**SUGGESTED PROBLEMS**

In Exercises 1-15, use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint, if such values exist.

1. \( f(x, y) = x + y, \ x^2 + y^2 = 1 \)

   We use the constraint to build the constraint function, \( g(x, y) = x^2 + y^2 \). We then take all the derivatives, which will be needed for the Lagrange multiplier equations.

\[
\begin{align*}
  f_x &= 1 & g_x &= 2x \\
  f_y &= 1 & g_y &= 2y
\end{align*}
\]

Set up the Lagrange multiplier equations:

\[
\begin{align*}
  f_x &= \lambda g_x &\Rightarrow& & 1 &= \lambda 2x \quad (1) \\
  f_y &= \lambda g_y &\Rightarrow& & 1 &= \lambda 2y \quad (2) \\
  \text{constraint: } &\Rightarrow & & x^2 + y^2 &= 1 \quad (3)
\end{align*}
\]

Taking (1) / (2), (assuming \( \lambda \neq 0 \))

\[
\frac{1}{\lambda 2x} = \frac{x}{y} \\
\text{so } y = x
\]

Sub into (3) to find

\[
2x^2 = 1 \quad \Rightarrow \quad x = \pm \sqrt{1/2}
\]

Combining with \( y = x \), we get the solutions \((x, y) = (\sqrt{1/2}, \sqrt{1/2}) \) and \((-\sqrt{1/2}, -\sqrt{1/2})\). Since our constraint is closed and bounded, we can simply compare the value of \( f \) at these two points to determine the maximum and minimum values of \( f \) subject to the constraint.

\[
\begin{align*}
  f(\sqrt{1/2}, \sqrt{1/2}) &= 2\sqrt{1/2} \\
  f(-\sqrt{1/2}, -\sqrt{1/2}) &= -2\sqrt{1/2}
\end{align*}
\]

From this, the maximum of \( f \) on \( x^2 + y^2 = 1 \) is at \((\sqrt{1/2}, \sqrt{1/2})\) and the minimum is at \((-\sqrt{1/2}, -\sqrt{1/2})\)
3. \( f(x, y) = xy, \ 4x^2 + y^2 = 8 \)

\[
\begin{align*}
  f_x &= y \\
  f_y &= x \\
  g_x &= 8x \\
  g_y &= 2y
\end{align*}
\]

Set up the Lagrange multiplier equations:

\[
\begin{align*}
  f_x &= \lambda g_x \quad \Rightarrow \quad y = \lambda 8x \\
  f_y &= \lambda g_y \quad \Rightarrow \quad x = \lambda 2y \\
  \text{constraint:} & \quad \Rightarrow \quad 4x^2 + y^2 = 8
\end{align*}
\]

Taking (4) / (5), (assuming \( \lambda \neq 0 \))

\[
\frac{y}{x} = \frac{\lambda 2x}{\lambda 2y} = \frac{8x}{2y}
\]

so \( y^2 = 4x^2 \) or \( y = \pm 2x \)

Sub into (6) to find

\[
4x^2 + 4x^2 = 8 \quad \Rightarrow \quad x = \pm 1
\]

Combining with \( y = \pm 2x \), we get the solutions \((x, y) = (1, 2), (1, -2), (-1, 2) \) and \((-1, -2)\).

Since our constraint is closed and bounded, we can simply compare the value of \( f \) at these four points to determine the maximum and minimum values of \( f \) subject to the constraint.

\[
\begin{align*}
f(1, 2) &= 2 \\
f(1, -2) &= -2 \\
f(-1, 2) &= -2 \\
f(-1, -2) &= 2
\end{align*}
\]

From this, the maximum of \( f \) on \( x^2 + y^2 = 1 \) is at two points, \((1, 2)\) and \((-1, -2)\). The minimum of \( f \) occurs at \((1, -2)\) and \((-1, 2)\).

6. \( f(x, y) = x^2 + y, \ x^2 - y^2 = 1 \)

\[
\begin{align*}
  f_x &= 2x \\
  f_y &= 1 \\
  g_x &= 2x \\
  g_y &= -2y
\end{align*}
\]
Set up the Lagrange multiplier equations:

\[ f_x = \lambda g_x \Rightarrow 2x = \lambda(2x) \quad (7) \]
\[ f_y = \lambda g_y \Rightarrow 1 = \lambda(-2y) \quad (8) \]

constraint: \[ x^2 - y^2 = 1 \quad (9) \]

From (7), we must have \( \lambda = 1 \) or \( x = 0 \)

- If \( \lambda = 1 \), then (8) gives \( 1 = (1)(-2y) \), or \( y = -\frac{1}{2} \), and from (9) \( x^2 - (\frac{-1}{2})^2 = 1 \), so \( x = \pm \sqrt{1 + \frac{1}{4}} = \pm \sqrt{\frac{5}{4}} \)
- If \( x = 0 \), then (9) gives \( 0^2 - y^2 = 1 \), but this has no solution! In other words, no point with \( x = 0 \) belongs to the constraint, so we won’t get any candidate points from this option.

The solutions to the Lagrange Multiplier equations are therefore \( (x, y) = (\sqrt{\frac{5}{4}}, -\frac{1}{2}) \), and \( (-\sqrt{\frac{5}{4}}, -\frac{1}{2}) \).

The associated function values at these points are:

- \( f(\sqrt{\frac{5}{4}}, -\frac{1}{2}) = x^2 + y = \frac{5}{4} - \frac{1}{2} = \frac{3}{4} \)
- \( f(-\sqrt{\frac{5}{4}}, -\frac{1}{2}) = x^2 + y = \frac{5}{4} - \frac{1}{2} = \frac{3}{4} \)

Since the constraint is not bounded, it is not trivial to demonstrate that these values are minimums of \( f \) on the constraint. However, with a little mathematical insight it can be done in just a few steps:

\[ f(x, y) = x^2 + y, \]

but we are limited to the constraint \( x^2 - y^2 = 1 \), or \( x^2 = y^2 + 1 \)

Substituting this into \( f \), we get

\[ f(x, y) = (y^2 + 1) + y = y^2 + y + 1 \text{ on the constraint} \]

Completing the square gives

\[ f(x, y) = \left(y + \frac{1}{2}\right)^2 + \frac{3}{4} \]

Since squared values are always positive, we can say that

\[ f(x, y) = \left(y + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} \text{ on the constraint curve} \]

Therefore, the values we found \( (f = \frac{3}{4}) \) are minimums of \( f \) on the constraint.
11. \( f(x, y) = x^2 + 2y^2, x^2 + y^2 \leq 4 \)

Note that we are dealing with an inequality for the constraint. We can consider any point \textbf{in or on} the boundary of a circle with radius 2. To look \textbf{on} the boundary, we use Lagrange multipliers. To look at the \textbf{interior}, we identify the critical points of \( f(x, y) \).

We’ll start with the Lagrange multipliers:

\[
\begin{align*}
    f_x &= 2x \\
    f_y &= 4y
\end{align*}
\]

\[
\begin{align*}
    g_x &= 2x \\
    g_y &= 2y
\end{align*}
\]

Set up the Lagrange multiplier equations:

\[
\begin{align*}
    f_x &= \lambda g_x \quad \Rightarrow \quad 2x = \lambda 2x & \quad (10) \\
    f_y &= \lambda g_y \quad \Rightarrow \quad 4y = \lambda 2y & \quad (11) \\
\text{constraint:} & \quad \Rightarrow \quad x^2 + y^2 = 4 & \quad (12)
\end{align*}
\]

From (10), either \( x = 0 \) or \( \lambda = 1 \). If \( x = 0 \), then (12) says \( y = \pm 2 \). Alternatively, if \( \lambda = 1 \), then (11) means \( y = 0 \), so \( x = \pm 2 \). Our solutions are

\[
(x, y) = (0, 2), (0, -2), (2, 0) \text{ and } (-2, 0)
\]

At these points,

\[
\begin{align*}
    f(0, 2) &= 8 \\
    f(0, -2) &= 8 \\
    f(2, 0) &= 4 \\
    f(-2, 0) &= 4
\end{align*}
\]

Before we can say these are global max or mins, we need to look for critical points in the interior of the circle \( x^2 + y^2 \leq 4 \).

Set \( f_x = 0 \) \quad \Rightarrow \quad 2x = 0 \]

and \( f_y = 0 \) \quad \Rightarrow \quad 4y = 0

The only critical points is \( (0, 0) \), and this is in the interior of the circle. The value of \( f(0, 0) = 0 \).

Combining the results on the boundary with the only critical point we see:

- \( f(0, 2) \) and \( f(0, -2) \) are global maxes with values of \( f = 8 \)
- \( f(0, 0) \) is the global min on the region, with \( f = 0 \).
A contour diagram showing the region and contours of $f$ is included below to illustrate the solution.

![Contour Diagram](image)

17. (a) Draw contours of $f(x, y) = 2x + y$ for $z = -7, -5, -3, -1, 1, 3, 5, 7$.

(b) On the same axes, graph the constraint $x^2 + y^2 = 5$.

(c) Use the graph to approximate the points at which $f$ has a maximum or a minimum value subject to the constraint $x^2 + y^2 = 5$.

(d) Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = 2x + y$ subject to $x^2 + y^2 = 5$.

(a) The contours of $f$ are straight lines with slope $-2$, as shown below.

![Contour Graph](image)

(b) Overlaying the constraint, we are allowed to move on a circle of radius $\sqrt{5}$.

![Overlay Graph](image)

(c) From the graph, the maximum values occur where the constraint circle just touches the $f = 5$ contour line, at $(x, y) = (2, 1)$. The minimum value is $f = -5$, which occurs on the opposite side of the circle, at $(-2, -1)$.
(d) Compare your answers to parts (b) and (c)

\[ f_x = 2 \quad g_x = 2x \]
\[ f_y = 1 \quad g_y = 2y \]

Set up the Lagrange multiplier equations:

\[ f_x = \lambda g_x \quad \Rightarrow \quad 2 = \lambda 2x \] (13)
\[ f_y = \lambda g_y \quad \Rightarrow \quad 1 = \lambda 2y \] (14)

constraint: \quad \Rightarrow \quad x^2 + y^2 = 5 \quad (15)

Taking (13) / (14), (assuming \( \lambda \neq 0 \))

\[ \frac{2}{1} = \frac{\lambda 2x}{\lambda 2y} = \frac{x}{y} \]

so \( 2y = x \)

Sub into (15) to find

\[ 4y^2 + y^2 = 5 \quad \Rightarrow \quad y = \pm 1 \]

Combining with \( 2y = x \), we get the solutions \( (x, y) = (2, 1) \) and \( (-2, -1) \). These are the same points we found in (c), and knowing their \( z \) values, we know that \( f(2, 1) \) is a maximum while \( f(-2, -1) \) is a minimum on the constraint.

18. A firm manufactures a commodity at two different factories. The total cost of manufacturing depends on the quantities, \( q_1 \) and \( q_2 \), supplied by each factory, and is expressed by the joint cost function,

\[ C = f(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500 \]

The company’s objective is to produce 200 units, while minimizing production costs. How many units should be supplied by each

We want to minimize

\[ C = f(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500 \]

subject to the constraint \( q_1 + q_2 = 200 \) (so \( g(q_1, q_2) = q_1 + q_2 \)).

Since \( \nabla f = (4q_1 + q_2, 2q_2 + q_1) \) and \( \nabla g = (1, 1) \), setting \( \nabla f = \lambda g \) gives

\[ 4q_1 + q_2 = \lambda 1 \]
\[ q_1 + 2q_2 = \lambda 1 \]

Solving, we get

\[ 4q_1 + q_2 = q_1 + 2q_2 \]
so

\[ 3q_1 = q_2. \]

We want

\[ q_1 + q_2 = 200 \]
\[ q_1 + 3q_1 = 4q_1 = 200 \]

Therefore,

\[ q_1 = 50 \]

and

\[ q_2 = 150 \]

From the problem statement, we can conclude that this production level will minimize the total manufacturing cost, given the desired size of production run.

24. The director of a neighborhood health clinic has an annual budget of $600,000. He wants to allocate his budget so as to maximize the number of patient visits, \( V \), which is given as a function of the number of doctors, \( D \), and the number of nurses, \( N \), by

\[ V = 1000D^{0.6}N^{0.3} \]

A doctor’s salary is $40,000; nurses get $10,000.

(a) Set up the director’s constrained optimization problem.

(b) Describe, in words, the conditions which must be satisfied by \( \frac{\partial V}{\partial D} \) and \( \frac{\partial V}{\partial N} \) for \( V \) to have an optimum value.

(c) Solve the problem formulated in part (a)

(d) Find the value of the Lagrange multiplier and interpret its meaning in this problem.

(e) At the optimum point, what is the marginal cost of a patient visit (that is, the cost of an additional visit)? Will that marginal cost rise or fall with the number of visits? Why?

(a) The problem is to maximize

\[ V = 1000D^{0.6}N^{0.3} \]

subject to the budget constraint that

\[ 40,000 \ D + 10,000 \ N \leq 600,000 \]

This will be easier to deal with if we divide by 10,000, so

\[ 4D + N \leq 60 \]

Since there our function, \( V \), always grows larger with larger \( N \) and \( D \), there is no point in using less than our budget, so we want to find the maximum value of \( V \) on the line \( 4D + N = 60 \). In this problem, then, our constraint function is \( g(D, N) = 4D + N \).
(b) We want $\nabla V$ to point in the same direction as $\nabla g$, or mathematically that

$$V_D = \lambda g_D$$
$$V_N = \lambda g_N$$

while satisfying the constraint $D + N = 60$

(c)

$$V_D = 1000(0.6)D^{-0.4}N^{0.3}$$
$$V_N = 1000(0.3)D^{0.6}N^{-0.7}$$

$g_D = 4$
$g_N = 1$

Set up the Lagrange multiplier equations:

$$V_D = \lambda g_D \implies 1000(0.6)D^{-0.4}N^{0.3} = \lambda 4 \quad (16)$$
$$V_N = \lambda g_N \implies 1000(0.3)D^{0.6}N^{-0.7} = \lambda 1 \quad (17)$$
$$\text{constraint:} \implies 4D + N = 60 \quad (18)$$

Taking (16) / (17), (assuming $\lambda \neq 0$)

$$\frac{1000(0.6)D^{-0.4}N^{0.3}}{1000(0.3)D^{0.6}N^{-0.7}} = \frac{4}{1}$$
$$2D^{-1}N = 4$$
$$N = 2D$$

Sub into (18) to find

$$4D + 2D = 60 \implies D = 10$$

Combining with $N = 2D$, we get the solution that Nurses ($N$) = 20 and Doctors ($D$) = 10. This results in $V = 1000(10^{0.6})(20^{0.3}) \approx 9,779$ visits per year.

(d) From (c), we can find $\lambda a$ using (2):

$$\lambda = 1000(0.3)(10)^{0.6}(20)^{-0.7} \approx 146$$

Since $\lambda = \nabla V / \nabla g$, it represents the change of visitors for each change in budget. Since our units for the budget constraint were $10,000$ (remember, we divided by $10,000$ to simplify in (c)), this means that increasing the budget by one unit (of $10,000$) will result in handling $\approx 146$ more visits. More generally, we can handle approximately $0.0146$ more patients per dollar increase in budget.

(e) The marginal cost (dollars per patient) is the inverse of the quantity in (d) (patients per dollar). Thus each new patient costs roughly $1 / (0.0146 \text{ patients/dollar}) \approx $68.5 per patient.

The analysis of how this marginal rate depends on $D$ and $N$ is beyond the scope of this course.
26. Each person tries to balance his or her time between leisure and work. The tradeoff is that as you work less your income falls. Therefore each person has indifference curves which connect the number of hours of leisure, l, and income, s. If, for example, you are indifferent between 0 hours of leisure and an income of $1125 a week on the one hand, and 10 hours of leisure and an income of $750 a week on the other hand, then the points \( l = 0, s = 1125 \), and \( l = 10, s = 750 \) both lie on the same indifference curve. Table 15.3 gives information on three indifference curves, I, II, and III.

(a) Graph the three indifference curves.

(b) You have 100 hours a week available for work and leisure combined, and you earn $10/ hour. Write an equation in terms of \( l \) and \( s \) which represents this constraint.

(c) On the same axes, graph this constraint.

(d) Estimate from the graph what combination of leisure hours and income you would choose under these circumstances. Give the corresponding number of hours per week you would work.

Table 15.3

<table>
<thead>
<tr>
<th>Weekly income</th>
<th>Weekly leisure hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td>1125</td>
<td>1250</td>
</tr>
<tr>
<td>750</td>
<td>875</td>
</tr>
<tr>
<td>500</td>
<td>625</td>
</tr>
<tr>
<td>375</td>
<td>500</td>
</tr>
<tr>
<td>250</td>
<td>375</td>
</tr>
</tbody>
</table>

(a) The graphs are shown, along with the constraint from part (c), below.

(b) Since you’re earning $10 per hour, and \( s \) is your income, \( s/10 \) is the number of hours worked. To limit yourself to 100 hours per week, you must satisfy \( l + s/10 = 100 \)

(c) See the graph from (a)

(d) Since the constraint line just touches the indifference curve II at \( t = 50, s = 500 \), we can’t achieve a higher level of satisfaction than level II. To achieve that level of satisfaction, we should split our time into \( t = 50 \) hours of leisure, and \( s/10 = 500/10 = 50 \) hours of work.
30. A mountain climber at the summit of a mountain wants to descend to a lower altitude as fast as possible. The altitude of the mountain is given approximately by

\[ h(x, y) = 3000 - \frac{1}{10,000} (5x^2 + 4xy + 2y^2) \text{ meters} \]

where \( x, y \) are horizontal coordinates on the earth (in meters), with the mountain summit located above the origin. In thirty minutes, the climber can reach any point \((x, y)\) on a circle of radius 1000 m. In which direction should she travel in order to descend as far as possible?

The mountain climber can reach anywhere in the circle \( x^2 + y^2 \leq 1000^2 \) in the half hour.

We want to find the minimum value of \( h \) on (or inside) that circle.

To identify any local minima inside the boundary, we first look for critical points:

\[
\begin{align*}
    h_x &= \frac{10x + 4y}{10,000} \\
    h_y &= \frac{4y + 4x}{10,000}
\end{align*}
\]

Setting both equal to zero,

\[
\begin{align*}
    \frac{10x + 4y}{10,000} &= 0 \\
    5x &= -2y
\end{align*}
\]

\[
\begin{align*}
    \frac{4y + 4x}{10,000} &= 0 \\
    y &= -x
\end{align*}
\]

The only solution to these equations is \( x = 0, y = 0 \). This we were already told is the local (and global) maximum at the origin, so we can ignore this in our solution.

Since there are no local minima within the circle the hiker can reach, we now look on the boundary of their 1,000 m reachable circle to determine the lowest she can travel, using Lagrange multipliers. The constraint is \( x^2 + y^2 = 1000^2 \), or \( g(x,y) = x^2 + y^2 \):

\[
\begin{align*}
    h_x &= \frac{10x + 4y}{10,000} \\
    h_y &= \frac{4y + 4x}{10,000} \\
    g_x &= 2x \\
    g_y &= 2y
\end{align*}
\]

Set up the Lagrange multiplier equations:

\[
\begin{align*}
    h_x &= \lambda g_x &\Rightarrow \quad \frac{10x + 4y}{10,000} &= \lambda 2x \\
    h_y &= \lambda g_y &\Rightarrow \quad \frac{4y + 4x}{10,000} &= \lambda 2y \\
    \text{constraint:} &\Rightarrow \quad x^2 + y^2 = 1000^2
\end{align*}
\]
Taking (19) / (20), (assuming \( \lambda \neq 0 \))

\[
\begin{align*}
\frac{10x + 4y}{4y + 4x} &= \frac{\lambda 2x}{\lambda 2y} = \frac{x}{y} \\
\text{so } (10x + 4y)y &= x(4y + 4x) \\
5xy + 2y^2 &= 2xy + 2x^2 \\
2y^2 + 3xy - 2x^2 &= 0
\end{align*}
\]

Factorizing: \((2y - x)(y + 2x) = 0\)

So either \(2y = x\) or \(y = -2x\). Using the constraint equation, that gives the possibilities

\[
\begin{align*}
2y = x & \implies 4y^2 + y^2 = 1000^2 \quad y \approx \pm 447, x \approx \pm 894 \ (x, y \text{ with same signs}) \\
y = -2x & \implies x^2 + 4x^2 = 1000^2 \quad x \approx \pm 447, y \approx \pm 894 \ (x, y \text{ with opposite signs})
\end{align*}
\]

Substituting these values into the height function, \(h(x, y)\), we find that the points with the lowest \(h\) values are \((894, 447)\) and \((-894, -447)\), giving a height of 2,400 meters. The other points give higher heights, of around 3,900 meters. This means that the hiker should leave the point \((0,0)\) heading towards either of the points \((894, 447)\) or \((-894, -447)\). At the end of the half hour, she will be as low as she can be, given her walking speed and the shape of the mountain.

37. For each value of \(\lambda\) the function \(h(x, y) = x^2 + y^2 - \lambda(2x + 4y - 15)\) has a minimum value \(m(\lambda)\).

(a) Find \(m(\lambda)\).

(b) For which value of \(\lambda\) is \(m(\lambda)\) the largest and what is that maximum value?

(c) Find the minimum value of \(f(x, y) = x^2 + y^2\) subject to the constraint \(2x + 4y = 15\) using the method of Lagrange multipliers and evaluate \(\lambda\).

(d) Compare your answers to parts (b) and (c).

(a) When we are looking for \(m(\lambda)\), it means that we can treat \(\lambda\) as a constant in our function, since it will be provided later. That means we need to optimize over \(x\) and \(y\), given \(\lambda\).

There is no \((x, y)\) constraint in this problem, so we simply look for critical points of \(h\), treating \(\lambda\) as a constant.

\[
\begin{align*}
h_x &= 2x - 2\lambda \\
h_y &= 2y - 4\lambda
\end{align*}
\]

Setting both equal to zero, we get

\[
\begin{align*}
0 &= 2x - 2\lambda \\
0 &= 2y - 4\lambda
\end{align*}
\]

Solving gives \(x = \lambda\) and \(y = 2\lambda\)
So there is only one critical point, at \((x, y) = (\lambda, 2\lambda)\). We can determine the type of critical point with the second derivative test:

\[
h_{xx} = 2, h_{yy} = 2, h_{xy} = 0
\]

so \(D = (2)(2) - 0^2 = 4 > 0\)

and \(h_{xx} > 0\) (concave up)

meaning \((x, y) = (\lambda, 2\lambda)\) is a local minimum.

To find the actual value of \(h\) at the critical point, we sub in the coordinates of the critical point into the original formula:

\[
h(\lambda, 2\lambda) = \lambda^2 + (2\lambda)^2 - \lambda(2\lambda + 4(2\lambda) - 15)
\]

\[
= \lambda^2 + 4\lambda^2 - 2\lambda^2 - 8\lambda^2 + 15\lambda
\]

\[
= -5\lambda^2 + 15\lambda
\]

\[
= 5\lambda(3 - \lambda)
\]

so \(m(\lambda) = 5\lambda(3 - \lambda)\)

(b) Now we get to select \(\lambda\) to make \(m\) as large as possible. Since \(m\) is only a 1D function, we can simply differentiate and set the derivative equal to zero. (Alternatively, we could notice that \(m\) is a parabola in \(\lambda\), and will have its maximum halfway between its roots of \(\lambda = 0\) and \(\lambda = 3\). That trick only works because \(m\) is quadratic, though, so we’ll use the more general derivative approach.)

\[
\frac{dm}{d\lambda} = -10\lambda + 15
\]

Set derivative equal to zero: \(0 = -10\lambda + 15\)

\[
\lambda = \frac{15}{10} = 1.5
\]

From \(m\) being a quadratic with negative \(\lambda^2\) coefficient, we know this value of \(\lambda\) gives a maximum of \(m\). The value of \(m(1.5) = 11.25\).

(c) Optimize \(f(x, y) = x^2 + y^2\) subject to \(g(x, y) = 2x + 4y = 15\).

\[
f_x = 2x \\
f_y = 2y \\
g_x = 2 \\
g_y = 4
\]

Set up the Lagrange multiplier equations:

\[
f_x = \lambda g_x \quad \Rightarrow \quad 2x = 2\lambda \quad (22)
\]

\[
f_y = \lambda g_y \quad \Rightarrow \quad 2y = 4\lambda \quad (23)
\]

constraint: \(\Rightarrow \quad 2x + 4y = 15 \quad (24)\)
Taking (22) / (23), (assuming $\lambda \neq 0$)

$$\frac{2x}{2y} = \frac{\lambda^2}{\lambda^4}$$

so $x = \frac{y}{2}$

Sub into (24) to find

$$2 \left( \frac{y}{2} \right) + 4y = 15 \implies y = 3$$

Combining with $x = y/2$, we get the solutions $(x, y) = (1.5, 3)$. The value of $\lambda$ is then $\lambda = x = 1.5$.

Showing that this is a minimum of $f$ requires only noticing that if we move away from this point, $x^2 + y^2$ will grow larger towards infinity. Thus, we must be at a local (and global) minimum of $f$ given the constraint.

(d) We notice that the solutions to both of these problems, (a,b) and (c), are identical. This indicates that there may be alternative ways to set up or interpret constrained optimization problems. The details of these relationships are beyond the scope of this course, though.