Week 11 & 12: Optimization and Modeling, Trigonometric Functions

Goals:

- Discuss optimizations problems in economics
- Review trigonometric functions
- Study the derivatives of trigonometric functions.

Suggested Readings: Chapter 13: §13.6, and Trigonometric Functions (on-line notes)

- Review: 7, 11, 17, 23, 27, 33, 35*, 49, 55, 61, 63, 67, 69
- Practice questions in the on-line notes - Trigonometric Functions.

Note: the function given in question 35 is different in two editions.

35. (12th) \( f(x) = (x^2 + 1)e^{-x}. \)

35. (13th) \( f(x) = x \ln x. \)
Applied Maxima and Minima (continued)

To prepare for this topic, please read section §13.6 in the textbook.

Example 1: (The discussion at the end of §13.6) Recall that the total profit function \( P \) is defined as \( P(x) = R(x) - C(x) \), where \( R \) is the total revenue and \( C \) is the total cost when \( x \) units of product produced and sold.

(a) Show that at the level of production \( x_0 \) that yields the maximum profit for the company, the following two conditions are satisfied:

\[
R'(x_0) = C'(x_0) \quad \text{and} \quad R''(x_0) < C''(x_0)
\]

If \( x_0 \) is a max., then it must be a critical value, so

\[
P'(x_0) = R'(x_0) - C'(x_0) = 0 \quad \text{i.e.} \quad R'(x_0) = C'(x_0). \quad \text{or} \quad \frac{MR}{x_0} = \frac{MC}{x_0}
\]

\( x_0 \) is a max. so \( P''(x_0) < 0 \)

\[
P''(x_0) = R''(x_0) - C''(x_0) < 0
\]

It follows \( R''(x_0) < C''(x_0) \) or \( MR'(x_0) < MC'(x_0) \)

(b) Interpret the two conditions in part (a) in economic terms and explain why they make sense.

If \( R' \geq C' \), i.e. \( MR \geq MC \), we want to increase production to increase profit. If \( R' < C' \), i.e. \( MR < MC \), we want to decrease production to prevent decline of profit. So max profit happens when \( R' = C' \), i.e. \( MR = MC \), and at which the rate of change of \( MR \) is less than the rate of change of \( MC \). To see this, if \( x_0 \) is a critical pt., we have \( MR = MC \) on the left of \( x_0 \). On the right we have \( MR > MC \) and on the right of \( x_0 \), we have \( MR < MC \). So it follows that either \( MR \) dec or \( MC \) inc or both. So \( MR' < MC' \).
Example 2: The marginal revenue and marginal cost for a certain item are graphed below. Do the following quantities maximize profit for the company?

\[
\text{at } x=a \text{ & } x=b \quad MR = MC
\]

So there are two critical points.

Since \( MR' = 0 \) and at \( x=a \) \( MC' < 0 \), we have

at \( x=a \) \( MR' > MC' \). So \( x=a \) is not the max.

at \( x=b \). \( MC' > 0 \). So \( MR' < MC' \) at \( x=b \).

So \( x=b \) the profit is max at \( x=b \).
Next we look at some examples in which a function is given graphically and the optimum values are read from a graph.

Optimization without formulae

Example 3: (December Exam 2004) You are starting your own business doing tutoring for high school kids. If you spend \( x \) dollars on advertising, the number of hours \( H \) of business you receive is given by the graph below.

![Graph showing a function \( H(x) \) with a tangent line and a point (600, 50) with slope 1/2.

Suppose that you change $12 per hour.

(a) What is your profit function? How much should you spend on advertising to maximize your profit?

Revenue: \( 12H(x) \)

Cost: \( x \)

Profit: \( P = 12H(x) - x \)

Goal: \( \max P \)

- Find critical value(s):
  \[ P' = 12H'(x) - 1 = 0 \]
  \[ H'(x) = \frac{1}{12} \]
  This is to say that at the critical value the tangent of \( H(x) \) is \( \frac{1}{12} \).
  By the graph, \( x \approx 380 \) and it should give us the max profit.
  To verify we can use the 2nd derivative test.

- Verify
  \[ P'' = 12H''(x) \]
  Since \( H(x) \) is concave down, \( H'' < 0 \).
  So \( P'' < 0 \). So at \( x = 380 \), \( P \) is a max.
(b) Suppose a tax of 10% is applied on your revenue. How much should you spend on advertising to maximize your profit?

\[
\text{Revenue: } 12H(x) \\
\text{Cost: } x + 0.1(12H(x)) = x + 1.2H(x) \\
\text{Profit: } P = 12H(x) - [x + 1.2H(x)] = 10.8H(x) - x
\]

**Goal:** \(\max P\):

- Find all critical point(s).

\[P' = 10.8H'(x) - 1 = 0 \Rightarrow H'(x) = \frac{1}{10.8}\]

To find the point we draw a line from \((0,0)\) with slope \(\frac{1}{10.8}\). For example, we may take point \((24,30)\) and connect it with \((0,0)\). Then find a tangent of \(H(x)\) which is parallel to the line. By the graph @ \(x=400\), \(H'(x) = \frac{1}{10.8}\). So \(x = 400\) is the critical pt.

And the profit is a max at \(x=400\). To verify it, we may use 2nd derivative test similar to what we did in part (a).
Example 4: (December Exam 2004) Your scheme to raise money for the field trip is to make scarves in the school colours and sell them to students. You intend to spend the long weekend producing scarves of a fixed width. Your only cost is the wool at $4 per meter of scarf length. The question is, how long should you make each scarf to maximize your profit? The total number of meters you can produce is a fixed number L, so the longer you make each scarf, the fewer scarves you will get, but the more you can sell each scarf for. The following graph indicates the relationship between the length x of a scarf (in meters) and its selling price S (in dollars). Note that no one will buy a scarf less than 50 cm in length, and scarves that are too long are unwieldy.

Assume you have no problem to sell all your scarves.

(a) Write an expression for your total profit function.

\[
\text{Revenue: } \quad S(x) \cdot \frac{L}{x} \quad \text{cost: } \quad \#L
\]

\[
\text{Profit: } \quad P = S(x) \cdot \frac{1}{x} - \#L
\]

(b) Find a construction on the graph above which maximizes your profit.

The profit function is given by:

\[
P'(x) = -\frac{1}{x^2} \left( S(x) - S(x) \right) = -\frac{1}{x^2} \left( S(x) - S(x) \right)
\]

Let \( P' = 0 \). We have \( S'(x) = S(x) \). Geometrically, \( S'(x) \) is the slope of the tangent of \( S(x) \) and note that \( \frac{S'(x)}{x} = \frac{S(x) - 0}{x - 0} \). It is the slope of the line through \((0,0)\) and \((x,S(x))\); so the critical point is the point on \( S(x) \) at which the tangent of \( S(x) \) passes through \((0,0)\).

By the graph, \( x \approx 1.7 \), so the profit is a max if each scarf is made 1.7 meters long. It can be verified by the 2nd derivative test (which is omitted here).
(c) Suppose you have to pay your school $5 per scarf sold for the use of the school colors, and you decide to keep your price the same as before. What is your new profit function? If you wish to maximize your profit, how long should you make each scarf?

\[ S(x) = \frac{1}{x} \]

\[ \text{Cost} : \quad 4L + 5 \cdot \frac{1}{x} \]

\[ \text{Profit} : \quad P = S(x) - 4L - 5 \cdot \frac{1}{x} = L \cdot \frac{S(x) - 5}{x} - 4L \]

Goal: \[ \max P \]

- Find all critical points:

\[ P' = L \cdot \frac{\left(S'(x) - 0 \right) x - \left[S(x) - 5\right] \cdot 1}{x^2} - 0 = L \cdot \frac{xS'(x) - [S(x) - 5]}{x^2} \]

Let \[ P' = 0 \], we have \[ xS'(x) - [S(x) - 5] = 0 \] that is

\[ S'(x) = \frac{S(x) - 5}{x} \]

Geometrically, \[ S'(x) \] is the slope of the tangent of \( S(x) \) and also note that \[ \frac{S(x) - 5}{x} = \frac{S(x) - 5}{x - 0} \] is the slope of the line through \( (0, 5) \) and \( (x, S(x)) \) a point on the \( S(x) \) curve. Some critical point satisfies \[ S'(x) = \frac{S(x) - 5}{x} \] that is, at the critical point, (the slope of the tangent of \( S(x) \) is the same as the line through \( (0, 5) \) and the pt \( (x, S(x)) \)). The tangent line of \( S(x) \) must also pass through \( (0, 5) \). By the graph, \( x \approx 2 \). The profit is \( \text{max} \) if the scarf is 2 meters long.
Unit Circle and Angles

The **unit circle** is the circle of radius 1 centered at the origin on the \(xy\)-plane. The equation of the unit circle is \(x^2 + y^2 = 1\). An **angle** is formed when one ray in the plane is rotated into the other about their common endpoint. The angle generated is a positive angle if the ray is rotated counterclockwise, and a negative angle if the ray is rotated clockwise. The size of an angle can be represented in either degrees or radians.

**Relationship between Degrees and Radians**

\[
180^\circ = \pi \text{ radians}, \quad 1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ, \quad 1^\circ = \frac{\pi}{180} \text{ radians}
\]

To convert degrees to radians, multiply by \(\frac{\pi}{180}\). To convert radians to degrees, multiply by \(\frac{180}{\pi}\).

**Example 5:**

<table>
<thead>
<tr>
<th>Radians</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\pi}{4})</td>
<td>45°</td>
</tr>
<tr>
<td>(\frac{\pi}{6})</td>
<td>30°</td>
</tr>
<tr>
<td>(\frac{\pi}{2})</td>
<td>90°</td>
</tr>
<tr>
<td>(\frac{2\pi}{3})</td>
<td>120°</td>
</tr>
<tr>
<td>(\frac{5\pi}{6})</td>
<td>150°</td>
</tr>
</tbody>
</table>

**Length of a Circular Arc**

In a circle of radius \(r\), the length \(s\) of an arc that subtends a central angle of \(\theta\) radians is

\[
s = r\theta
\]
Trigonometric Functions

Right Triangle Definition

\[
\sin \theta = \frac{a}{c}, \hspace{1cm} \csc \theta = \frac{c}{a} = \frac{1}{\sin \theta}
\]

\[
\cos \theta = \frac{b}{c}, \hspace{1cm} \sec \theta = \frac{c}{b} = \frac{1}{\cos \theta}
\]

\[
\tan \theta = \frac{a}{b}, \hspace{1cm} \cot \theta = \frac{b}{a} = \frac{1}{\tan \theta}
\]

Example 6: Find trigonometric ratios of \( \theta = 0, \frac{\pi}{6}, \frac{\pi}{2} \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \csc \theta )</th>
<th>( \sec \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
<td>1</td>
<td>( \infty )</td>
</tr>
<tr>
<td>\frac{\pi}{6}</td>
<td>\frac{1}{2}</td>
<td>\frac{\sqrt{3}}{2}</td>
<td>\frac{\sqrt{3}}{3}</td>
<td>2</td>
<td>\frac{2}{\sqrt{3}}</td>
<td>\frac{\sqrt{3}}{2}</td>
</tr>
<tr>
<td>\frac{\pi}{2}</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
<td>1</td>
<td>( \infty )</td>
<td>0</td>
</tr>
</tbody>
</table>

Unit Circle Definition

Let \( t \) be any real number. Let’s mark off a distance \( \theta \) along the unit circle, starting at the point \((1,0)\) and moving in a counterclockwise direction if \( \theta \) is positive or in a clockwise direction if \( \theta \) is negative. In this way we arrive at a point \( P(x, y) \) on the unit circle. The point \( P(x, y) \) is called the terminal point determined by the number \( \theta \). We define

\[
\sin \theta = y, \hspace{1cm} \cos \theta = x, \hspace{1cm} \tan \theta = \frac{y}{x}, \hspace{1cm} \csc \theta = \frac{1}{y}, \hspace{1cm} \sec \theta = \frac{1}{x}, \hspace{1cm} \cot \theta = \frac{x}{y}
\]

Example 7: Evaluate the trigonometric functions \( y = \sin t, \ y = \cos t \) and \( y = \tan t \) at \( t = \frac{\pi}{4} \)

\[
\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}
\]

\[
\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}
\]

\[
\tan \frac{\pi}{4} = 1
\]
Special Values of the Trigonometric Functions

<table>
<thead>
<tr>
<th>$\theta$ in radians</th>
<th>$0$</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
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<th>$\frac{3\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>0</td>
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<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>0</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>1</td>
<td>$\sqrt{3}$</td>
<td>$\infty$</td>
<td>$-\sqrt{3}$</td>
<td>-1</td>
<td>$-\frac{1}{\sqrt{3}}$</td>
<td>0</td>
</tr>
</tbody>
</table>
The Graphs of sine, cosine and tangent Functions

The graphs of sine, cosine and tangent functions are as follows.

![Graphs of sine, cosine, and tangent functions](image)

Transformations of the Graphs of Trigonometric Functions

\[
\sin(x + \frac{\pi}{2}) = \cos x \quad \sin(x + \pi) = -\sin x
\]
Similarly we have
\[ \cos(t + \frac{\pi}{2}) = -\sin t \quad \text{and} \quad \cos(t + \pi) = -\cos t \]

**Inverse Trigonometric Functions**

By their nature of being periodic functions, trigonometric functions are not one-to-one. However, we can restrict the domain so that the trigonometric function is one-to-one and then we can find its inverse function.

<table>
<thead>
<tr>
<th>Trigonometric Inverse Functions</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \arcsin x )</td>
<td>([-1, 1])</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
</tr>
<tr>
<td>( y = \arccos x )</td>
<td>([-1, 1])</td>
<td>([0, \pi])</td>
</tr>
<tr>
<td>( y = \arctan x )</td>
<td>([-\infty, \infty])</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
</tr>
</tbody>
</table>

**Fundamental Trigonometric Identities**

\[ \sin^2 t + \cos^2 t = 1, \quad \tan^2 t + 1 = \sec^2 t, \quad \tan t = \frac{\sin t}{\cos t} \]

\( \sin(\arcsin x) = x \quad \text{for} \quad -1 \leq x \leq 1, \quad \arcsin(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \)

\( \cos(\arccos x) = x \quad \text{for} \quad -1 \leq x \leq 1, \quad \arccos(\cos x) = x \quad \text{for} \quad 0 \leq x \leq \pi, \)
**Derivatives of Trigonometric Functions**

In this section we develop rules for differentiating trigonometric functions. To derive the rule for differentiating the sine function, we need the following two important limits.

\[
\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0
\]

**Example 8:** Find the derivative of the function \( y = \sin x \) at \( x = 0, \frac{\pi}{2}, \) and \( \pi \)

\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin h - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h - 0}{h} = \sin 1
\]

\[
f'\left(\frac{\pi}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2} + h\right) - f\left(\frac{\pi}{2}\right)}{h} = \lim_{h \to 0} \frac{\sin \left(\frac{\pi}{2} + h\right) - \sin \frac{\pi}{2}}{h} = \lim_{h \to 0} \frac{\cos h - 1}{h} = -\sin \frac{\pi}{2} = 0
\]

\[
f'(\pi) = \lim_{h \to 0} \frac{f(\pi + h) - f(\pi)}{h} = \lim_{h \to 0} \frac{\sin (\pi + h) - \sin \pi}{h} = \lim_{h \to 0} \frac{-
\sin h - 0}{h} = -\sin 1
\]

**Example 9:** Draw the graph of the derivative function of \( y = \sin t \).
From our graphical intuition, we are lead to the following elegant result:

The derivative of the sine function $y = \sin x$ is $y' = \cos x$, i.e.,

$$\frac{d}{dx} \sin x = \cos x$$

---

**Example 10:** *Differentiate.*

(a) $y = \sin 3x$

$$y' = (\cos 3x) \cdot 3 = 3 \cos 3x$$

(b) $y = \cos x$

$$\text{Since } \cos x = \sin \left(\frac{\pi}{2} + x\right)$$

$$(\cos x)' = \left( \sin \left(\frac{\pi}{2} + x\right) \right)' = \cos \left(\frac{\pi}{2} + x\right) \cdot 1 = -\sin x$$

(c) $y = \cos(x^2)$

$$y' = -\sin(x^2) \cdot 2x = -2x \sin(x^2)$$

---

**Example 11:** *Prove that $(\tan x)' = \frac{1}{\cos^2 x}$.*

Since $\tan x = \frac{\sin x}{\cos x}$

$$\left(\tan x\right)' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$
The derivative of the cosine and tangent functions
\[ \frac{d}{dx} \cos x = -\sin x \]
\[ \frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x \]

Example 12: The revenue of a fish farm is approximately
\[ R(t) = 2 \left( 5 - 4 \cos \frac{\pi t}{6} \right) \quad 0 \leq t \leq 12 \]
during the t-th month, where R is measured in thousands of dollars.
(a) Find the marginal revenue function MR.

(b) Find the values of MR at t = 3 and t = 9.
Now we derive the derivative of the inverse trigonometric functions.

**Example 13:** Prove that \( \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \) where \( |x| < 1 \).

Since \( \sin(\arcsin x) = x \)

diff. both sides

\[
\cos(\arcsin x) \cdot \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}}
\]

\[
\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}
\]

The derivative of the inverse trigonometric functions

\[
\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}, \quad |x| < 1
\]

\[
\frac{d}{dx} \arctan x = \frac{1}{1+x^2}
\]