just little longer than the estimate of 5.102 we found using the parabola insight.

7. The height of a sand dune (in centimeters) is represented by $f(t) = 800 - 5t^2$ cm, where t is measured in years since 1995. Find the values $f(8)$ and $f'(8)$, including units, and determine what each means in terms of the sand dune.

Since $f(t) = 800 - 5t^2$ cm, $f(8) = 800 - 5(8)^2 = 480$ cm. Since $f'(t) = -10t$ cm/yr, we have $f'(8) = -10(8) = -80$ cm/yr. In the year 2003, the sand dune was 480 cm high and it was eroding at a rate of 80 centimeters per year.

8. With a yearly inflation rate of 5%, prices are given by

$$P(t) = P_0(1.05)^t$$

where P_0 is the price in dollars when $t = 0$ and t is time in years. Suppose $P_0 = 1$. How fast (in cents per year) are prices rising when $t = 10$?

$P = P_0(1.05)^t$. If $P_0 = 1$, then $P'(t) = (1.05)^t \ln(1.05)$. At $t = 0$, $P'(0) = (1.05)^0 \ln(1.05) = \ln(1.05) \approx 0.0488$ dollars/year.

The prices are increasing at a rate of approximately 0.0488 dollars per year initially (at $t = 0$), or 4.88 cents per year.

At $t = 10$, $P'(10) = (1.05)^{10} \ln(1.05) =\approx 0.0795$ dollars/year, or 7.95 cents per year.

9. With t in years since January 1st, 1990, the population P of a small US town has been given by

$$P = 35,000(0.98)^t$$

At what rate was the population changing on January 1st, 2010, in units of people/year?

$P(t) = 35000(0.98)^t$, so $P'(t) = 35000(0.98)^t \ln(0.98)$

Since $t = 0$ is January 1st 1990, and t is in years, Jan 1st 2010 represents $t = 20$.

$P'(20) = 35000(0.98)^{20} \ln(0.98) \approx -472$ people/year. The rate is negative, indicating that people are leaving the town.

10. The value of an automobile can be approximated by the function

$$V(t) = 25(0.85)^t,$$

where t is in years from the date of purchase, and $V(t)$ is its value, in thousands of dollars.

(a) Evaluate and interpret $V(4)$.

(b) Find an expression for $V'(t)$.

(c) Evaluate and interpret $V'(4)$.

(a) $V(4) \approx 13.05$. 4 years after purchase, the car will be worth approximately 13 thousand dollars.

(b) $V'(t) = 25(0.85)^t(\ln(0.85)) \approx -4.06(0.85)^t$.

(c) $V'(4) = -2.12$ means that, 4 years after purchase, the car will be losing value at a rate of roughly 2 thousand dollars per year.

11. The quantity, q of a certain skateboard sold depends on the selling price, p, in dollars, with $q = f(p)$. For this particular model of skateboard, $f(140) = 15,000$ and $f'(140) = -100$.

    (a) What do $f(140) = 15,000$ and $f'(140) = -100$ tell you about the sales of this skateboard model?

    (b) The total revenue, R, earned by the sale of this skateboard is given by

    $$R = (\text{price per unit})(\text{quantity sold}) = p \cdot q$$

    Find the **rate of change** of revenue, $\frac{dR}{dp}$ when $p = 140$. This is sometimes written as

    $$\left.\frac{dR}{dp}\right|_{p=140}$$

    (c) From the sign of $\left.\frac{dR}{dp}\right|_{p=140}$, decide whether the company would actually increase revenues by increasing the price of this skateboard from \$140 to \$141.

(a) $f(140) = 15,000$ says that at a selling price of \$140, 15,000 skateboards will be sold.
$f'(140) = -100$ indicates that if the price is increased from \$140, for each increase of \$1 in selling price, sales will drop by roughly 100 skateboards (drop b/c of the negative derivative value).

(b) $R = pq$, so $\frac{dR}{dp} = \frac{dp}{dp} \cdot q + p \cdot \frac{dq}{dp}$ by the product rule.
But $\frac{dp}{dp} = 1$, and $\frac{dq}{dp} = f'(p)$, since f gives the quantity sold q as a function of p.
Therefore, at the selling price of \$140,

$$\begin{aligned}\left.\frac{dR}{dp}\right|_{p=140} &= \frac{dp}{dp} \cdot q + p \cdot \frac{dq}{dp} \\ &= 1 \cdot f(140) + (140) \cdot f'(140) \\ &= 15000 + 140(-100) \\ &= 1000\end{aligned}$$

(c) The sign of $\left.\frac{dR}{dp}\right|_{p=140}$ is positive.
This indicates that if p is increased by a small amount (e.g. \$140 to \$141), revenues will increase. This happens because, even though unit sales decrease, the loss of sales is offset by the price increase, so the net revenue effect is positive.

12. The theory of relativity predicts that an object whose mass is m_0 when it is at rest will appear heavier when moving at speeds near the speed of light. When the object is moving at speed v, its mass m is given by

    $$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}, \text{ where } c \text{ is the speed of light}$$

    (a) Find $\frac{dm}{dv}$.

    (b) In terms of physics, what does $\frac{dm}{dv}$ tell you?

$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$, and m_0 and c are constants. We can rewrite this using powers as $m = m_0 \left[1 - \frac{1}{c^2}v^2\right]^{-1/2}$

(a)

$$\begin{aligned}\frac{dm}{dv} &= m_0 \frac{-1}{2}\left[1 - \frac{1}{c^2}v^2\right]^{-3/2}\left(\frac{-1}{c^2}(2v)\right) \\ &= \frac{m_0}{c^2}v\left[1 - \frac{1}{c^2}v^2\right]^{-3/2}\end{aligned}$$

(b) The derivative $\frac{dm}{dv}$ represents how the (relativistic) mass of an object changes as its velocity changes. Note that for small v ($v \ll c$ or "*v much smaller than c*"), this rate is almost zero, since relativistic effects are only pronounced when our speed approaches the speed of light.

13. A museum has decided to sell one of its paintings and to invest the proceeds. If the picture is sold between the years 2000 and 2020 and the money from the sale is invested in a bank account earning 5% interest per year, compounded annually, then the balance in the year 2020 depends on the time t when the painting is sold. Let $P(t)$ be the price the painting will sell for if sold in year t, and let $B(t)$ be the balance from the sale (after interest) in 2020, again depending on the year of sale, t.

 If $t = 0$ in the year 2000, so $0 \leq t \leq 20$, then

$$B(t) = P(t)(1.05)^{20-t}$$

 (a) Explain why $B(t)$ is given by this formula.

 (b) Show that the formula for $B(t)$ is equivalent to

$$B(t) = (1.05)^{20}\frac{P(t)}{(1.05)^t}$$

 (c) Find $B'(10)$, given that $P(10) = 150,000$ and $P'(10) = 5000$.

 (d) Decide whether, if the museum still has the painting in 2010 (at $t = 10$), it should continue to hold on to the painting, or sell it immediately if its goal is to maximize its financial future.

(a) The form for $B(t)$ is an exponential growth given 5% interest. The only odd part is the exponent, $20 - t$. Note that the value that goes in the exponent is the *length of time* for the deposit, **not** always the variable t. (Usually the two are the same because we start counting the deposit time from $t = 0$).

If the painting is sold at year t, then in the 20-year period there will be (20-t) years left for the sale price to gain interest in the bank. The sale price is $P(t)$, and it becomes the amount of the initial deposit.

(b)

$$\begin{aligned}\text{If } B(t) &= P(t)(1.05)^{20-t}\\ \text{then } B(t) &= P(t)(1.05)^{20}(1.05)^{-t}\\ &= (1.05)^{20}\frac{P(t)}{(1.05)^t}\end{aligned}$$

(c)

$$B' = (1.05)^{20}\frac{P'(t)(1.05)^t - P(t)\cdot(1.05)^t\ln(1.05)}{((1.05)^t)^2}$$

If $P(10) = 150,000$ and $P'(10) = 5000$,

$$\begin{aligned}B'(10) &= (1.05)^{20}\frac{5000(1.05)^{10} - 150000\cdot(1.05)^{10}\ln(1.0}{((1.05)^{10})^2}\\ &\approx -3776.63\end{aligned}$$

(d) Since $B'(10)$ is negative, if the museum has waited until year 10 to sell the painting, then they should sell immediately, because their final profit is decreasing at a rate of \$3776 per year. This is because the value of the painting is only growing at $5000/15000 = 0.03$ or roughly 3% per year, while they could be making 5% interest if they sold it and invested the proceeds.

14. Let $f(v)$ be the gas consumption (in liters/km) of a car going at velocity v (in km/hr). In other words, $f(v)$ tells you how many liters of gas the car uses to go one kilometer, if it is travelling at velocity v. You are told that

$$f(80) = 0.05 \text{ and } f'(80) = 0.0005$$

 (a) Let $g(v)$ be the distance the same car goes on one liter of gas at velocity v. What is the relationship between $f(v)$ and $g(v)$? Find $g(80)$ and $g'(80)$.

 (b) Let $h(v)$ be the car's gas consumption in liters per hour. In other words, $h(v)$ tells you how many liters of gas the car uses in one hour if it is going at velocity v. What is the relationship between $h(v)$ and $f(v)$? Find $h(80)$ and $h'(80)$.

 (c) How would you explain the practical meaning of the values of these function and their derivatives to a driver who knows no calculus?

$f(v)$ gives the consumption in l/km at speed v in km/hr.
$f'(80) = 0.05$ l/km and $f'(80) = 0.0005$ l/km for each km/hr increase above 80 km/hr.

(a) $g(v)$ is the distance the car goes in one km at speed v. By considering the units, $g(v)$ (km/l) $= \dfrac{1}{f(v)\text{ l/km}}$. So $g(v) = [f(v)]^{-1}$.

 $g(80) = 1/f(80) = 1/0.05 = 20$ km/l.

 Finding g' requires the chain rule:

$$
\begin{aligned}
g' &= (-1)(f(v))^{-2}f'(v)\\
\text{so } g'(80) &= (-1)(f(80))^{-2}f'(80)\\
&= (-1)\frac{1}{0.05^2}0.0005\\
&\approx -0.2 \text{ km/l per km/h speed above 80}
\end{aligned}
$$

(b) If $h(v)$ is the consumption in l/hr, then we need a combination of the speed and consumption rate:

$$
\begin{aligned}
h(v) \text{ l/hr} &= [f(v) \text{ (l/km)}] \times [v \text{ (km/hr)}]\\
h(v) &= f(v)\cdot v\\
\text{so } h(80) &= f(80)\cdot 80\\
&= 0.05\cdot 80 = 4 \text{ l/hr}
\end{aligned}
$$

Careful with the derivative of h: our variable is v, so $\frac{d}{dv}f = f'$, and $\frac{d}{dv}v = 1$.

$$
\begin{aligned}
h'(v) &= \left(\frac{d}{dv}f(v)\right)\cdot v + f(v)\cdot\left(\frac{dv}{dv}\right)\\
&= f'(v)v + f(v)\\
\text{so } h'(80) &= f'(80)\cdot 80 + f(80)\\
&= .0005\cdot 80 + .05\\
&= 0.09 \text{ l/hr for each km/h above 80 km/h}
\end{aligned}
$$

(c) Part (a) tells us that the car can go 20 km on 1 liter. Since the first derivative is negative, it means going faster *reduces* the distance the car can travel on each liter.

Part (b) tells us that at 80 km/hr, the car uses 4 l/hr. Since the derivative is positive, going at higher speeds will use up more gas per hour.

15. If $P(x)$ is a polynomial, and it can be factored so that $P(x) = (x-a)^2Q(x)$, where $Q(x)$ is also a polynomial, we call $x = a$ a *double zero* of the polynomial $P(x)$.

  (a) If $x = a$ is a double zero (i.e. $P(x)$ can be written as $(x-a)^2Q(x)$), show that both $P(a) = 0$ and $P'(a) = 0$.

  (b) Show that if $P(x)$ is a polynomial, and both $P(a) = 0$ and $P'(a) = 0$, then we can factor $P(x)$ into the form $(x-a)^2Q(x)$.

(a) Clearly $P(a) = 0$: $P(a) = (a-a)^2Q(a) = 0$
To find the derivative, we use the product rule & chain rule,

$$
\begin{aligned}
&P'(x) = 2(x-a)Q(x) + (x-a)^2Q'(x)\\
\text{so at } x = a,\quad &P'(a) = 2(a-a)Q(a) + (a-a)^2Q'(a)\\
\text{so}\quad &P'(a) = 0
\end{aligned}
$$

We can be sure of this because Q is a polynomial, so $Q'(a)$ has a nice finite value (the only way $P'(a)$ could *not* equal zero would be if $Q'(a)$ did not exist).

(b) If both $P(a)$ and $P'(a)$ equal zero, and $P(x)$ is a polynomial, then $(x-a)$ must be a factor in both P and P' (this is how you find & check factors by hand).

Since $x = a$ is a root of $P(x)$, we can write $P(x) = (x-a)R(x)$, where $R(x)$ is another polynomial. From this,

$$
\begin{aligned}
&P'(x) = 1\cdot R(x) + (x-a)R'(x)\\
\text{but } P'(a) = 0, \text{ so}\quad &P'(a) = R(a) + (a-a)R'(a) = 0\\
&\text{so } R(a) = 0
\end{aligned}
$$

This means $R(a)$ also has an $(x-a)$ factor in it: $R(x) = (x-a)T(x)$, where $T(x)$ is yet another polynomial. Substituting this form back into P, we get

$$P(x) = (x-a)R(x) = (x-a)[(x-a)T(x)] = (x-a)^2T(x)$$

so $P(x)$ must have a double root at $x = a$ if both $P(a) = P'(a) = 0$.

16. If $g(x)$ is a polynomial, it is said to have a *zero of multiplicity* m at $x = a$ if it can be factored so that

$$g(x) = (x-a)^m h(x)$$

where $h(x)$ is a polynomial such that $h(a) \neq 0$.

Explain why having a polynomial having a zero of multiplicity m at $x = a$ must then satisfy $g(a) = 0, g'(a) = 0, \ldots,$ and $g^{(m)}(a) = 0$. (Note: $g^{(m)}$ indicates the m-th derivative of $g(x)$.)

If $g(x) = (x-a)^m h(x)$, then repeated derivatives will give

$$
\begin{aligned}
g'(x) &= m(x-a)^{m-1}h(x) + (x-a)^m h'(x)\\
g''(x) &= m(m-1)(x-a)^{m-2}h(x) + \ldots \text{[terms with high}\\
&\vdots\\
g^{(m-1)\dagger}(x) &= m(m-1)\ldots 2(x-a)^1 h(x) + \ldots \text{[terms with hig}
\end{aligned}
$$

Note that, since every term has an $(x-a)$ factor, when we evaluate the function or its derivatives at $x = a$, we get zero.

17. (a) Find the *eighth* derivative of $f(x) = x^7 + 5x^5 - 4x^3 + 6x - 7$. Look for patterns as you go...

  (b) Find the *seventh* derivative of $f(x)$.

(a) Since each derivative removes one power from a polynomial, the eighth derivative of any seventh-degree polynomial will be zero.

(b) Only the derivative of the x^7 term will survive seven derivatives. Looking at the pattern,

$$
\begin{aligned}
f'(x) &= 7x^6 + \dots \\
f''(x) &= 7 \cdot 6x^5 + \dots \\
&\vdots \\
f^{(7)} &= 7 \cdot 6 \cdot 5 \cdot \dots \cdot 1x^0 = 7! \text{ (7 factorial) } = 5040
\end{aligned}
$$

18. (a) Use the formula for the area of a circle of radius r, $A = \pi r^2$, to find $\dfrac{dA}{dr}$.
    (b) The result from part (a) should look familiar. What does $\dfrac{dA}{dr}$ represent geometrically?
    (c) Use the difference quotient definition of the derivative to explain the observation you made in part (b).

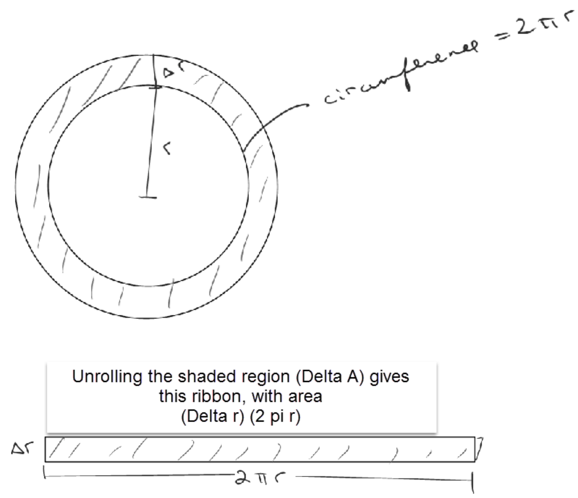
(a) If $A = \pi r^2$, then $\dfrac{dA}{dr} = 2\pi r$.

(b) This rate looks like the formula for the circumference of a circle, $2\pi r$.

(c) Imagine a circle with radius r. Then we enlarge it by just the tiniest bit, increasing the radius by a very small Δr. The derivative relationship tells us that the resulting change in area of the circle will be approximated by

$$
\begin{aligned}
\frac{dA}{dr} &\approx \frac{\Delta A}{\Delta r} = 2\pi r \\
\text{so } \Delta A &\approx (2\pi r)\Delta r
\end{aligned}
$$

We can see that this product of the circumference and the change in radius provides a good approximation to the total change in area of the circle with a diagram:



Linear Approximations and Tangent Lines

19. Find the equation of the tangent line to the graph of f at (1,1), where f is given by $f(x) = 2x^3 - 2x^2 + 1$.

$f(x) = 2x^3 - 2x^2 + 1$.
In general, the slopes of the function are given by $f'(x) = 6x^2 - 4x$
At the point $(1, 1)$ (which you should check is actually on the graph of $f(x)$!), the slope is
$f'(1) = 6 - 4 = 2$
Using the point/slope formula for a line (or the tangent line formula), a line tangent to the graph of $f(x)$ at the point $(1, 1)$ is

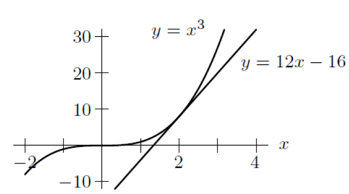
$$
\begin{aligned}
y &= f'(1)(x-1) + f(1) \\
&= 2(x-1) + 1 \\
\text{or } y &= 2x - 1
\end{aligned}
$$

20. (a) Find the equation of the tangent line to $f(x) = x^3$ at $x = 2$.
    (b) Sketch the curve and the tangent line on the same axes, and decide whether using the tangent line to approximate $f(x) = x^3$ would produce *over*- or *under*-estimates of $f(x)$ near $x = 2$.

(a) $f(x) = x^3$, so $f'(x) = 3x^2$.
At $x = 2$, $f(2) = 8$ and $f'(2) = 12$, so the tangent line to $f(x)$ at $x = 2$ is

$$y = 12(x-2) + 8$$

(b)



From the graph of $y = x^3$, it is clear that the tangent line at $x = 2$ will lie *below* the actual curve. This means that using the tangent line to estimate $f(x)$ values will produce *underestimates* of $f(x)$.

21. Find the equation of the line tangent to the graph of f at $(3, 57)$, where f is given by $f(x) = 4x^3 - 7x^2 + 12$.

Differentiating gives $f'(x) = 12x^2 - 14x$, so $f'(3) = 66$. Thus the equation of the tangent line is $y = y_0 + f'(x_0)(x - x_0) = 57 + 66(x - 3)$.

22. Given a power function of the form $f(x) = ax^n$, with $f'(3) = 16$ and $f'(6) = 128$, find n and a.

Since $f(x) = ax^n$, $f'(x) = anx^{n-1}$. We know that $f'(3) = (an)3^{n-1} = 16$, and $f'(6) = (an)6^{n-1} = 128$. Therefore,

$$\frac{f'(6)}{f'(3)} = \frac{128}{16} = 8.$$

But

$$\frac{f'(6)}{f'(3)} = \frac{(an)6^{n-1}}{(an)3^{n-1}} = 2^{n-1},$$

so $2^{n-1} = 8$, and so $n = 4$.

Substituting $n = 4$ into the expression for $f'(3)$, we get $4a3^3 = 16$, so $a = \frac{4}{27}$.

23. Find the equation of the line tangent to the graph of f at $(2, 1)$, where f is given by $f(x) = 2x^3 - 5x^2 + 5$.

Differentiating gives $f'(x) = 6x^2 - 10x$, so $f'(2) = 4$. Thus the equation of the tangent line is $y - 1 = 4(x - 2)$, or $y = 1 + 4(x - 2)$.

24. Find all values of x where the tangent lines to $y = x^8$ and $y = x^9$ are parallel.

Let $f(x) = x^8$ and let $g(x) = x^9$. The two graphs have parallel tangent lines at all x where $f'(x) = g'(x)$.

$$f'(x) = g'(x)$$

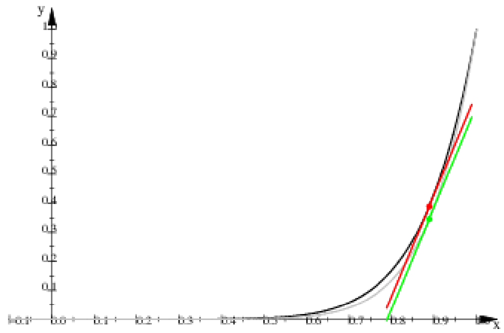
$$8x^7 = 9x^8$$

$$8x^7 - 9x^8 = 0$$

$$x^7(8 - 9x) = 0$$

hence, $x = 0$ or $x = \frac{8}{9}$.

The point at $x = 0$ is easy to visualize (both graphs are flat there). Here is a graph showing the parallel tangents at $x = 8/9$.



25. Consider the function $f(x) = 9 - e^x$.

(a) Find the slope of the graph of $f(x)$ at the point where the graph crosses the x-axis.

(b) Find the equation of the tangent line to the curve at this point.

(c) Find the equation of the line perpendicular to the tangent line at this point. (This is the *normal* line.)

(a) $f(x) = 9 - e^x$ crosses the x-axis where $0 = 9 - e^x$, which happens when $e^x = 9$, so $x = \ln 9$. Since $f'(x) = -e^x$, $f'(\ln 9) = -9$.

(b) $y = -9(x - \ln(9))$.

(c) The slope of the normal line is the negative reciprocal of the slope of the tangent, so $y = \frac{1}{9}(x - \ln(9))$.

26. Consider the function $y = 2^x$.

(a) Find the tangent line based at $x = 1$, and find where the tangent line will intersect the x axis.

(b) Find the point on the graph $x = a$ where the tangent line will pass through the origin.

(a) We find the linearization using $f(x) = 2^x$, so $f'(x) = 2^x \ln(2)$ (non-e exponential derivative rule).

At the point $x = 1$, $f(1) = 2^1 = 2$ and $f'(1) = 2^1 \ln(2)$, so the linear approximation to $f(x)$ is $L(x) = 2 + (2\ln(2))(x - 1)$.

Solving for where this line intersects the x axis (or the $y = 0$ line), we find the x intercept is approximately -0.4427.

(b) This question is more general. Instead of asking for a linearization at a specific point, it is asking "at what point would the linearization pass through the origin?" Let us give the point a name: $x = a$ (as opposed to $x = 1$ used in part (a)).

From the function and the derivatives, the linearization at the point $x = a$ is given by:

$$L_a(x) = \underbrace{2^a}_{f(a)} + \underbrace{2^a \ln(2)}_{f'(a)}(x - a)$$

That is true in general, but we want the point $x = a$ where the linearization will go through $(0, 0)$, i.e. for which $L_a(0) = 0$:

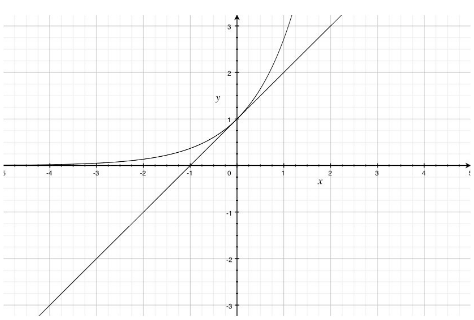
$$
\begin{aligned}
0 &= 2^a + 2^a \ln(2)(0 - a) \\
\text{Solving for } a, \quad 0 &= 2^a(1 - a\ln(2)) \\
0 &= 1 - a\ln(2) \\
a\ln(2) &= 1 \\
a &= \frac{1}{\ln(2)} \approx 1.442
\end{aligned}
$$

At that x point, the graph of $y = 2^x$'s tangent line will pass exactly through the origin.

27. (a) Find the tangent line approximation to $f(x) = e^x$ at $x = 0$.
 (b) Use a sketch of $f(x)$ and the tangent line to determine whether the tangent line produces over- or under-estimates of $f(x)$.
 (c) Use your answer from part (b) to decide whether the statement $e^x \geq 1 + x$ is always true or not.

(a) $f(x) = e^x$, so $f'(x) = e^x$ as well.
To build the tangent line at $x = 0$, we use $a = 0$ as our reference point: $f(0) = e^0 = 1$, and $f'(0) = e^0 = 1$. The tangent line is therefore $l(x) = 1(x - 0) + 1 = x + 1$

(b)



Since the exponential graph is concave up, it curves upwards away from the graph. This means that the linear approximation will always be an underestimate of the original function.

(c) Since the linear function will always underestimate the value of e^x, we can conclude that

$$1 + x \leq e^x$$

, and they will be equal only at the tangent point, $x = 0$.

28. The speed of sound in dry air is

$$f(T) = 331.3\sqrt{1 + \frac{T}{273.15}}\text{m/s}$$

where T is the temperature in degrees Celsius. Find a linear function that approximates the speed of sound for temperatures near 0^o C.

$$
\begin{aligned}
f(T) &= 331.3\sqrt{1 + \frac{T}{273.15}} \\
f'(T) &= 331.3\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{1 + \frac{T}{273.15}}}\right)\frac{1}{273.15}
\end{aligned}
$$

so at $T = 0^o$ C, $\quad f(0) = 331.3 \quad f'(0) = \dfrac{331.3}{(2)(273.15)} \approx 0.606$

Thus the speed of sound for air temperatures around 0^o C is

$f(T) \approx 0.606(T - 0) + 331.3$, or $f(t) \approx 0.606T + 331.3$ m/s

29. Find the equations of the tangent lines to the graph of $y = \sin(x)$ at $x = 0$, and at $x = \pi/3$.
 (a) Use each tangent line to approximate $\sin(\pi/6)$.
 (b) Would you expect these results to be equally accurate, given that they are taken at equal distances on either side of $\pi/6$? If there is a difference in accuracy, can you explain it?

(a) At $x = 0$, the tangent line is defined by $f(0) = 0$ and $f'(0) = 1$, so

$$y = 1(x - 0) + 0 = x$$

is the tangent line to $f(x)$ at $x = \dfrac{\pi}{3}$.
At $x = \dfrac{\pi}{3}$, the tangent line is defined by $(\pi/3) = \frac{\sqrt{3}}{2}$ and $f'(\pi/3) = \frac{1}{2}$, so

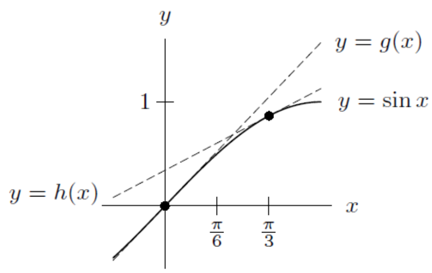
$$y = \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$$

is the tangent line to $f(x)$ at $x = 0$.
The estimates of each tangent line at the point $x = \pi/6$ would be

- Based on $x = 0$ tangent line, $f(x) \approx x$, so $f(\pi/6) \approx \pi/6 \approx 0.5236$.
- Based on $x = \pi/3$ tangent line, $f(x) \approx \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$,
 so $f(\pi/6) \approx \frac{1}{2}\left(\pi/6 - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2} \approx 0.6042$
- The **actual** value of $f(x) = \sin(x)$ at $x = \pi/6$ is $\sin(\pi/6) = 0.5$.

(b) From these calculations, the estimate obtained by using the tangent line based at $x = 0$ gives the more accurate prediction for $f(x)$ at $x = \pi/6$

A sketch might help explain these results.



In the interval $x \in [0, \pi/6]$, the function stays very close to linear (i.e. does not curve much), which means that the tangent line stays a good approximation for a relatively long time.

The function is most curved/least linear around its peak, so the linear approximation around $x = \pi/3$ is less accurate even over the same Δx.

30. Find the **quadratic** polynomial $g(x) = ax^2 + bx + c$ which best fits the function $f(x) = e^x$ at $x = 0$, in the sense that

$$g(0) = f(0), g'(0) = f'(0), \text{ and } g''(0) = f''(0)$$

$$\begin{aligned} f(x) &= e^x \\ \text{so } f'(x) &= e^x \\ \text{and } f''(x) &= e^x \\ \text{Evaluating at } x &= 0, \\ f(0) = f'(0) &= f''(0) = 1 \end{aligned}$$

Comparing with the derivatives of the quadratic,

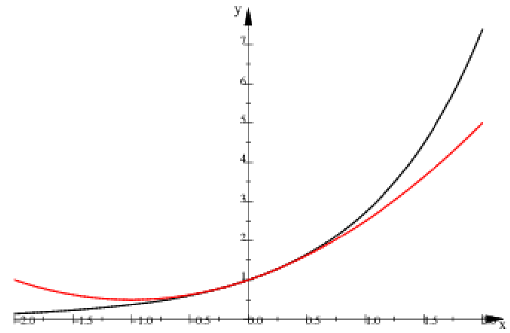
$$\begin{aligned} g(x) &= ax^2 + bx + c \\ \text{so } g'(x) &= 2ax + b \\ \text{and } g''(x) &= 2a \\ \text{Evaluating at } x &= 0, \\ g(0) = c, g'(0) &= b, \text{ and } g''(0) = 2a \end{aligned}$$

For $g(x)$ to fit the shape of $f(x)$ near $x = 0$, we would then pick $c = 1$, $b = 1$ and $2a = 1$ or $a = 0.5$, so

$$g(x) = 0.5x^2 + x + 1$$

would be the best fit quadratic to $f(x) = e^x$ near $x = 0$.

Here is a graph of the two functions, $f(x) = e^x$ in black, $g(x) = 0.5x^2 + x + 1$ in red. Notice how similar they look near their intersection.



31. Consider the graphs of $y = \sin(x)$ (regular sine graph), and $y = ke^{-x}$ (exponential decay, but scaled vertically by k).

If $k \geq 1$, the two graphs will intersect. What is the smallest value of k for which two graphs will be *tangent* at that intersection point?

Let $f(x) = \sin(x)$ and $g(x) = ke^{-x}$. They intersect when $f(x) = g(x)$, and they are tangent at that intersection if $f'(x) = g'(x)$ as well. Thus we must have

$$\sin(x) = ke^{-x} \qquad \text{and } \cos(x) = -ke^{-x}$$

We can't solve either equation on its own, but we can divide one by the other:

$$\begin{aligned} \frac{\sin(x)}{\cos(x)} &= \frac{ke^{-x}}{-ke^{-x}} \\ \tan(x) &= -1 \\ x &= \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots \end{aligned}$$

Since we only need one value of k, we try the first value, $x = 3\pi/4$.

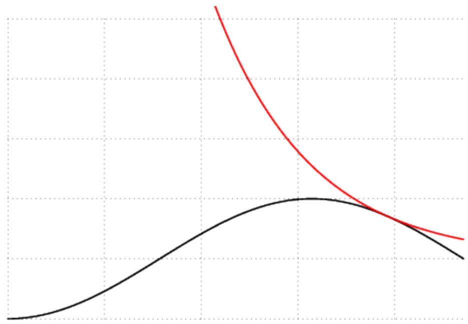
$$\begin{aligned} \sin(3\pi/4) &= ke^{-3\pi/4} \\ \frac{1}{\sqrt{2}} e^{3\pi/4} &= k \\ k &\approx 7.46 \end{aligned}$$

We confirm our answer by verifying both the values and derivatives are equal at $x = 3\pi/4$,

$$\begin{aligned} \sin(3\pi/4) &= 7.46e^{-3\pi/4} \approx 0.7071 \text{ (same } y\text{: intersection)} \\ \text{and } \cos(3\pi/4) &= -7.46e^{-3\pi/4} \approx -0.7071 \text{ (same derivative)} \end{aligned}$$

The actual point of tangency is at $(x, y) = \left(\dfrac{3\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$.

A sketch is shown below.



32. (a) Show that $1 + kx$ is the local linearization of $(1 + x)^k$ near $x = 0$.
    (b) Someone claims that the square root of 1.1 is about 1.05. Without using a calculator, is this estimate about right, and how can you decide using part (a)?

(a)

$$\begin{aligned} & f(x) = (1+x)^k \quad & f'(x) = k(1+x)^{k-1} \\ \text{so at } x = 0, \quad & f(0) = 1^k = 1 \quad & f'(0) = k(1^{k-1}) = k \end{aligned}$$

so the tangent line at $x = 0$ will be

$$y = k(x - 0) + 1 \text{ or } y = 1 + kx$$

(b) As an estimate for the square root of 1.1, we could note that $\sqrt{1.1} = (1 + 0.1)^{1/2}$. This matches exactly the form of $f(0.1)$ if we choose $k = \frac{1}{2}$. From our linearization above,

$$f(0.1) \approx 1 + \frac{1}{2}(0.1) = 1.05$$

so yes, a good approximation for $\sqrt{1.1}$ is 1.05. (Calculator gives the value of ≈ 1.0488.

33. (a) Find the local linearlization of e^x near $x = 0$.
    (b) Square your answer to part (a) to find an approximation to e^{2x}.
    (c) Compare your answer in part (b) to the actual linearization to e^{2x} near $x = 0$, and discuss which is more accurate.

(a) e^x has a tangent line/local linearization near $x = 0$ of $y = x + 1$ (slope 1, point (0, 1)).

(b) Multiplying this approximation by itself, we get $(e^x)(e^x)$ or $e^{2x} \approx (x + 1)(x + 1) = x^2 + 2x + 1$

(c) To compare with the actual linearization of $g(x) = e^{2x}$, we find its derivative and value at $x = 0$,

$$\begin{aligned} g(x) &= e^{2x} \qquad & g(0) = 1 \\ g'(x) &= 2e^{2x} \qquad & g'(0) = 2 \end{aligned}$$

so a linearization of $g(x) = e^{2x}$ near $x = 0$ is $y = 2(x - 0) + 1$ or

$$y = 2x + 1$$

Note that his is the same as the approximation we obtained before, except that our product version had an additional term, x^2.

These approximations give the same straight-line estimate of the function, but I would expect the first (multiplication) version to be more accurate because it contains more information (the squared term that the pure linear approximation was missing).

We will see more of this idea in Taylor polynomials and Taylor series.

34. (a) Show that $1 - x$ is the local linearization of $\dfrac{1}{1+x}$ near $x = 0$.
    (b) From your answer to part (a), show that near $x = 0$,
    $$\frac{1}{1+x^2} \approx 1 - x^2.$$
    (c) Without differentiating, what do you think the derivative of $\dfrac{1}{1+x^2}$ is at $x = 0$?

(a) Let $f(x) = 1/(1 + x)$. Then $f'(x) = \frac{-1}{(1+x)^2}$.
At $x = 0$, $f(0) = 1$ and $f'(0) = -1$.
So near $x = 0$, $f(x) \approx -1x + 1 = -x + 1$

(b) For small x values (i.e. x near zero), we can approximate $1/(1+x)$ with $1-x$. Replace the variable x with y (because the name doesn't matter),

$$1/(1 + y) \approx 1 - y$$

If we choose y small but equal to x^2, then

$$\frac{1}{1+x^2} \approx 1 - x^2$$

(c) The linearization of $1/(1+x^2)$ is the linear part of $1 - x^2$, or just 1. Since the derivative at $x = 0$ is the coefficient for x in the linear part, this means $\dfrac{d}{dx}\dfrac{1}{1+x^2}$ at $x = 0$ must equal zero.

35. (a) Find the local linearization of

$$f(x) = \frac{1}{1+2x}$$

near $x = 0$.

(b) Using your answer to (a), what quadratic function would you expect to approximate $g(x) = \dfrac{1}{1+2x^2}$?

(c) Using your answer to (b), what would you expect the derivative of $\frac{1}{1+2x^2}$ at $x = 0$ to be even without doing any differentiation?

(a) We know $f(0) = 1$ and $f'(0) = -\frac{2}{(1+2(0))^2} = -2$, so the local linearization is $f(x) \approx 1 - 2x$.

(b) Next, $g(x) = f(x^2)$, so we expect that $g(x) \approx 1 - 2x^2$.

(c) Noting that the approximation we found in (b) is downward opening parabola with its vertex at $x = 0$ (i.e. its local maximum or a critical point), we expect that the derivative of $g(x)$ at $x = 0$ will be zero.

Newton's Method

36. Consider the equation $e^x + x = 2$. This equation has a solution near $x = 0$. By replacing the left side of the quation by its linearization near $x = 0$, find an approximate value for the solution.

    (In other words, perform one step of Newton's method, starting at $x = 0$.)

Our equation is

$$\underbrace{e^x + x}_{f(x)} = 2$$

To find the linearization of $f(x)$ near $x = 0$, we need f and its derivative f', both evaluated at $x = 0$.

$$\begin{aligned} f(x) &= e^x + x & \text{so } f(0) &= e^0 + 0 = 1 \\ f'(x) &= e^x + 1 & \text{so } f'(0) &= e^0 + 1 = 2 \end{aligned}$$

The linearization is then

$$e^x + x \approx \underbrace{(2)}_{f'(0)}(x-0) + \underbrace{1}_{f(0)} = 2x + 1$$

We now replace the original (unsolvable) equation

$$\begin{aligned} & e^x + x = 2 \\ \text{with the simpler approximation:} \quad & 2x + 1 = 2 \end{aligned}$$

Solving this second version is straightforward, yielding $x = 0.5$. This is actually a fair approximation to the solution, since $e^{0.5} + 0.5 \approx 2.149$ which is close to the RHS value of 2.

(If we continued our linearizations and their approximations, we would get values even closer to the real solution, which is (to 4 decimal places) 0.4429.)

37. Use Newton's Method with the equation $x^2 = 2$ and initial value $x_0 = 3$ to calculate x_1, x_2, x_3 (the next three solution estimates generated by Newton's method).

Moving everything to the left side, we get

$$\underbrace{x^2 - 2}_{f(x)} = 0$$

Differentiating, we have $f'(x) = 2x$. Therefore:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 2}{2x_0} = 3 - \frac{3^2 - 2}{2 * 3} = 1.83333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.83333 - \frac{1.83333^2 - 2}{2 \times 1.83333} \approx 1.46212$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.46212 - \frac{1.46212^2 - 2}{2 \times 1.46212} \approx 1.415$$

This sequence provides successive approximations to the exactly solution,which would equal
$\sqrt{2} \approx 1.4142$

38. Use Newton's Method with the function $x^3 = 5$ and initial value $x_0 = 1.5$ to calculate x_1, x_2, x_3 (the next three solution estimates generated by Newton's method).

Moving everything to the left side, we get

$$\underbrace{x^3 - 5}_{f(x)} = 0$$

Differentiating, we have $f'(x) = 3x^2$. Therefore:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - 5}{3x_0^2} = 1.5 - \frac{1.5^3 - 5}{3 * 1.5^2} = 1.74074$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.74074 - \frac{1.74074^3 - 5}{3 \times 1.74074^2} \approx 1.71052$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.71052 - \frac{1.71052^3 - 5}{3 \times 1.71052^2} \approx 1.70998$$

This sequence provides successive approximations to the third root of 5, $\sqrt[3]{5} \approx 1.70997$

39. Use Newton's Method to approximate $4^{\frac{1}{3}}$ and compare with the value obtained from a calculator.

    (Hint: write out a simple equation that $4^{\frac{1}{3}}$ would satisfy, and use Newton's method to solve that.)

We need to find an approximation to $x = 4^{\frac{1}{3}}$ using Newton's Method. Since we are assuming we don't have a cube-root function, we need to translate this into an equivalent form. Cubing both sides we get

$$x^3 = 4$$

To use Newton's method, we have to move everying to one side to get the form $f(x) = 0$:

$$\underbrace{x^3 - 4}_{f(x)} = 0$$

With $f(x) = x^3 - 4$, we then get $f'(x) = 3x^2$.
A good initial guess could be $x_0 = 8^{1/3} = 2$. Thus:

$$x_1 = x_0 - \tfrac{f(x_0)}{f'(x_0)} = 2 - \tfrac{2^3 - 4}{3 \times 2^2} = 1.66667$$

$$x_2 = x_1 - \tfrac{f(x_1)}{f'(x_1)} = 1.66667 - \tfrac{1.66667^3 - 4}{3 \times 1.66667^2} \approx 1.59111$$

$$x_3 = x_2 - \tfrac{f(x_2)}{f'(x_2)} = 1.59111 - \tfrac{1.59111^3 - 4}{3 \times 1.59111^2} \approx 1.58741$$

$$x_4 = x_3 - \tfrac{f(x_3)}{f'(x_3)} = 1.58741 - \tfrac{1.58741^3 - 4}{3 \times 1.58741^2} \approx 1.5874$$

Using a calculator to find $4^{\frac{1}{3}}$, we get: $4^{\frac{1}{3}} \approx 1.58740$.