Unit #7 : Optimization, Optimal Marginal Rates

Goals:

- Formalize the first derivative test and the second derivative test for identifying local maxima and minima.
- Distinguish global vs. local extrema.
- Practice optimization word problems.
- Study generalized optimization problems related to marginal rates.

Identifying Types of Critical Points

Earlier we found critical points by looking for points on the graph where

- $f'(x) = 0$, or
- $f'(x)$ was undefined (but $f(x)$ *was* defined).

We will now formalize two ways to determine if a critical point is a local min, max, or neither. This avoids the need for a sketch of the entire graph.

First Derivative Test

One way to decide whether at a critical point there is a local maximum or minimum is to examine the sign of the derivative on opposite sides of the critical point. This method is called the **first derivative test**.

Complete this table:

	Sketch	f' sign left of c	f' sign right of c
local minimum at c			
local maximum at c			
neither local max nor min			

Example: *Find the critical points of the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$. *Use the first derivative test to show whether each critical point is a local maximum or a local minimum.*

Using your answer to the preceding question, determine the number of real solutions of the equation $2x^3 - 9x^2 + 12x + 3 = 0$.

Second Derivative Test

You may also use the Second Derivative Test to determine if a critical point is a local minimum or maximum.

- The first derivative test uses the **first** derivative **around** the critical point.
- The second derivative test uses the **second** derivative **at** the critical point.
- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$ then the test is inconclusive.

Example: *A function f has derivative $f'(x) = \cos(x^2) + 2x - 1$. Use the second derivative test to determine whether it has a local maximum, a local minimum, or neither at its critical point $x = 0$.*

Global vs. Local Optimization

The first and second derivative tests only give us *local* information in most cases. However, if there are multiple local maxima or minima, we usually want the **global** max or min. The ease of determining when we have found the global max or min of a function depends strongly on the properties of the question.

Local vs Global Extrema

A **local max** occurs at $x = c$ when $f(c) > f(x)$ for x values **around** c.

A **global max** occurs at $x = c$ if $f(c) \geq f(x)$ for **all** values of x in the domain.

It is possible to have several global maxima if the function reaches its peak value at more than one point.

Corresponding definitions apply for local and global minima.

Example: *Give an example of a simple function with multiple global maxima.*

Example: *Give an example of a simple function with a single global maximum, but no global minimum.*

Example: *Give an example of a simple function with neither a global maximum nor a global minimum.*

Example: *Earlier we worked with the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$. *If we limit the function to the interval* $x \in [0, 2.5]$, *what are the* **global max** *and* **global minimum** *values on that interval?*

Global Extrema on Closed Intervals

A continuous function on a closed interval will **always** have a global max and a global min value. These values will occur at either

- a critical point *or*
- an end point of the interval.

To find which value is the global extrema, you can compute the original function's values at all the critical points and end points, and select the point with the highest/lowest value of the function.

Global Extrema on Open Intervals

A function defined on an open interval may or may not have global maxima or minima.

If you are trying to demonstrate that a point is a global max or min, and you are working with an open interval, including the possible interval $(-\infty, \infty)$, proving that a particular point is a global max or min requires a careful argument. A recommendation is to look at either:

- values of f when x approaches the endpoints of the interval, or $\pm\infty$, as appropriate; or

- if there is only one critical point, look at the sign of f' on either side of the critical point.

With that information, you can often construct an argument about a particular point being a global max or min.

Example: *Consider the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$.
Does $f(x)$ *have a global max?*

(a) Yes, $f(x)$ has a global maximum somewhere in its domain.

(b) No, $f(x)$ does not have a global maximum.

For the function $f(x) = 2x^3 - 9x^2 + 12x + 3$, *does* $f(x)$ *have a glocal minimum?*

(a) Yes, $f(x)$ has a global minimum somewhere in its domain.

(b) No, $f(x)$ does not have a global minimum.

Example: *Does the function* $g(x) = (x-2)^4$ *have a global maximum?*

(a) Yes, $g(x)$ has a global maximum somewhere in its domain.

(b) No, $g(x)$ does not have a global maximum.

Example: *Does the function* $g(x) = (x-2)^4$ *have a global minimum?*

(a) Yes, $g(x)$ has a global minimum somewhere in its domain.

(b) No, $g(x)$ does not have a global minimum.

Optimization

An optimization problem is one in which we have to find the maximum or minimum value of some quantity. In principle, we already know how to find the maximum and minimum values of a function if we are given a formula for the function and the interval on which the maximum or minimum is sought. Usually the hard part in an optimization problem is interpreting the word problem in order to find the formula of the function to be optimized.

Example: *A farmer wants to build a rectangular enclosure to contain livestock. The farmer has 120 meters of wire fencing with which to build a fence, and one side of the enclosure will be part of the side an already existing building (so there is no need to put up fence on that side). What should the dimensions of each side be to maximize the area of the enclosure?*

What is the quantity to be maximized in this example?

(a) Length of wire.

(b) Area enclosed.

(c) Side lengths beside barn.

What are the variables in this question, and how are they related? You may want to draw a picture.

Express the quantity to be optimized in terms of the variables above. Try to eliminate all but one of the variables.

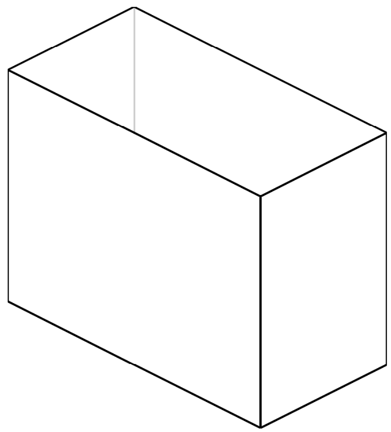
What is the domain on which the one remaining variable makes sense?

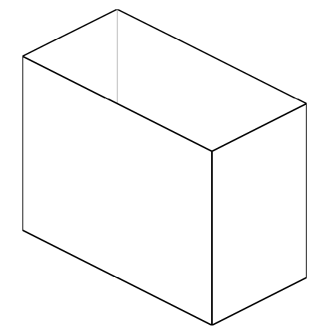
Use the techniques learned earlier in the course to maximized the function on this domain. Give reasons explaining why the answer you found is the **global** *maximum.*

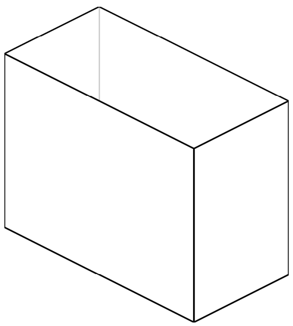
(Fencing example continued)

Example: *(Storage Container)*

A rectangular storage container with an open top is to have a volume of 10 m^3*. The length of its base is to be twice its width. Material for the base costs \$10.00 per m*^{2}*, and material for the sides costs \$6.00 per m*^{2}*. Determine the cost of the material for the cheapest such container.*

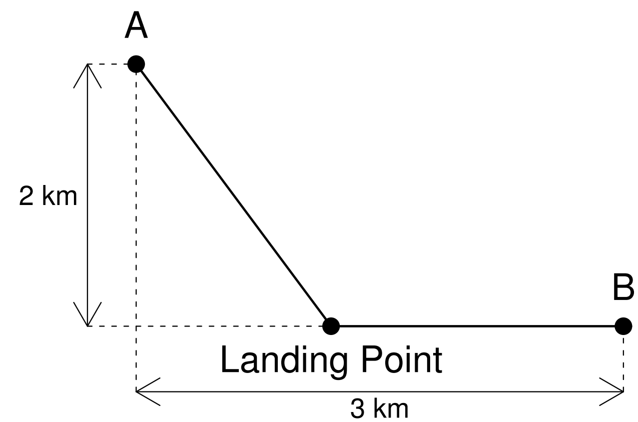




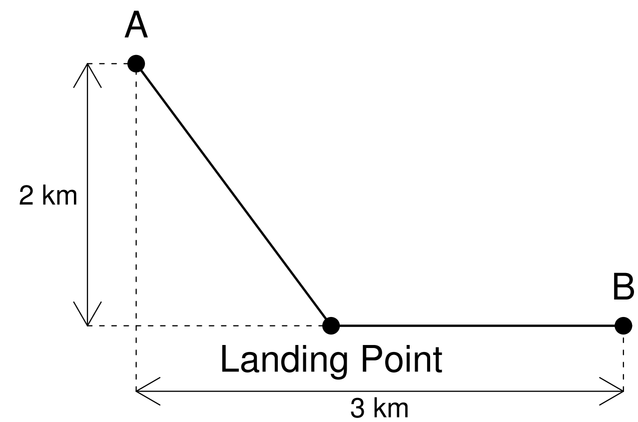


Example (Taken from 2004 Dec Exam)

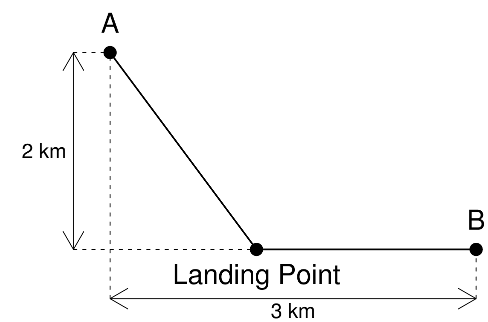
A fisher is in a boat at point A, which is 2 km from the nearest point on the shoreline. He is to go to a lighthouse at point B, which is 3 km down the coast (see figure below).

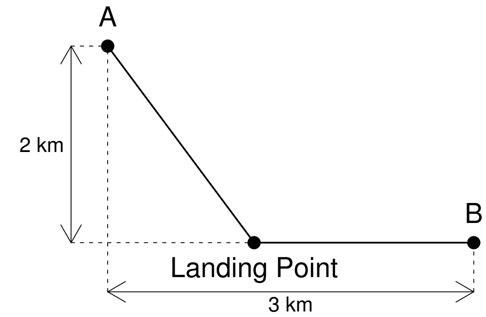


If the fisher can row at 4 km per hour, and walk at 5 km per hour, find an expression for $T(x)$, the travel time if the fisher lands the boat x km down the shore from the nearest approach.



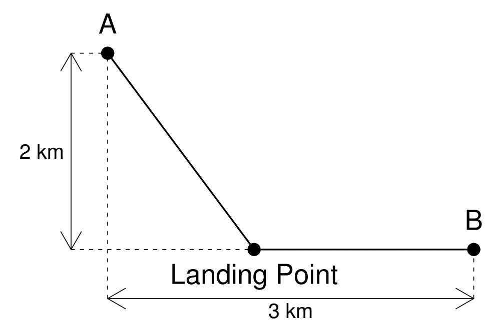
If the fisher can row at 4 km per hour, and walk at 5 km per hour, how far from the point B should he land the boat to minimize the time it takes to get to the lighthouse? Make sure to indicate how you know your answer is the global minimum.

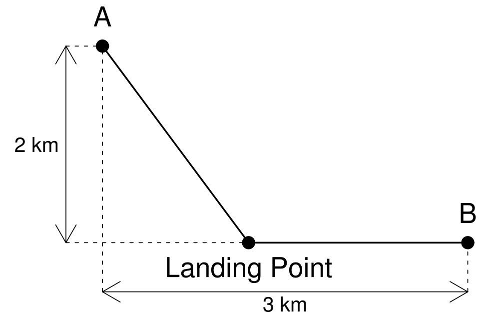



A
2 km
B
Landing Point
3 km

Often, the numerical values in an optimization problem are somewhat arbitrary, or estimated using best guesses. It is often more important to discover the response in the solution to a *range* of possible problem values. In that vein, we now suppose the fisher has a motor that will drive his boat at a speed of v km per hour.

If the fisher's walking speed is still 5 km per hour, for what values of v will it be fastest to simply drive the boat directly to the lighthouse (i.e. do no walking)?

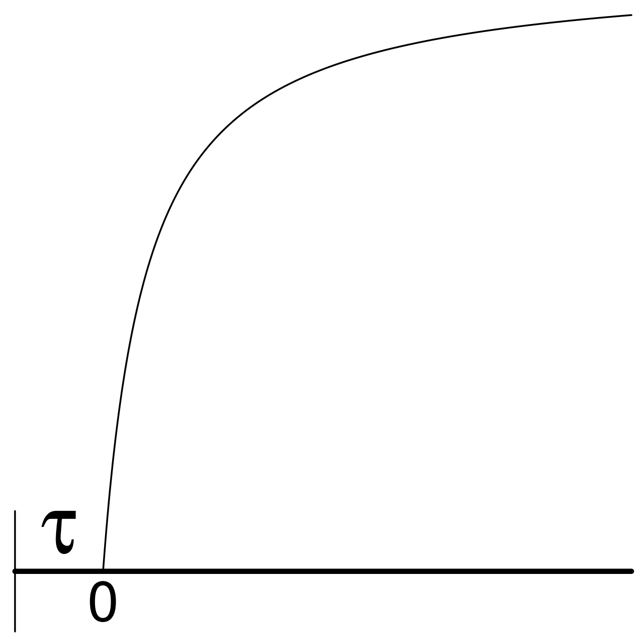



A
2 km
B
Landing Point
3 km

Optimization Without Formulas

In the preceding examples, we used derivative calculations on explicit formulas to find critical points. Unfortunately, such formulas aren't always available. Sometimes, the function we are given isn't the one we need to optimize directly. The following example showcases an important class of problems from both biology and economics.

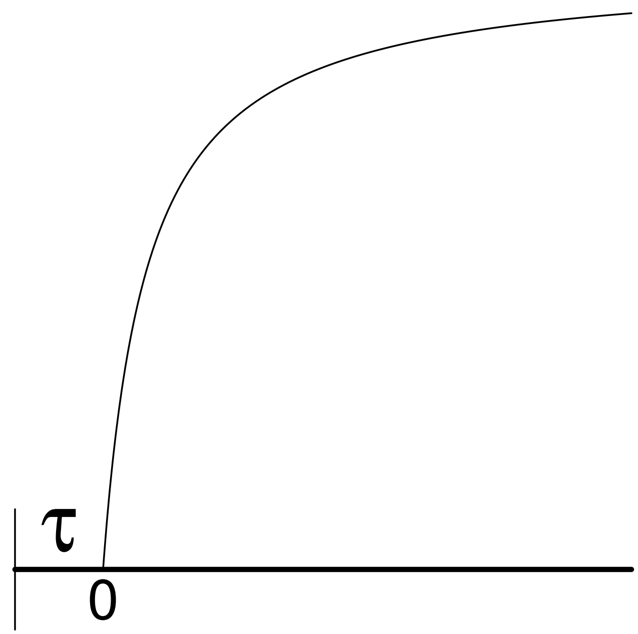
Example: *A chipmunk stocks its nest with acorns for the winter. The chipmunk wants to be as efficient as possible while foraging. There is a relationship between the number of acorns in the chipmunk's mouth and the time it takes to pack another into its cheeks. Clearly, when the cheeks are already very full, it becomes harder to add another acorn without dropping one already collected. So, we can imagine that the graph of "load size" versus time looks like:*



This graph shows $F(t)$, the amount of food collected in collection time t. Also shown is the **travel time**, τ, before the chipmunk reaches the collection site. (Total trip time is $t + \tau$.)

How would we compute the **average collection rate** *based on collecting nuts for time* t*? (call this average rate* $R(t)$*).*

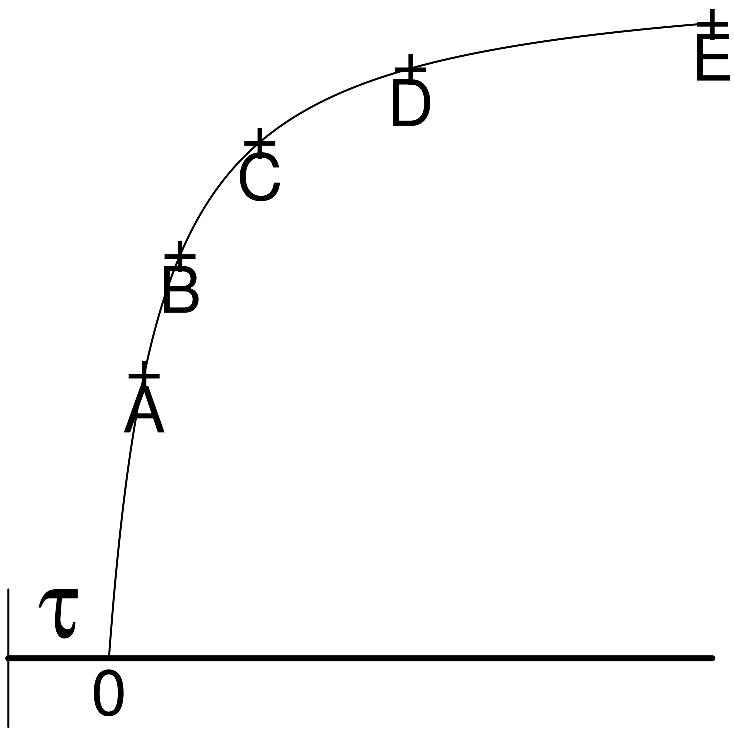
An important consequence of this formula is that the value of the function $R(t)$ can be seen as a **slope** *in the $F(t)$ graph. Sketch it on the graph.*



While there is some merit in studying this example using particular numbers, an important insight can be arrived at by studying the problem using an arbitrary function $F(t)$.

Without knowing the feeding function, $F(t)$, find the derivative of $R(t)$, and find its maximum value.

Question: Based on those calculations, which of the following points maximizes the value of $R(t)$?



This result, that the optimal time to stop harvesting occurs when the

current instantaneous rate of harvest equals the
current average rate of harvest,
is known as the
Marginal Value Theorem.

It applies just as well to mining, oil wells, and crop harvesting as it does to biology.